Proceedings of the 15th International Conference on Technology in Mathematics Teaching (ICTMT 15)

Making and Strengthening "Connections and Connectivity" for Teaching Mathematics with Technology

Danish School of Education, Aarhus Unviersity

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Editors

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ABOUT THESE PROCEEDINGS

Editors

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INTRODUCTION TO THE PROCEEDINGS OF ICTMT 15

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FACTS AND FIGURES

The Fifteenth International Conference on Technology in Mathematics Teaching (ICTMT 15) took place on September 13–16, 2022, in the Danish School of Education, Aarhus University, located on campus Emdrup, in the Northwestern district of Copenhagen, Denmark. There were a total of 66 participants from 15 different countries.

The scientific programme consisted of plenary lectures, paper sessions, a poster session and hands-on workshops. The four plenary lectures related to each of the four themes of the conference. Thirty-one papers were presented and discussed throughout the week. The poster session included eight contributions. Furthermore, participants had the chance to join one or two of the 11 workshop activities.

The conference also included a rich social programme in the city of Copenhagen. On Monday, 13 September, the poster session was accompanied by a wine reception sponsored by Maplesoft. On Tuesday, 14 September afternoon, participants went for a walk-and-talk near campus, leading to a visit to Grundtvig's Church. The conference excursion was a bus and ferry tour combo throughout the streets and canals of Copenhagen on Wednesday, 15 September. The trip's last stop was Christianshavn, within walking distance of the conference dinner venue, namely the restaurant Spiseloppen, located in the free town Christiania.

CONNECTIONS AND CONNECTIVITY

ICTMT 15 certainly focused on the impacts that the coronavirus pandemic has had on global mathematics education. However, it looked at the impacts of digital technology from a much wider perspective. In particular, the conference aimed to highlight how technology facilitates the multiple "Connections and Connectivity" between us all to achieve the goals of purposeful mathematics education in the early 21st century.

By "Connections" we mean the interrelationships between researchers, teachers, students, parents, policymakers, and industry (big and small). "Connectivity" includes oral, aural, textual and gestural communications as mediated by the internet, learning environments and classroom activities. Together, "Connections and Connectivity" describes the relationships between people, between different ideas and strategies to teach, and between people and environments. It offers a frame through which to interpret assessment in mathematics education as a more formative process from the point of view of both teachers and students.

Within the overarching frame of "Connections and Connectivity", the conference concerned four themes that give structure to these proceedings and which we describe in the following paragraphs.

Designing technology

The first theme addressed the design of technology for mathematical learning and its assessment—a focus on theoretical or actual 'designs' with contributions from researchers, industry and teachers.

Chronis Kynigos gave his plenary lecture situating concrete digital tool designs within strong theoretical underpinnings of post-normal science. He challenges the role of mathematics and mathematics education in a world surrounded by wicked problems. As a response, his talk focuses on "Choices with consequences" (ChoiCo), a digital tool where students have the chance to grapple with mathematical aspects inside socio-scientific games.

Some contributions to this theme reported students' experiences as part of the design of digital learning environments for engaging with mathematical ideas. Some examples are: the case of an algebraic modelling web tool for relational thinking (Oldenburg), a computer-based learning environment for mathematical modelling (Frenken), metaphor-based animations for algebra (Bos & Renkema) outdoors and home versions of an applet for math trails (Jablonski et al.; Larmann et al.), an applet for preformal proving (Platz), and error-inducing interactive videos (Schirmer et al.). Some had a particular focus on feedback, by making sense of it in a multimodal algebra learning system (Reid et al.) and producing it semi-automatically for handwritten tasks (Moons & Vandervieren).

Others focused on designs of mathematical tasks making use of digital tools. These included tasks using GeoGebra's algebra view (Gregersen), silent video tasks (Kristinsdóttir et al.), tasks for integrating programming and computational thinking (Elicer & Tamborg), and creative tasks with digital-media (Diamantidis & Kynigos).

Making sense of 'classroom' practice

The second theme aimed at making sense of 'classroom' practices with and through technology—a focus on the work of teachers and lecturers, where the classroom might be geographically located or mobile. Again, contributions could be both theoretical and practical.

In her plenary lecture, Anna Baccaglini-Frank set the scene in the distance-teaching context in Italy resulting from the Covid-19 pandemic. She re-examined Ruthven's (2012) claim that technologies "are not strongly framed in didactic terms (…); nevertheless, in practice, they are often appropriated to a reproductive didactic" (p. 629). In that sense, she challenged the theme by advocating for a shift from teaching mathematics *with* technology to teaching mathematics *through* technology.

In this theme, most contributions focused on the role of digital technologies for different purposes of students' development, including the mathematical thinking competency (Thomsen & Jankvist; Pedersen), the notion of STEAM (Ferrara et al.) and Allgemeinbildung (Johansen).

Some authors looked at how technologies can prompt issues of classroom practice, such the effect of digital textbooks in the gender gap (Brnic & Greefrath), graphing calculators in connecting geometry and functions (Subtil et al.) and computer algebra systems in conjecturing and proving theorems (Szücs).

Another group of contributions focused on mathematics teachers' interactions with new technologies. Two studies took a professional development perspective concerning the inclusion of computational thinking (Nøhr et al.) and pre-service teachers' experiences at an online school (Tunç-Pekkan et al.). Another two studies zoomed into the orchestrating role of teachers mediated by a videogame (Vilchez & Lemmo) and a distance learning context (Faggiano & Mennuni).

Fostering mathematical collaborations

The third theme was concerned with the fostering of mathematical collaborations with and through technology—a focus on the communications aspect of technology, including assessment strategies.

Shai Olsher positioned his plenary talk as a concrete application of topic-specific learning analytics to foster collaborations between students *through* technology. By focusing on geometrical example-eliciting tasks, his research group defined automatic assessment-based recommendations for grouping students with different pedagogical purposes. His study displays content-informed group categories and implications for teaching.

Some contributions focused on connectivity issues, such as a platform for online teacher education (Tunç-Pekkan et al.), modularised mathematics courses for engineering (Kiliç et al.) and heuristic worked example videos in a collaborative setting (Wirth & Greefrath). Other contributions were centred on particular mathematical communicative aspects mediated by digital tools, including handwriting in tablet-computers with smartpens (Schüler-Meyer), and digital geometry environments (Bach & Bikner-Ahsbahs).

Innovating with technologies

The fourth theme dove into innovating with technologies for mathematical learning—a focus on highly innovative approaches in the early stages of development for constructive critique by the community.

As a sharp example of such innovations was given by Dan Meyer in his plenary presentation entitled "Pixels are pedagogy". Joining us virtually from California, he introduced the platform Desmos as a way of questioning two common beliefs; namely, that mathematics is a purely objective discipline and that technology is a morally neutral actor. The speaker described the pedagogical decisions that underpinned the design of the platform by enabling the audience to experience it first-hand.

Some contributions to this theme focused on digital learning environments for mathematical modelling (Frenken & Greefrath), linear functions (Barana) and deductive geometry (Ballin & Kouropatov). Others discussed how state-of-the-art technologies and constructs intertwine with mathematics teaching and learning, here among mobile devices (Ludwig et al.), virtual and mixed reality (Dilling & Sommer), data science (Podworny & Fleischer), machine learning (Fleischer & Podworny), computer-aided assessment (Fahlgren et al.; Klingbeil et al.), and programming and computational thinking (Tamborg et al.).

THE ICTMT SERIES

This biennial conference began in Birmingham, UK, in 1993, under the influential enterprise of Bert Waits from Ohio State University. The previous instance was held in Essen, Germany, in 2019. ICTMT 16 is set out to be organised and take place at the National and Kapodistrian University of Athens, Greece.

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Theme 1: Designing Technology

for mathematical learning and its assessment

EMBEDDING MATHEMATICS IN SOCIO-SCIENTIFIC GAMES: THE CASE OF THE MATHEMATICAL IN GRAPPLING WITH WICKED PROBLEMS

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This paper discusses the ways in which digitally enabled transformation in mathematics education could envisage a role for rationality in post-normal science and wicked problems. The scene is set firstly by reviewing the ways in which digital media have been designed and used in transformative mathematics education as a rationale for thinking about such media for wicked problem education. The problem is set in epistemological terms, can normal science approaches contribute to post-normal science. Taking into account the basic arguments regarding wicked problem education, I focus on the discussion of a specific constructionist digital tool called 'ChoiCo: Choices with Consequences', designed to embed mathematical ideas, facilitate mathematical reasoning, yet be about grappling with wicked problems. The final section discusses student discourse to set the scene for what such reasoning might look like in the context of grappling with wicked problems.

Keywords: Digital media, post-normal science, wicked problems.

WICKED PROBLEMS AS A CHALLENGE FOR TRANSFORMATION IN MATHEMATICS EDUCATION

In recent times, every one of us feels exposed to wicked problems, those universal ill-defined, controversial, complex, value-laden socio-scientific issues such as the pandemic and climate change. Our society is replete with individual and shared stress, denial and inertia, ultimately leading to exponential augmentation of risk for wide-ranging consequences. In Europe, at least, there is a push for educational transformation aiming to provide students with experience in grappling with such issues in a knowledgeable contributory way. In this paper, I discuss the potential role for using digital media to engage in rationality and mathematical thinking as a means of grappling with such issues. Pedagogical transformation is not new to mathematics education, albeit in different ways. So, could mathematical rationality in handling wicked problems be one of the transformation avenues worth addressing in mathematics education?

It has now been 50 years since Papert introduced the idea of fundamentally changing students' experiences with mathematical reasoning through the use of digital media to express, explore and generate mathematical meaning (Papert, 1972). The need for transformation in mathematics education has since then been widely argued from many angles beyond the advent of digital technologies. It has been generally portrayed as a need to move away from overbearing 'visiting the works' paradigms, as Chevallard (2012) would put it, where students are typically exposed to abstract mathematical truths in a rigid, control-oriented, time-bound setting aiming to strengthen their ability to respond to specially pre-designed tasks (Riling, 2020). Instead, the push has been to find ways to provide students with agency (Andersson & Norén, 2011), with experiences in mathematical reasoning for themselves, for meaning-making in personally relevant individual and discursive settings and digital media have been perceived as powerful tools to that end (Noss & Hoyles, 1996) [1]. This powerful way in which learners use digital media to structure mathematical knowledge-in-

use has been well researched and established as a goal and a challenge for transformation in mathematics education (Noss & Hoyles, 2017).

From an epistemological point of view, it is not the nature of mathematics that has been challenged but rather the ways in which the practice of mathematicians has been understood and communicated. Mathematical epistemology has not been fundamentally debated. Mathematics has been deservedly perceived as the ultimate scientific endeavor; it is the field where reasoning comes from, where rigor comes from, where the ability to make connections, to deduce and to prove, to generalize, to be certain, or to gain accuracy and develop a sophisticated language about uncertainty. This is a science where we know when something is true, and we question whether something is true in very rigorous ways (Davis & Hersh, 1981).

The transformation sought has thus to do with education, not the scientific paradigm. It is to provide learners with the opportunity to experience what it means to do mathematics, the same kind of experience mathematicians themselves go through. Mathematicians expose ideas and propositions to peer scrutiny and refutation attempts. They thus perceive mathematical ideas to be fallible, and only the ones which survive this scrutiny remain as mathematical certainty (Davis & Hersch, 1981; Lakatos, 1976). This means that most of the time they spend scrutinizing ideas by others or having their own ideas put in the frying pan so to speak. The scrutiny process is a fundamental part of doing mathematics, and transformational approaches in mathematics education argue that learners should be given much more space to engage in this kind of process.

Recently, however, and importantly highlighted by the era of the pandemic, we have realized that what has hit each of us in our society and everyday life is the engagement, pre-occupation and involvement at a personal level not with clean, potentially solvable, mathematical problems but with very complex issues and problems that are around us: climate crisis, sustainability, sustainable cities, pandemics, personal diet combining health and well-being. These kinds of issues do not really have a solution in them, and there is not any clever way in which we can find the way to deal with them, nor can we find how to cope with them and get rid of them in the end. Even though mathematicians produce endless models of such complex situations, none of them really explains the respective phenomenon in any comprehensive, resolving way. So, at the individual and social citizen level, what is required is that we become a little less stressed about these issues. How can we learn, as citizens, to grapple with them in order to survive within contexts where these issues apply. And how can mathematical reasoning and scrutiny maintain and enhance its perceived value in situations where it could play an important yet not primary role regarding the issues at hand.

In this paper, I address a question which I believe should be put to the mathematics education community:

- Is there a role for mathematics and mathematical thinking in coping with complex, contentious, socio-scientific issues?
- If yes, how can digital media be designed and used to introduce mathematical thinking and rationality in addressing and grappling with such issues?
- How can we think of pedagogical mathematical transformation with digital media to include grappling with wicked problems?

Such issues have played a central role in creating big currents pushing for change that affect educational systems in Europe and around the world. They have been connected to the ideas of cultivating 21st-century skills and action competence. Mathematics education researchers have connected those kinds of skills—such as creativity, computational thinking, collaboration and

communication, problem posing and solving—to mathematical reasoning. At the same time, however, the push for educational reform has originated from much wider educational and societal perspectives and is a part of the European strategy for education and equity. The EU strategic plan for 2021–2024 is oriented towards the twin green and digital transitions for a sustainable, fair and more resilient economy and society (European Commission [EC], 2021). The development of a high-performing education ecosystem through the digital transformation of educational paradigms is also one of the core goals of the EU Education Action Plan for 2021–2027 (EC, 2020) and of UNESCO's Education 2030 Framework for action.

So, is there a role for us as a mathematics education community to contribute to this wave of change, or is it better that we stay on the side in the hope that our siloed domain of mathematics and its beauty will remain and be respected as in the previous century? In other words, is there a role for mathematical thinking and rationality in post-normal science? If yes, what kind of digital media can be used for expressing mathematical reasoning while grappling with wicked problems? How can we design for added pedagogical value based on their use?

The following three sections provide a background to carefully address these questions. First, the ways in which digital media have been designed and used in transformative mathematics education are analyzed as a rationale for thinking about such media for wicked problem education. The next two sections set the ground with respect to epistemology, how can normal science approaches contribute to post-normal science. The third then sets the scene and basic arguments regarding wicked problem education. What follows is the description of a specific constructionist digital tool called 'ChoiCo: Choices with Consequences', designed to embed mathematical ideas, facilitate mathematical reasoning, yet be about grappling with wicked problems. The final section discusses student discourse to set the scene for what such reasoning might look like in the context of grappling with wicked problems.

DIGITAL MEDIA AS TOOLS TO TRANSFORM MATHEMATICS EDUCATION

Let us, in this section, look a little more deeply at the ways in which digital media have been perceived and designed to bring added pedagogical value in transforming mathematics education paradigms to cultivate mathematical reasoning (Bray & Tangney, 2017). Researchers seem to agree that there is particular value in digital media being used by students as tools with which to engage in mathematical reasoning, in putting mathematical concepts and ideas to use, in mathematical discourse and expressivity. When we have classrooms where students are given space to develop their own ideas and to work with these tools, we can see that the mathematical meanings that they develop and construct are unavoidably connected to the tools that they use. Researchers have witnessed this kind of reciprocal shaping of meanings and tools when mathematics is put to use to create and change mathematical models and representations (Noss & Hoyles, 2017; Artigue, 2012). The pedagogical value in meaning-making has been considered as important enough so as to address the connection between those meanings and the abstract curricular mathematical concepts as a necessary educational task in the context of students having built a positive disposition towards and experience with mathematical reasoning.

As learners create models and representations with these tools, they progressively create 'schemes of action' as Vergnaud (2009) would put it, i.e. individual and shared meanings of a tool's functionality and kinds of use together with the kinds of mathematics cultivated during such use. Thus the key aspects of focus in designing such tools and envisaging their usage are:

- Mathematical expression, augmenting the representational repertoire and interdependencies

- Engagement with mathematical thinking
- Putting concepts to use
- Reciprocal shaping, instrumentalization, constructionism and creativity

The task is to generate environments that are rich in opportunities for meaning-making, to perceive digital artifacts as media for expressing mathematical meaning, to access powerful mathematical ideas that are otherwise difficult or obscure with pencil and paper or with other representations and to engage teachers in taking part in the design of pedagogically added-value activities. The main concern in looking for pedagogical added value is thus connected to designing for innovation to adopt a transformative stance to education. The main thrust in this approach questions the way that mathematics is perceived and taught and the way that curricula are structured and looks for ways in which we can use technology within a transformation process. As researchers, we focus on technology for expression and meaning-making and we are developing theory on meaning-making processes and on teachers' knowledge and teachers' practices. Our main concerns have thus been to:

- generate environments rich in opportunities for mathematical meaning-making (Papert, 1972);
- perceive digital artefacts as expressive media for mathematical meaning-making, a new literacy (Noss & Hoyles, 1996);
- access powerful mathematical ideas otherwise obscured by traditional methods of expression (diSessa, 2000; Willensky & Papert, 2010);
- engage teachers in designing added-value media and activities and dealing with professional, institutional and societal traditions so as to generate such environments in the classroom (Ruthven, 2014); and
- develop media that is specially designed for questioning traditional practices and doing something different.

At the Educational Technology Lab we have been adopting a transformational approach by designing and using media for teachers and students to in turn design and tinker with models and representations. Over a period of more than 25 years, we have been engaged in design research to illuminate mathematical processes in respective educational practices and to contribute to the development of a 'framework for action' theory helping to both design and understand meaning-making (diSessa, & Cobb, 2004). In this venture, we found it most useful to combine and integrate diverse theoretical constructs, having been greatly influenced by our participation in the TELMA, ReMATH and M C Squared European Research projects whose main aim was precisely to forge connectivities amongst constructs lying in fragmentation on a 'theoretical landscape', to use Artigue's terms (Artigue & Mariotti, 2014). Our particular objective was not the practice of creating such connections per se but instead of considering how to best try to make sense of the environments we designed and studied. So, we found these particular four constructs, albeit widely diverse, to be pivotal in our approach.

1. Conceptual fields (Vergnaud, 2009)
2. Restructurations (Willensky & Papert, 2010)

3. Half-baked artefacts (Kynigos, 2008)
4. Reciprocal shaping of meaning and tool (Hoyles et al., 2004)

We found Vergnaud's idea of conceptual fields centrally useful in the sense that, for mathematics education, it diverts priority from a mathematical concept to be 'learned', to all that makes it useable and communicable. To think, i.e., of a concept in educational design, it is necessary to place it in the center of a dense circle of related concepts and a set of representations that become the basis for resolving a set of problem situations. So, in education, this is the way we should be thinking. We should not be thinking of whether students learn how to factorize or learn how to solve a quadratic equation, but rather of situations resolvable by dense sets of concepts around a central one.

Restructurations is the exercise of questioning the structure of the curriculum and the kinds of mathematics to best approach mathematical problems. The current curriculum structures have been decided, established and fixed in historical time before the advent of digital media and even before the advent of mathematics education research, for that matter. But mathematics is the discipline characterized by fluidity in the ways in which it makes sense to build structures; its nature is such that you can portray mathematical concepts in a very large number of different alternative structures. So now that we have technologies and we live in the technology world, it is time to rethink about what kind of structure of mathematical concepts is now amenable for children to engage in mathematical thinking with these tools. Imagine, for instance, a section on 'curvature', on periodicity, on rate of change, inflation, compound interest and approximation combined, on mathematical complexity, on gaming theory. Ask the question: which mathematical structures are good spaces for students to engage in meaning-making and mathematical reasoning, given digital media?

Half-baked artifacts; well, this is didactical design, or rather, engineering. It is when, from a pedagogical point of view, students are given problems, models or representations that are incomplete or have faults in them and then invited to identify and correct them. Behind this, there is the epistemological idea of fallibility, the idea of questioning and the idea of not perceiving mathematics as a game of absolute truths but perceiving mathematics as a field where reasoning and questioning prevails.

So, the main concerns of the research community at large, and of our Educational Technology Lab, have seen mathematics as useable intellectual processes and traits in diverse situations. Educational transformation in mathematics education has perceived digital media as a pivotal tool and digital transformation in society as the global situation in which this educational transformation may start to materialize. In this wake, mathematics curricula and curricular structures have come into scrutiny, asking the question: what kinds of structures can operate as fertile fields within which learners can develop mathematical reasoning. With respect to digital media, mathematical reasoning has been connected to constructionism, a kind of discursive low-stakes tinkering-style engineering co-evolving with computational thinking skills and competences. However, still in all cases, the mathematics education research community has understandably perceived mathematics as the priority and the end target. The situations, the tools and the restructurations have been a means to an end.

NORMAL VS POST-NORMAL SCIENCE

The problem is with normal science. Scientists are perceived by the wider society with diminishing credibility and relevance, as people who will give you facts and truths that are not so relevant when you think of the issues at hand. So what happens? So how else can we think of science? Well, recently, there has been a movement termed 'post-normal science'. It addresses complex issues of our time

where disciplinary fragmentation and traditional scientific conduct appear to lack the necessary capacity to allow an integrated understanding of the issues. Transmitting simple truths does not help policy makers, and that does not help citizens and the individuals. We need to draw on epistemological principles that recognize uncertainty and develop ways of dealing within uncertainty, within value-laden agency.

Most current science thinking and research, centrally including mathematics, has been built on epistemological assumptions developed along the deliberate aim to moderate complexity and minimize uncertainties in the world so that problems and issues can be ultimately modeled. These 'normal science' paradigms (Kuhn, 1962), supported by appropriate conceptual modes of representing reality and specialized codes for studying it, narrow down the focus of their enquiry within the boundaries of specialized disciplinary fields such as mathematics to address attentively defined (but eventually simplified) 'problems', with the intent to generate valid and generalizable evidence-based knowledge to feed decision-making. However, when it comes to complex issues of our time, disciplinary fragmentation and traditional scientific conduct as encouraged by 'normal science' seem to be lacking the necessary capacity to allow an integrated understanding of the issues. Moreover, transmitting simple truths to policy making is rather inadequate when dealing with multi-faceted issues carrying a great degree of uncertainty (Heazle, 2012). There are many open questions as to how science can contribute to fostering social innovation and change in as many social groups as possible, rather than providing only expert-based knowledge to policy makers. This is particularly the case with, for example, current crisis and sustainability challenges, recognized as complex, controversial, and value-laden issues by nature and, therefore, difficult to be dealt with in mono-disciplinary ways. Complexity stems from their multi-faceted character and the requirement to apply various perspectives to grasp them more holistically. Different interpretations may lead to different implementations based on the context and the situation. These features render such wicked problems difficult for normal scientific practice to address and deal with. To counteract these shortcomings, 'post-normal science' has emerged as an alternative paradigm of scientific enquiry and knowledge (Funtowicz & Ravetz, 1993). Drawing on epistemological principles that recognize uncertainty, value-laden agency, and context-specificity as intrinsic attributes of the contemporary, post-normal science promotes transdisciplinary approaches to framing and studying current complex issues and gaining an understanding of the world. Global crises, pandemics and sustainability issues are among those most characteristic examples the understanding of what necessitates the application of post-normal lenses and processes, such as the co-creation of diverse types of knowledge, the employment of participatory methods, designs, and tools, that facilitate the emergence of multiple representations and reflection to take place.

WICKED PROBLEMS AND WICKED PROBLEM EDUCATION

Consider the role of schooling to inspire lifelong citizen engagement with 'wicked' problems that can contribute to a democratic, socially engaging sustainable development practice, where experts and various groups of citizens with different perspectives engage in a dialogical inquiry on a complex, fuzzy, multi-faceted, contentious issue, such as sustainable living. This kind of issue has been called 'a wicked problem', i.e., a dysfunctionality within a complex system (Conklin, 2006). Wicked problems are difficult to contain and structure, are interconnected and interdependent, are ill-defined and dynamic as their parameters are continually in flux (Rittel & Webber, 1973; Coyne, 2005). Individuals often feel overwhelmed, develop denial and resignation to such a problem, followed by inertia due to a sense of determinism, which permeates societies (Lazarus, 2009; Hulme, 2009). Yet wicked problems need action at many levels, present inertia risks and exponential growth of the problem and its consequences at high stakes (Brown et al., 2010). For the individual, it is important

to engage in becoming sensitive and knowledgeable on the problem and to also engage in actions such as taking care of individual footprint, challenging own actions, beliefs and habits, being interested not only in individual action but also in contributing to collective action at a level of the city or municipality (Cantor et al., 2015). A paradigm shift is needed, from solving well-defined siloed problems to a post-normal science approach (Lehtonen et al., 2019). So, consider a transformational stance to schooling in an attempt to integrate such a post-normal science approach in teaching and learning, addressing and perceiving students as young citizens (McLaren, 2013). Consider the challenge of harnessing wicked problem education to become syntonic and integrated with the innovative educational push towards cultivating the eight key competences for lifelong learning.

- Agency, ability to make own decision, challenge not set by another (Kynigos & Diamantidis, 2021)
- Action-in-context, when the context is not necessarily about mathematics in school
- Beyond silo disciplinary approaches
- Beyond timed, solvable, regulated challenges
- Beyond integration: a competency (Geraniou & Jankvist, 2019) in the service of another

So what is a wicked problem exactly? It is a problem impossible to solve, and that is because it is not well defined. It is contradictory. Different people have different views on it. It changes all the time. It connects to different things. Conklin (2006) called it a dysfunctionality within a complex system. Such problems are difficult to contain and structure; they are interconnected, interdependent and ill-defined. And their parameters are continually in flux. Some examples of wicked problems are poverty; urban renewal; school curriculum design; education, environmental and natural resources policy; healthcare; climate change challenges; sustainable cities; diet; individual and social challenges in times of world crisis.

So, these wicked problems cause problems to the individual. They often develop a denial about the problem. Assertions such as ‘come on, pandemic corona-virus! It’s easy, it’s just a flu!’. They create resignation of the individual regarding the acknowledgment of the existence of the problem. We see such a point argued all over the media, ‘there’s no point in vaccinating since if you’re vaccinated you can still catch it’. Resignation is followed by inertia; ‘I’ll wait, I’ll wait for everybody else to get vaccinated and then see what happens’. But wicked problems need action at many levels because the inertia is a risk, and there is an exponential growth of the problem if people do not realize and do not start developing strategies. And the consequences are high stakes. So, for the individual, it is important to become sensitive and knowledgeable not to find a solution and also to engage in actions such as taking care of the individual footprint, but also perceiving that they are a member of societies at different levels who are collectively addressing the problem.

To date, transformation in mathematics education has hardly addressed the role of mathematics and mathematical thinking in post-normal science. What can a normal science such as mathematics provide to empower and support post-normal perspectives aiming to address controversial ill-defined, complex socio-scientific and value-laden issues? Fallibility in mathematics has been perceived in the context of the process to look for truth, for certainty, in a normal science setting. In post-normal science, however, the focus is not on the process of producing mathematical truth but could potentially be on the role such truth can play in, say, wicked problems. In this paper, I suggest that mathematical thinking and mathematical concepts do and should have a secondary but no less important role to play in post-normal science and that mathematics educators need to consider their

involvement in the recent pedagogical wave demanding schooling to afford wicked problem education.

DIGITAL MEDIA TO GRAPPLE WITH WICKED PROBLEMS: THE CASE OF CHOICO

The first section in this paper contains a discussion of how digital media have been designed as tools to help with the pedagogical wave of transforming the mathematics education paradigm from what Chevallard (2012) calls a 'visiting the works' paradigm to an experiential, questioning the world, creative and discursive paradigm. The transformations, however, have so far maintained mathematics as the primary educational objective of the enterprise where the focus is on the modeling of mathematical objects and representations outright or at most the modeling of objects and behaviors directly and importantly embedding mathematical concepts (Artigue, 2002; Kynigos, 2018; Sarama & Clements, 2002; Kaput et al., 2002; Sinclair & Freitas, 2014). Even in the case of media which is primarily focused on computational thinking and creativity to create games such as scratch, mathematics educators have shaped microworlds and modeling exercises with a focus on the mathematical concepts inherent within (Benton et al., 2016; Cader, 2018). The most well-known attempt to design a medium for students to engage with complex issues is NETLOGO (Willensky, 2020). Even there, however, the focus is on mathematics as a means to fully understand the phenomenon by modeling it, based on an albeit diverse kind of mathematics calling for a restructuration of our perception of mathematical curricula (Willensky & Papert, 2010). In this paper, the attempt is to consider digital media primarily in the role of tools to help grapple with wicked problems, yet, at the same time, embedding mathematical ideas and designed to cultivate rationality in such an enterprise.

It is in this context that we introduce a constructionist tool which we call 'ChoiCo', a digital medium specially designed for post-normal science education. ChoiCo is an acronym for 'Choices with Consequences' (Kynigos & Grizioti, 2020). It is a system for authoring games embedding socio-scientific issues. The system leaves the choice and definition of such an issue up to the user. It is based on the gaming idea that there is a single gamer making choices amongst objects placed on a geo-coded map. Every choice has consequences across a pre-set range of fields, yet there is no clean choice, i.e. one which has only positive or negative consequences. The game ends when the player crosses some pre-set value in one of the fields, i.e. crosses a 'red line'. So the gamer needs to navigate through a field of choices, the point of the game being to stay on the game as long as they can, avoiding 'red liness'. The more choices made, the better the player. Sustainability is key; the more the player can sustain making choices, the better. But most importantly, ChoiCo affords important transparency, leaving users, in the role of game creators or modifiers, to name as many fields as they wish, to set values for every choice, to program the starting values, the 'red lines' and a number of 'warning messages' and other rules via a block-based programming language. Field values can be numerical fixed or random, visible or hidden from the player (they can make the player need to infer the consequence of the field by observing some text, a video or a picture).

ChoiCo is thus not just about designing or playing a game, it affords the user to take on the role of a prosumer, someone who engages in-game modding as well as design and play, interchangeably. Each game can be considered as a 'living document' to use the term coined by Trouche and his colleagues in their theory of addressing educational practice through the continual re-design of educational resources (Guin & Trouche, 1998). The main features of ChoiCo are based on the following design principles.

- Constructionist games, games affording access to the content and rules of the game and providing tools to define and to change them

- Free climate – alternative reality, low stakes, so that users find a safe space to try out risk-free solutions to wicked problems and consider the consequences
- A framework – reference for discussion and debate, the idea is that pairs or groups of students engage in the modding process
- Rules and content of a game open to modding
- Gaming rules: sustainability, i.e. stay on the game as long as you can
- Interchangeable gameplay and game modding, i.e. the practice of adopting a binary role of player and designer of a game

So, let us start with an example of a game that was actually designed by postdoc researchers at the Lab (Grizioti et al., 2021). Consider a user in the role of the player; let us give her a name, 'Mary'. Mary is a citizen, and she lives in the covid pandemic era. She has some choices to make on what to do in her day. For instance, she can consider running. If she does choose to run, the game tells her what the consequences will be along a line of values. The values are 'physical capacity', 'it's fun', 'social', 'money' and 'risk of covid infection'. So the game tells Mary what is going to happen if she makes a choice before she decides to make it. If she chooses to engage in running, then the covid risk would be a random number from minus 15 to minus 20, i.e. an equal probability in the respective range of values. So it would not be much of a risk, but still, Mary would not be certain of what is going to happen. The 'physical condition' consequence is a function of how much physical condition Mary would have if she was walking. And the others are just numbers. So, upon clicking on the selection of the choice to go running, Mary observes the change of aggregate values in a respective panel on the screen. Then, she can try doing something else. For example, she can try going to the local store (Figure 1).

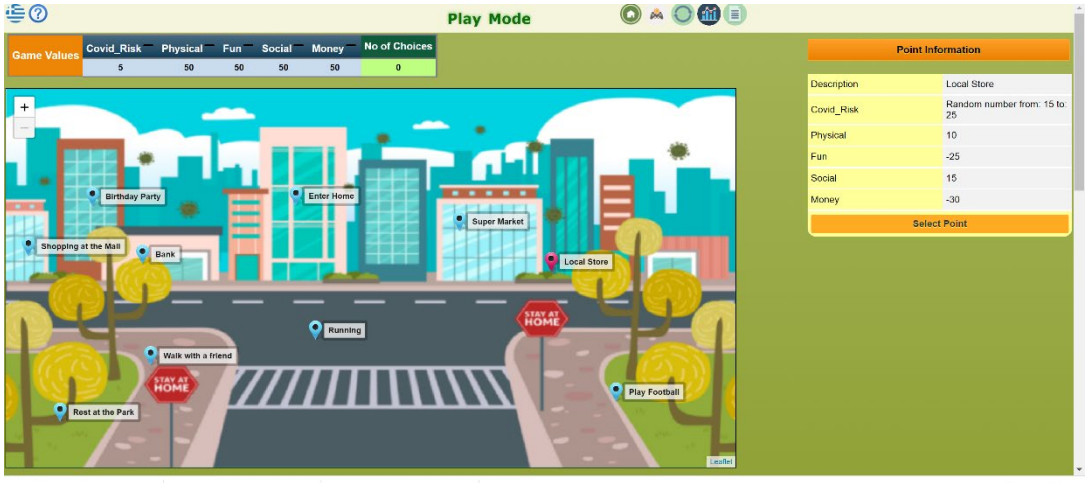

Figure 1. The consequences of going to the local store

That choice would result in a much larger covid infection risk, 15 to 25, and would result in spending money (-30). On the positive side, it would up the social and physical condition, a lot of walking up and down the isles you see. If she then chooses to go to the mall, she would be in danger of being thrown out of the game since the covid risk there would be immense. So maybe she might decide going to her home instead. Clicking on that choice brings up a different plane with things to do at home, e.g. work from home, shop online etc. For instance, if she decides to sleep and continues to select that choice effectively to sleep through the pandemic, the game quickly sends a warning sign 'You're unhappy. You need to do something.' and then if she continues, throws her out since the

‘fun’ value crosses the line and the game reports that she has become depressed. So the idea of the game is to stay on the game as long as possible, and there is no choice that has only positive or only negative consequences. Mary needs to find a way to navigate in order to stay in the game as long as possible. This is a game about a citizen dealing with a wicked problem.

And now about modding a game, taking on the role of a hacker. In the example of the covid game, everything mentioned so far can be changed by the player changing roles and becoming a game modder. A button click switches to the editing page where Mary herself or anyone else can make changes at all levels. Change the picture. Add new choices. A new choice appears on the map and also on a tabular representation of a relational database as a new record. Mary can give it a name and start putting values on the consequence fields. She can observe that there are diverse kinds of values she can allow for each of the field columns. For instance, the covid risk field has been defined as containing random values in a range set by Mary. There are other formulas she can use to define field values, simple functions or just straight numerical values. Mary can also easily change or add fields (Figure 2). Finally, she can switch to a Blockly programming feature in order to write programs to set the initial values, set the ‘red lines’, put up warning texts or sounds when a certain value gets close to a red line and anything else regarding the game rules. (Figure 3).

Figure 2. Making changes to map, choices and their consequences

All of these affordances have been designed to allow users to engage with a larger and more complex set of concepts, practices and values than one may find in like-minded authoring systems in education. There are three distinct but also interconnected areas for these. One is, of course, the socio-systemic issue embedded in a game; the other is the computational thinking, i.e. concepts and practices cultivated and employed, and the third one may involve mathematical concepts and rationale. Before we elaborate on the latter, it is important to say a few words about computational thinking since this too can be connected to mathematical thinking in various ways (Barr et al., 2011). ChoiCo affords the use of functionalities to do with geo-coded data with relational databases and with block-based programming giving some emphasis on event handling and boolean logic. This enables the main characteristics of computational thinking as originally defined by Wing (2006).

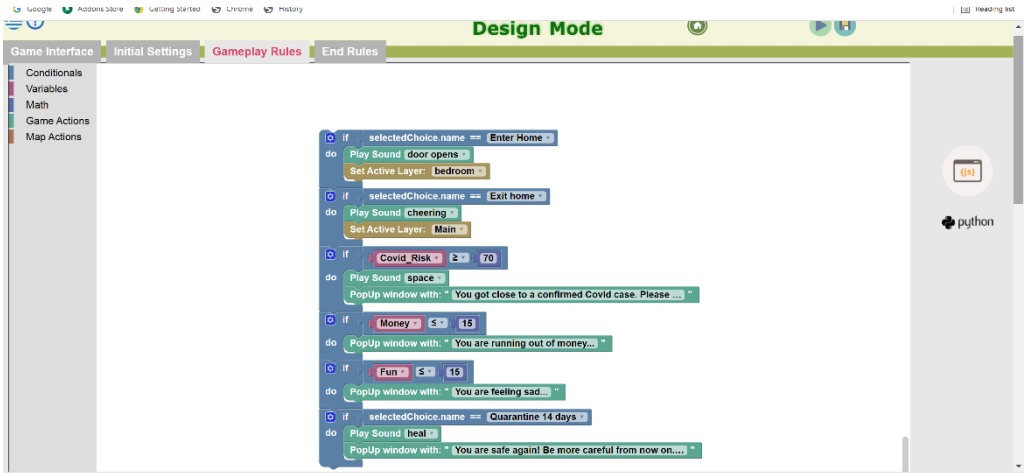

Figure 3. Changing the rules, initial values, warnings and red lines

In a study of students using ChoiCo in such a way (Kynigos & Grizioti, 2020), we analyzed their progressive process of exploration (play the game), deconstruction (break down the game structure), analysis (analyze the game elements), synthesis and construction. The process involved the integration of interacting with various affordances (graphics, story, rules, characters, etc.) and with the use and understanding of Blockly programming concepts such as conditionals, boolean logic, event handling, but also programming processes such as developing high thinking skills like iteration and refinement, debugging, error prediction, etc. (El-Nasr & Brian, 2006; Moshirinia, 2007; Salen 2007).

THE MATHEMATICAL IN GRAPPLING WITH WICKED PROBLEMS

So how can we think of the use of ChoiCo to develop dispositions to use rationality and mathematical thinking in order to grapple with wicked problems such as the consequences of the covid pandemic? Let us first consider the Covid game—and ChoiCo more widely—as a mathematical microworld (Healy & Kynigos, 2010). What mathematical concepts are or could be embedded in a game? And when students change the game, what mathematical reasoning could they engage in as they identify, question and modify values and relations between them? As implicitly discussed in the previous section, the Covid game embeds proportional thinking, functions, probability, mathematical issues related to programming. But, of course, these mathematical ideas adopt the status of affordances. When the game is put to use by learners, the identification of these ideas and the ways they may or may not be put to use is a process of instrumentation and instrumentalization (Artigue, 2012).

Game modding in education has mainly been connected to computational thinking, but if the games have embedded mathematical ideas, then this computational thinking becomes connected to thinking mathematically. ChoiCo games are thus seen as productions; they are artifacts designed to be used by somebody else. They are fun and have many different kinds of connections: connections to real issue debates, connections to gaming, connections to entrepreneurship. They could therefore be considered as useful resources in this new era of 21st-century skills and equity and wanting to change this silo domain structure of the education system. And they still retain some of the benefits that we have from teaching students domains and mathematics in specific.

ChoiCo games can be resources for learners to engage and to grapple with socio-scientific issues in the context of post-normal science and they can be specifically designed to invite students to make changes. They need not be designed for changes to be difficult and for students to be terribly savvy. They can be designed for students to find it easy to engage in modding. From a pedagogical point of

view, rather than resources designed to enhance student responsiveness to pure mathematical tasks, they can be considered as tools to cultivate mathematical disposition and competence, perceiving students as young citizens. Modding with games such as ChoiCo can then help students to recognize the value of mathematization, modelling, tinkering with models, using rationality to grapple with wicked problems

There are a number of freely available games already up there on the ChoiCo site, http://etl.ppp.uoa.gr/choico. Furthermore, a hitherto small number of ChoiCo games designed for more focus on mathematical ideas around shopping in a supermarket reside in a space with large-scale visibility and use in the Greek Education system (http://photodendro.edu.gr). This is an infrastructure based at the Ministry of Education called 'the digital school', which contains a large portal of digital artefacts for students to use in all subjects and at the same time has links to these artefacts residing inside the online version of the curriculum books from year 3 to 11. There are around 1600 such artefacts in mathematics, 1200 built with GeoGebra, 220 built with an ETL grown 3D dynamic programmable modeller called MaLT2 (http://etl.ppp.uoa.gr/malt2). There are six versions of the Supermarket ChoiCo game spread at the end of primary and beginning of secondary year books.

The links to ChoiCo games in the curriculum book is in a section about mathematical problems. Clicking onto the ChoiCo micro-experiment (Kynigos, 2020) gets you directly into the game, which is about doing things in a supermarket. Your values are a number of items, how much money you have, health and pleasure (Figure 4).

So if you buy chocolate then you get: price is 7E, number of items is 7 times more than honey, health is -5, pleasure is 4. So if you keep eating only chocolate, the game will warn you when your health is below 10 and throw you out when it is below zero on the grounds of poor health. If, instead, you buy yogurt or broccoli, your health gets better, but your pleasure is reduced. The idea here is that consequences do not only have direct values. Sometimes values need to be considered in connection to other things, so students need to think about units of measure and proportional relations and operations. The Supermarket game is an example of shaping the design of a game to fit the mathematics more directly within a silo mathematics curriculum. But still, even now, there are important value-laden issues such as taste, health, pleasure eating, balanced spending etc.

Figure 4. The Supermarket Game

The idea behind ChoiCo is what we at the Lab call 'black and white box design' (Kynigos, 2004). So there are some digital objects and functionalities that are black boxes to the user, such as the database. ChoiCo has not been designed to get users to reprogram a relational database. It is unlikely for a piece

of technology to contain programming and databases and a GIS at the same time. The design principle, however, is to prioritize a pedagogical perspective irrespective of whether it's easy to find the technological infrastructure. This is dealt with by finding available components keeping the development part mainly in gluing them together, and then building whatever else is needed on top (Kynigos, 2004).

WHEN MATHEMATICAL REASONING AND VALUES CO-EXIST

So what does mathematical reasoning in the context of grappling with wicked problems sound like? In the example elaborated here, a group of three 13-year-olds are jointly modding a ChoiCo game designed by the researchers to include questionable perceptions about what it means for a citizen to live a life-supporting sustainability in their city. The students played the game to start with and then began to discuss the ideas embedded within. R is the researcher, S(x) is a student.

R: So who is the winner?

S(8): We all are! We all finished the game.

S(1): We won because we have the largest amount of money: 1200.

S(8): Who said that money was most important for the game? All you did was to go from work to home and vice versa.

S(5): I think we won 'cause we have the highest energy levels and the highest social status.

S(11): Wait, wait. The winner is the one who has the highest values in all these: money, energy, fun, social status, health, hygiene.

R: What kind of life do you have to live in order to achieve that?

S(16): You have to do a lot of everything: have a lot of fun, have a lot of money, do not neglect your social life… This is all too much.

S(3): You have to be a freak to live like this; you won't have a moment of peace.

In the above dialogue generated by the researcher, the students discussed over a gaming idea—what does it mean to win—connecting it primarily to the wicked problem at hand, is a citizen's life worth living if they go for high achievement in all aspects all the time. Within this cycle, the students considered numerical values and their aggregates, implicitly keeping in mind that each of the available choices had at least one undesirable value, a negative number in this respect. The underlying problem was how to increase the values of all the fields even though every choice would unavoidably bring a decrease in at least one of them.

S1) I am not sure about not having cars in the city.

S(2) I am telling you it has been done in Freiburg. Cars are related to pollution.

S(1) Yes, but imagine how much more time you need if you go to work by bicycle. You need to wake up at least one hour earlier.

S(3) Ok then, we will add time in the indicators. Taking the bicycle should have reduced pollution but raised time.

In discussing the pros and cons of ways to go to work, the students identified that pollution and time were conversely dependent and thus decided to insert another field, time. Here again, their focus was

on the pollution problem primarily, and the inverse proportion idea was a tool with which to think about and argue for the citizen's choice to take the car or the bicycle to work.

DISCUSSION

Admittedly this paper has opened up many issues simultaneously; can mathematics as a normal science be considered as a tool, skill and competence in situations better understood through post-normal science, such as individuals and collectives against up against wicked problems? How can the design of digital media for wicked problem education incorporate agendas from transformative mathematics education, such as those mentioned in the first section? What kinds of skills, competences and dispositions can be cultivated with the help of mathematical reasoning? Can mathematical reasoning be cultivated in the context of making one's own decisions, being creative and adopting an individual and social active stance to wicked problems?

The ChoiCo 'citizen in the Covid-19 Era' game example and the short excerpts from students modding a 'citizen in a sustainable city' game were used as contexts with which to bring some elaboration of potential educational and research endeavors. The mathematical ideas embedded in the two example games ranged from simple operations to proportions and linear functional relationships to probability. The range of ideas that may be useful in wicked problem education has yet to be understood, but most of all, the interesting question is what kind of rationality may grow using mathematical reasoning and, in particular, how can this rationality be used in the quest to understand and grapple with wicked problems. In the excerpts, we saw students reasoning at different levels, from articulating local arguments of aggregating or calculating values to re-considering the 'big issues' such as 'what is the value of being a high scorer in everything'. There is much more to be learned about how reasoning can be used in such situations and how and when mathematical ideas and concepts might become useful. The Covid ChoiCo game could embed a larger range of mathematical concepts such as quadratic or exponential or periodic functional relations, and a pendulant course could be designed with respect to attention and focus from citizen habits to the mathematics underlying the consequences. In any case, the paper aimed to call upon the mathematics education community to consider the role of mathematical reasoning for wicked problems and the challenge to develop an argument for cultivating such reasoning in this kind of transformative educational context.

NOTES

1. It is, of course, the case that diverse approaches and tools have been developed and tried out in mathematics education, some of them designed to maintain and enhance drill and practice effectiveness in a traditional exposition to abstract mathematics curricula.

PLENARY ADDRESS VIDEO

https://www.youtube.com/watch?v=kRr0gTcOt_s&ab_channel=FacultyofArts%2CAarhusUniversitet

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METAPHOR-BASED ALGEBRA ANIMATION

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We investigated whether dynamical animations with visualizations based on metaphorical linking are more effective for grasping algebraic manipulations than static animations. The question is whether and how ideas from conceptual metaphor theory, in particular, embodied cognition, can be drawn to positively contribute to the design of effective animations for visualizing algebraic manipulations. In classroom tests, grade-7 students watched an animated video on algebraic manipulation, either with dynamic visualization, a visualized person dynamically performing the manipulations, or a more static video not based on those ideas. For higher-level students with some pre-knowledge of algebra, we found a small positive effect for dynamic and dynamic embodied videos. For lower-level students with no pre-knowledge of algebra, the embodied animation turned out to be adverse effective.

Keywords: Dynamic visualization, embodied simulation, object collection metaphor.

INTRODUCTION

In the last few years, a revolution in informal mathematics education took place, to our best knowledge, ignored by the mathematics education community. Through his YouTube channel 3Blue1Brown, mathematics educator Grant Sanderson reached tens of millions of students worldwide with dynamically animated videos covering a wide range of topics, including linear algebra, calculus, probability, and machine learning (3Blue1Brown, n.d.). The responses to those videos are jubilant – students claiming to "finally understand the topic". A natural question is whether viewers are simply overawed by the smooth animations, or do animated effects really contribute to better learning outcome. In this study, we focus on one specific feature of these videos: Dynamically animated algebraic manipulations. If one encounters algebraic manipulations in a video, is it beneficial for these to be dynamically animated? We propose a theoretical basis for such animations and study the effect.

Wittmann and collaborators (2013) observed students discussing algebraic manipulations as if the terms were physical objects moving in a landscape. As they explain using Conceptual Metaphor Theory, one might argue such metaphorical language is grounded in spatial embodied experiences (Lakoff & Núñez, 2000): Reasoning about moving physical objects is linked to reasoning about algebra through metaphors. Nicaud and Maffei (2013) explore how this theory leads to an interactive algebra environment where terms can be dragged across the screen. We study the same phenomenon in the non-interactive flat environment of animated video.

No literature has brought such an approach in mathematics education, but recent inquiries by educational psychology researchers have explored the role of embodiment in educational videos (Castro-Alonso et al., 2018; De Koning & Tabbers, 2013; Pouw et al., 2016). More generally, the issue is whether dynamic animations are more effective than static animations, possibly using embodiment (Berney & Bétrancourt, 2016). Dynamic animations use continuous movement and deformation, whereas static animations are more like a traditional slide show portraying a discrete set of images.

In this study, we compare in a quantitative way the effect of a short instruction video on the topic of elementary algebra with respect to three conditions: static animation, dynamic animation, and

embodied dynamic animation. The aim is to gain insight in whether these conditions have any quantitatively measurable difference in effect on the learning outcome.

THEORETICAL BACKGROUND

Abrahamson and Lindgren wonder whether "learning environments (can) be designed to foster grounded learning, in which students sustain a tacit sense of meaning from corporeal activity even as they are guided to re-think this activity formally" (Abrahamson & Lindgren, 2014, p.3). They state that "manipulating symbolic notation is cognitively quite similar to physically moving objects in space" (p.3). We consider an instruction video as such a learning environment in which moving the algebraic terms in the plane might support being able to follow and understand algebraic manipulations.

Lakoff and Núñez's theory states that mathematical cognition is organized through certain linking processes (Lakoff & Núñez, 2000). These linking processes, called metaphors, consist of mappings from one conceptual domain – the source domain – to another conceptual domain – the target domain, usually more concrete. For this study, the source-path-goal metaphor, the object collection arithmetic grounding metaphor, and the arithmetic-algebra linking metaphor are important. The source-path-goal metaphor is in play when the changes of position of a term are mapped onto a path of the term as an object in the plane. Such a term is interpreted to have a source—the original position—to move along a path, and a goal—the final position in the equation. The arithmetic grounding metaphor interprets arithmetical operations as manipulations on a collection of objects. The cognitive schemas supporting object collection are image and motor schemas, in particular the hypothesized containment schema, that deals with the metaphorical use of "container" as an object that envelops a collection (Lakoff & Núñez, 2000). For example, addition is then mapped onto the experience of joining the content of two containers. A final metaphor of importance to this study is the arithmetic-algebra linking metaphor: a metaphor that links algebra to arithmetic. Algebraic rules map onto essential characteristics of arithmetic. The hypothesis central to Lakoff and Núñez's theory is that a metaphor transfers the inferential structure of the source domain to the target domain. Applied to our case: the (aspiring) mathematician reasons about arithmetic by mapping reasoning in object collection onto the arithmetic domain; and likewise, they reason about algebraic manipulation by mapping reasoning and experiences in arithmetic computations onto the algebra domain, and about the repositioning of terms as those terms traveling along paths. These metaphors and the way they transfer reasoning inspired a dynamical visual "language" for animating the algebraic manipulation in the dynamic and embodied-dynamic condition.

In a recent meta-analysis of 140 pair-wise comparisons, Berney and Bétrancourt (2016) found a significant advantage for dynamic animations over static in learning outcome. However, in only 31% of the studies dynamic animation was superior, compared to 10% where static was superior, and 59% with no significant difference. For the eleven studies on mathematics videos in their meta-analysis, they found a small negative effect for dynamic animation, but this did not include the topic of algebra. Ayres and Paas find advantages and disadvantages for learning through dynamic animation (2007). An advantage of dynamic animation is that movement and changes can be visualized in a life-like way. Being able to see instead of having to infer the motion and changes helps students follow reasoning steps. Another advantage is that dynamic animation facilitates cueing: movement can be used to direct students' attention. A disadvantage is that information tends to be transient: Information comes and goes, and remembering and integrating it all is a challenge.

Our research question is what the effect is of dynamic videos—where animations are based on the discussed metaphors—on learning outcome, compared to static visualization, showing the algebraic

manipulations line by line. The next question is what the effect on learning outcome is of showing a part of the body performing the dynamically animated manipulations.

METHOD AND MATERIALS

We performed a pilot study, a first experiment, and a second experiment. The dynamics and embodiment might influence both the retention and the understanding of the information in the video (Pouw et al., 2016). The retention of information influences the performance on reproductive tasks, whereas processing and understanding the information influences performance on more challenging tasks. In the second experiment, we measured performance on reproductive and on more challenging tasks separately. Pouw et al. (2016) show that embodied animation can be more effective for students with a lower level of achievement in the subject of the video. Therefore, we performed the experiment on two populations: Higher-level mathematics students with little prior knowledge of algebra in the first experiment; lower-level mathematics students with no prior knowledge of algebra in the second experiment. The level was assessed by the teachers of the students.

For the experiments, there were two experimental groups and a control group. We designed three versions of the same video (see https://tinyurl.com/anialg). For each 92 seconds video, we used the same audio, as well as the same algebraic expressions, fonts and font size, colours, timing, outlay and design. The video shows five worked examples about basic manipulations in algebra: Scalar multiplication by a positive integer as repeated addition: $a + a + a + a + a = 5a$; addition of similar terms: $5a + 3a = 8a$; commutativity: $5a + 3b = 3b + 5a$; a combination of those: $5a + 3b + 3a = 8a + 3b$; distributivity: $3(5a + 3b) = 15a + 9b$. In Table 1, we present screenshots from the three versions of the intervention video. In the *Static* video, the algebraic expressions appear line by line at the moment they are addressed in the audio. The *Dynamic* video and the *Embodied dynamic* video use dynamic animations based on the discussed metaphors for algebra. The difference between *Dynamic* and *Embodied dynamic* is that, in the embodied dynamic version, a person is visualized performing the manipulations (upper body in examples 1, 2 and 3 and only the hands in examples 4 and 5).

Table 1. Comparing visualization in intervention videos

Time	Static	Dynamic	Embodied dynamic
0:17	$a+a+a+a+a$ $5a$	$a+a+a+a+a$ $a\ a\ a\ a\ a$	$a+a+a+a+a$ 
0:45	$5a+3b$ $3b+5a$	$5a+3b$ $3b+5a$	$5a+3b$ 
1:05	$5a+3b+3a$ $5a+3a+3b$ $8a+3b$	$5a+3b+3a$ $5a+3a+3b$ $5a3a+3b$	$5a+3b+3a$ $5a+3a+3b$ $(5a3a)+3b$
1:27	$3(5a+3b)$ $5a+3b$ $5a+3b$ $5a+3b$ $15a+9b$	$3(5a+3b)$ $5a+3b$ $5a+3b$	$3(5a+3b)$ 

Table 2 explains for each algebraic manipulation how the metaphorical mappings lead to a choice of dynamical visualization. The first column contains the algebraic manipulation rule, the second column the linking metaphor of the algebraic rule in arithmetic, the third column the two grounding metaphors, and the fourth the choice of visualization based on these grounding metaphors. We discuss here the reasoning behind the visualization of *commutativity*: The rule states that the algebraic terms in a sum can be interchanged, e.g., $5a + 2b = 2b + 5a$. This rule is an abstraction of the experience in arithmetic that numbers in a sum can be exchanged, e.g., $7 + 8 = 8 + 7$. These experiences and the rule are related by a linking metaphor, where the algebraic rule represents an *essential* property of arithmetic (so-called metonymy). The arithmetic can metaphorically be linked to the physical model of heaps of objects. The containment scheme allows one to interpret a heap of objects as one unit, a container, with an individual position. This can also be seen as a form of metonymy: the heap itself is not contained in an object, but one has a natural notion of *in the heap* and *not in the heap*. Inspired on the source-path-goal schema, commutativity is hence linked to moving heaps of objects as a whole: Interchanging the positions of the heaps does not influence the result of joining them. Therefore, in the dynamic visualization of commutativity, the terms $5a$ and $2b$ are treated as containers with a position that can interchange position by moving. To refer to these schemes more directly, the embodied dynamic animation presents a person moving the terms as if they were physical containers.

Table 2. How algebraic rules are visualized based on metaphorical mappings

Algebra over positive integers	Arithmetic	Object collection and source-path-goal metaphor	Dynamic visualization
Scalar multiplication	Multiplication as repeated addition	Swiping the similar objects on a heap Or the other way around: spreading the similar object on a heap out over space.	In a swiping gesture, the same variables in the summation together join into a number-times-variable expression In a spreading gesture away from the number-times-variable expression, the separate occurrences of the variable appear
Adding similar terms	Addition (distributivity: $3a + 2a = (2 + 3)a$)	Swiping heaps of similar objects together on a new heap	In a swiping gesture, the similar number-times-variable terms are joined together and there appears a new number-times-variable expression where the number is the sum of the previous numbers
Commutativity	Commutativity of number addition	Moving heaps of objects (in a container) around to make them interchange position	Hands move the number-times-variable terms around the plus sign to make them interchange position.
Distributivity	Distributivity in arithmetic	Reorganizing heaps of objects. More precisely: one has a number of heaps of two different object types and joins all objects of same type together on heaps	In a spreading gesture, the scalar multiplication of the terms between the brackets separates into occurrences of these terms (as in scalar multiplication) Then a swiping gesture joins the similar number-times-variable terms (as in adding similar terms)

The design included a pilot study in two classes, the first experiment in five classes and the second experiment in four classes. For each class, the teacher divided the students into three levels, based on their own assessment: below average, average, above average. The division of students over the conditions was such that each condition has about the same number of students of each math level. The pilot study took place in two 7th grade classes of a Dutch pre-university level secondary school to test the set-up and have indicative results. In one of the pilot classes—like the classes in our first experiment of pre-university level—we found an effect of $d = 0.81$ (Cohen's d) of the embodied dynamical condition. A power analysis suggested the size for each group in the first experiment to be 38 for a power of $1 - \beta = 0.8$. After that, we performed the first experiment, involving 132 seventh graders (boys and girls, age 12 - 13) of a Dutch pre-university level secondary school. The students had prior knowledge of what a variable is and had had a very limited primer on algebraic manipulation. The second experiment involved 42 seventh graders (age 12 -13) of one Dutch pre-vocational secondary school (*VMBO*) and 49 seventh graders (age 12 -13) of another Dutch pre-vocational secondary school (*VMBO*). These students had no prior knowledge of algebra.

In the first experiment, the written pre-test and post-test consisted of three items in line with the worked examples, e.g., simplify $3a + 2b + 2a$. For the second experiment, the tests had sixteen instead of three items: eight reproductive items and eight more challenging items: an improvement implemented to avoid the ceiling effect that could have played a role in the first experiment. In the first tests, students had as much time as they needed, whereas, in the second, they had precisely five minutes to complete as many of the sixteen items as they could. All tests were hand-marked by the researchers using a complete and strict marking model. In the first experiment, maximally 7 points could be scored. In the second experiment, 2 points per item could be scored, which leads to a maximum of 32 points. These pre- and post-test scores were taken as variables and analyzed statistically using SPSS version 25. For the first experiment we conducted an ANCOVA test to determine a statistically significant difference between the control, dynamical and embodied dynamical conditions on the post-test score controlling for the pre-test score. For the second experiment, we conducted one-way ANOVA tests to compare the post-test results on the reproductive and challenging items between control, dynamical and embodied dynamical conditions

RESULTS

For the first experiment, we carried out Levene's test and normality checks on the post-test data and the assumptions were met. The ANCOVA test revealed no significant effect of video-type on the post-test controlling for the pre-test score ($F(2, 128) = 0.459$, $p = 0.633$). Estimates of the means, standard errors and 95% confidence intervals are presented in Table 3 and Figure 1.

Table 3. Estimates for the post-test scores in the first experiment (maximum score is 7)

Condition	N	Mean	Std. error	Lower bound	Upper bound
Control	43	5.607	0.274	5.065	6.148
Dynamic	45	5.887	0.268	5.357	6.416
Embodied	44	5.955	0.271	5.419	6.491

Note: the covariate pre-test score was evaluated at 1.659

The effect of the dynamic condition (compared to control) was small, $d = 0{,}15$; and for the embodied condition also small, $d = 0{,}20$.

In the second experiment, the pre-test scores were zero for all candidates; not surprisingly, since they had no prior knowledge of algebra. The ANOVA test revealed no statistically significant difference between groups ($F(2, 88) = 0.824, p = 0.442$). The means, standard errors and 95% confidence intervals are presented in Table 4.

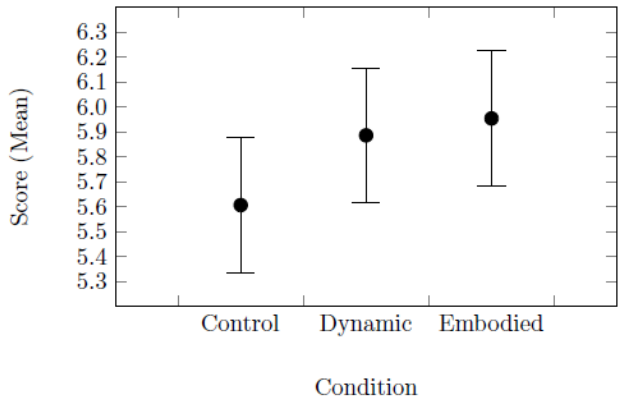

Figure 1. Estimates for the post-test scores in the first experiment (maximum score is 7)

Table 4. Post-test scores on the reproductive items in the second experiment (maximum score is 16)

Condition	N	Mean	Std. error	Lower bound	Upper bound
Control	31	6.419	0.772	4.843	7.996
Dynamic	29	6.828	0.878	5.030	8.626
Embodied	31	5.371	0.834	3.668	7.074

We conducted a one-way ANOVA test to compare the post-test result on the challenging (transfer) items between control, dynamical and embodied dynamical conditions. There was no statistically significant difference between groups ($F(2, 88) = 0.265, p = 0.768$). The means, standard errors and 95% confidence intervals are presented in Table 5.

Table 5. Post-test scores on the challenging items in the second experiment (maximum score is 16)

Condition	N	Mean	Std. error	Lower bound	Upper bound
Control	31	1.790	0.423	0.927	2.654
Dynamic	29	1.414	0.354	0.688	2.140
Embodied	31	1.855	0.559	0.713	2.996

Figure 2 presents the total scores for the post-test. There was no effect of the dynamic condition, $d = 0.01$; and for the embodied condition, there was a small adverse effect, $d = -0.14$.

DISCUSSION

From the ANCOVA and ANOVA tests, we may conclude that there are no significant effects of the dynamic and embodied dynamic condition on learning outcome as measured by the tests. This result adds to the 59% of no-significant-difference-results of Berney and Bétrancourt (2016). Even though this puts some weight in the scale in favor of "no added value of dynamics", for us, there is a lesson in the methods used, since we believe there may have been an effect that we failed to pick up to a significant level. The power-analysis based on the pilot seems to have had an outcome too much in

favor of the dynamic conditions leading to a rather small n. Moreover, a 91-second video is a very short intervention. Combined with a possible ceiling effect in the test, we feel a more precise method is needed to measure the effect in a significant way.

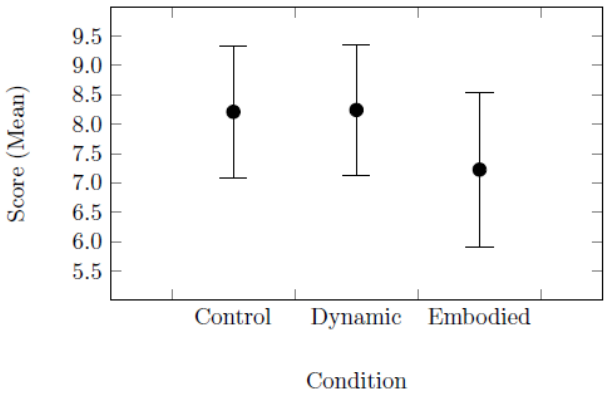

Figure 2. Total post-test scores in the second experiment (maximum score is 32)

Positive learning effects of embodied simulation, as observed in the first experiment, depend on whether the goal of the movement is understood (Van Gog et al., 2009). For the higher-level students of the first experiment (with a bit of prior knowledge of algebra), the goal of the algebraic manipulations, namely simplifying the expressions, was probably clear. On the contrary, the lower-level students of the second experiment may have struggled to grasp this goal from the visualized manipulations. The local goal of the movements themselves must have been clear to all students, though since the voiceover explained, for example, "we now join these terms" or we now "switching these terms". But possibly this is not enough: the global goal of where the algebraic manipulations are leading (i.e., a simplification) may have to be clear as well. The adverse effect found for the embodied condition in the second experiment (with lower-level students) is in line with the results of Castro-Alonso et al. (2018), who find that the effectiveness of dynamic animations reduces when showing hands.

The animations in this study are based on a chain of metaphors, first, a linking metaphor linking arithmetic to algebra, and next to grounding metaphors: the source-path-goal metaphor and the object collection metaphor. We questioned whether these metaphors could be supportive in animated algebra instruction videos. The measured small effects give rise to some optimism, but in particular for higher-level students with some prior knowledge of algebra. Presumably, the arithmetic grounding metaphorical link needs to be well-understood by students for the fundamental metonymy of commutative algebra—that letters take the role of numbers—to be accepted and "recognized" in the video. There may also be concern about the *transitivity* of the two metaphorical links: the object collection metaphor supports arithmetic reasoning, and arithmetic reasoning may support algebraic reasoning, but does that imply that the object collection metaphor supports algebraic insight? In each metaphorical link, part of the inferential structure is preserved, and part is lost. We believe enough supportive inferential structure is maintained, but the outcome of this research might imply that reality is more complicated.

A limitation of watching a video is that it is based on *motor mirroring* (embodied simulation) and not on *motor execution* (enactment). The motoric or bodily engagement is of low level (Duijzer et al., 2019). Also, the level of immediacy is low: Students watching the dynamic and, in particular, the embodied dynamic video have to rely on embodied simulation of previously acquired sensorimotor experiences. Including enactment in the intervention may increase the effect of the video.

This study focused on the effect of dynamical animation on being able to grasp and follow algebraic manipulations. In many videos—like those of 3Blue1Brown—algebraic manipulative skills themselves do not form the learning goal, rather those skills form a prerequisite, applied in a step to reach an interesting result. To make another step in our understanding of dynamic animation of algebra in such videos, our follow-up study investigates whether, in those cases, animated algebra contributes to learning outcomes.

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DIGITAL MEDIA AS TOOLS FOSTERING TEACHER CREATIVITY ON DESIGNING TASKS AROUND AN AREA OF MATHEMATICAL CONCEPTS

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While most research on mathematical creativity focuses on students, in this paper, based on previous research, we move our focus from students to teachers. This is a case study of two teachers that adapted an online task that involved a digital medium for the learning of mathematics and made their own version of it, while a researcher, as a participant observer studied their discourse, trying to shed light on indicators of creativity and to elaborate the potential of this procedure. We used the framework of Social Creativity in a broader sense to draw connections between education design and creativity in this context.

Keywords: Digital media, social creativity, teacher as designer.

INTRODUCTION

The potential of infrastructures like libraries and repositories of digital learning objects as expressive means for teachers to realize their pedagogical and didactical agendas has been discussed in previous research (Kynigos, 2017); where a teacher adapted online digital media embedded in the official educational resource portals for mathematics in Greece and transformed these to be suitable for use in his lessons. The analysis of the adapted resource and of incidents where students engaged in investigation using these media elaborated our view not only of the infrastructure's affordances but of the teachers' professional agenda as well. In the present paper, we focus on the procedure that leads to the adaptation of these resources, the purposeful transformation of the medium, which could be seen as teachers' creative professional activity. According to Sriraman (2004, p. 4), design for mathematics teaching and learning can be a fruitful field for teachers' mathematical creativity. While most research on creativity focuses on students, there are a few studies that make remarks on teachers' creativity as part of their professional engagement–not explicitly mathematical creativity–focusing on the product of teachers' instructional design, i.e. learning tasks that they give to their students (Vale et al., 2012; Vale & Barbosa, 2015). However, according to the literature on creativity, from Rhodes' "Four Ps'" model (Rhodes, 1961) to Plucker et al.'s (2004) approach, there is a consensus on the critical role of the process that leads to a product. As we mentioned above, in our study, we focus on teachers' adaptation of a task that exploits a digital resource; we are searching for aspects of creativity in teachers' efforts to define the mathematical content that is relevant with this resource in order to use it in their teaching. Building on previous research (Kynigos & Daskolia, 2021; Kynigos & Kolovou, 2017) on teachers as designers with the use of digital media, we move our focus to teachers' discourse during their collaborative adaptation design of a new digital medium for teaching mathematics. We aim to shed light on this process, in terms of creativity in general, as a first step to gain insight into the investigation of teachers' creativity. Teachers in this study collaborated on

designing a task with a digital tool that required no special skills in the use of ICT and which affords instructional design within a specific area of mathematics.

THEORETICAL FRAMEWORK

In research on creativity, the focus is mostly on the achievements of the subjects with regards to the mathematical content and certain resources that provoke mathematical activity; e.g. Leikin used multiple solution tasks (MST), which students were explicitly asked to address using more than one solution, to 'measure' aspects of their creativity (Leikin, 2013). In this study, we move our focus from students to teachers. We built on the assumption that mathematical creativity may emerge through a teacher's instructional design, especially through the reframing of the content in favor of a teaching and learning agenda (Sriraman 2004, p. 4). In other words, design for learning can be an aspect of mathematical creativity. With this approach, although instructional design mostly comes to an end and is realized through a product, we expected that creativity should be apparent in the process of design. So, we were interested in analyzing the process where teachers designed tasks for their students, perhaps using something analogous to MST as a means to provoke and analyze teachers' creativity.

In former studies about teachers designing digital artefacts for teaching and learning, creativity was evident in line with the Social Creativity (SC) framework (Fischer, 2004). This framework has been used to study the teachers as designers of educational resources through the lens of creativity (Kynigos & Kolovou, 2017). The SC framework creativity is compatible with the description of creative actions, procedures or products in a social context where people are working together using technology. According to SC, creative actions, procedures or products have characteristics of originality, are expressive, socially appreciated and socially evaluated by a group (Fischer, 2004); in the case of teachers, creativity emerges when they mutually take part actively in a process of designing or redesigning a product, e.g., a learning task, following their own agenda related to teaching and learning. SC was evident in cases where teachers engaged in boundary crossing (Akkerman & Bakker, 2011); teachers with different expertise modified their usual practices as they collaborated to design educational resources to provoke students' creativity towards a shared practice that led to a mutually accepted, appreciated and feasible outcome. In some cases that these studies refer to, the resources that teachers designed aimed to promote and foster investigation of mathematical concepts following the approach of a 'half-baked microworld' (Kynigos, 2007). The purpose was to engage students in a meaning-making process while they used a digital medium, i.e. to fix a dynamic shape, which is 'buggy by design'. From this starting point, in this study, we focused on a case of teachers working together to redesign a 'micro-experiment'; which includes a digital artefact that simulates the layout of an experiment addressing a challenge, provoking students to engage and be involved in inquiry (Kynigos & Grizioti, 2018). Micro-experiments are resources that involve a half-baked artifact that contains some closed questions readily matching equivalent questions from traditional curriculum exercises and one or two open-ended, constructionist or exploratory questions at the end (Kynigos, 2020). The layout focuses on a small set of concepts, e.g., related to 'sums of numbers' in Arithmetic, and the starting point of the challenge is a question, or a well-defined problem. Using the micro-experiment, students may pose their own questions, that lead

to further investigation. Based on previous research (Kynigos, 2017; Kynigos & Kolovou, 2017) and assuming that the adaptation of micro-experiments could be revealing for the teachers' agenda, our question was about the creative aspects of this professional activity.

In line with previous studies on creativity regarding the teacher's role as a designer (Kynigos & Daskolia, 2021; Kynigos & Kolovou, 2017), we followed the principles of 'meta-design' (Fischer & Giaccardi, 2006), a component of the SC framework, to design our intervention. Meta-design describes the conditions that foster the emergence of creativity in terms of SC and refers to a process of co-design in a digital environment, i.e., a digital medium for teaching and learning mathematics. The medium should be accessible by the teachers, meaning that they do not need to be experts to use it and to adapt it. The teachers' belief that the medium promotes their teaching agenda or that the medium is open to include it in their future teaching plans also provokes creativity. Moreover, the technical characteristics of the medium should facilitate the communication of ideas and support discourse among the teachers that co-design. Under these circumstances, SC is likely to emerge, while evidence for this emergence includes ways of using the medium for teaching and learning purposes, or versions of the medium which are 'new' for the teachers that took part in the process of design, with regards to their experiences and practices. This description of SC's emergence could be related to situations that 'everyday creativity' is apparent. In this approach, our choice to use 'meta-design' was justified.

THE DESIGN AND METHOD OF THE STUDY

For our research, we employed a digital authoring tool for mathematics teachers to use ready-made or make their own micro-experiments, called 'Abacus', which, in our view, meets the description of 'meta-design' framework. Abacus can be used by a teacher for task design focused on teaching and learning of numbers, their properties, and their representations in the decimal number system. It consists of a digital abacus simulation, where the user/designer can change the number of rods and put up to 9 beads in each rod to represent a number (Figure 1). There is also an option to use a decimal point so that some of the rods represent decimals like tenths, hundredths, etc. It is feasible to make new tasks/micro-experiments or to adapt existing ones by transforming them. Abacus can be used as a medium for communicating ideas between teachers as they are exchanging or discussing on different versions of a task, which is crucial for the meta-design framework. Decisions such as how many rods should be visible, or whether students should have access to change the number of the rods or the position of a decimal point changing at the same time the place value of each rod could raise critical issues related to the teaching and learning agenda of the designers.

There are 84 Abacus' micro-experiments among a large number (over two thousand) of micro-experiments that have been developed and are embedded in the digital textbooks of Mathematics in Greece (Kynigos, 2020). These micro-experiments are accessible online, provided by the Greek Ministry of Education in Photodentro (http://photodentro.edu.gr/lor/?locale=en), which is a repository, an institutionalised infrastructure available for all students and teachers in Greece, or anyone that is interested in education. The implementation of Photodentro's redesigned digital tools have been studied in classroom settings, focusing on the provoked learning and the underlying teaching agenda of the teacher that redesigned it and used it (Kynigos, 2017). In this paper, we present

a case study (Yin, 2014) of two experienced teachers in Greece that frequently use digital artifacts in their lessons and, in this case, they collaborated to adapt one of Photodentro's micro-experiments or to make a new one drawing ideas from Photodentro. We kept our focus before the implementation phase on the shared redesign of an Abacus micro-experiment by two teachers, which we assumed 'encountered' Abacus as a legitimate medium to use in their classes. So, under these assumptions, we were interested in analyzing the discourse that took place in a co-design process between the two teachers, using the Abacus. M is a primary school teacher and D is a mathematics teacher in junior high school. They had three conversations about the micro-experiment that they designed, which we observed, around twenty minutes each. The first conversation took place before they started to design the micro-experiment, while the next two were conducted to make reviews of their first efforts and decide what should be the final product. The researcher had the role of the participant observer to the discussions, and the corpus of data consisted of notes taken and voice recordings of the conversation, emails and all the digital artefacts produced from their first to their last version. One methodological hypothesis was that the participants and the researcher shared some common views around the use of micro-experiments, which transfuses characteristics of ethnography to our methodology (Hammersley & Atkinson, 2010). From here onwards, we refer to the participant teachers as 'designers.' Aiming for a concrete view of teachers' design for teaching and learning as a creative process in the Abacus environment, our goal was to describe any visible connections of the design process with aspects of designers' creativity and to shed light on the potential of the adaptation of digital media like the micro-experiments.

Figure 1: The original micro-experiment, on the left-hand side. The rods and beads, as they appeared on a draft of the final version of the micro-experiment, on the right-hand side.

RESULTS

The task given to the designers, for the needs of the present study, was to 'design a micro-experiment in Abacus, to make a good use of its affordances' within a week. The only restriction given was that the researcher should be present during task discussions to collect all the data produced and that all their related email communication should be 'cc'-ed to us. What follows are four extracts of interest from their three discussions. The first one is taken from their first discussion.

1 D: So, what do you think, should we make a micro-experiment from scratch?

2 M: I think we should look at the 84 that are designed with the Abacus.

3 D: Ok, you can take the first 40, and I will take the rest. I do not know much about them, anyway.

4 M: I think I will take a look at all of them. I use most of them every year.

Since all the 84 Abacus micro-experiments are intended for primary school students, M felt that it could be helpful if she took over an initial review of the Abacus micro-experiment. In the meantime, D, who needed a better overview of the Abacus' micro-experiments, began 'taking a tour' to be ready for the next step. The next extract is from the first minutes of their second conversation.

5 M: I think that most of the micro-experiments are not representative of the Abacus affordances.

6 D: Don't you use them regularly?

7 M: Yes, but I make some modifications 'on the fly' to be more challenging.

8 D: I have made one myself, to practice. It is about a divisibility rule.

All the 84 micro-experiments designed with Abacus are related to addition and its properties. However, D tried to make one from scratch, incorporating divisibility rules, which on the one hand, is a subject related to Arithemtic, but on the other hand, is something unusual for the use of Abacus. The next extract is from the last minutes of the designers' second conversation, where it seems that the idea of including divisibility rules in the repertoire of Abacus, was abandoned.

9 D: You know, I think this is not the case for Abacus.

10 M: Yes, me too; I think that division is out of its range!

11 D: But, what about addition? Some of the micro-experiments are boring. I could make that addition, with paper and pencil, too.

12 M: I agree; the point is not just on doing operations with Abacus.

In the interim between the second and the final discussion, which was three days, there was a consensus on the kind of micro-experiment that they should use as a sparker. M sent an email to D, suggesting that they could use a micro-experiment about a fishing boat and the total weight of its haul. The student was supposed to add 3.6 tons to 0.42 tons and 792 kilos of different kinds of fish, to calculate their total amount (Figure 1). The given rods on the Abacus were those of Ones, Tens, Hundreds and Thousands. The key point, as M said, was that in this micro-experiment, the student should put the beads for 3.6 and 0.42 on these rods, so a transformation from tons to kilos had to take place beforehand. The following extract is from their final discussion, where D had a different approach.

13 D: I think that the transformation from tons to kilos before putting the beads on the rods is not a good use of Abacus. Could we change the micro-experiment for the students to make the transformation on the Abacus?

14 M: Which way?

15 D: We could add two rods, one for Tenths, and another one for Hundreds so that students can put beads for 3.6 and 0.42 on the Abacus, and then move them three rods to the left to make the transformation and after that, the addition of 792 with new beads. What do you think?

16 M: I think that this is good! It is about place value, not addition! I tend to believe that even addition is irrelevant with Abacus!

Finally, after this discussion, designers completed the adaptation of the micro-experiment, with a transformation as described above.

DISCUSSION

Starting with one of our methodological assumptions, in the first extract, the designers seemed to think of Abacus as a legitimate tool for teaching, since they decided to select it among the existing micro-experiments. However, they initially failed to specify the domain of mathematics that Abacus' affordances focus on; D was impulsive to go beyond the 'boundaries' of the 84 Abacus' micro-experiment, trying to make something new integrating divisibility (line 8) which could lead to a major modification of a micro-experiment, while he had no detailed view of these micro-experiments (line 3). M was also supportive of major changes, since she seemed not too satisfied with the ready-made micro-experiments; she used to modify them to be in line with her teaching agenda (line 7). During the second discussion, it seemed that D changed his mind about divisibility (line 9), while M agreed with him, saying that Abacus has a limited 'range', referring to the mathematical content that Abacus is focused on (line 10). The agenda of their discourse was to define the Abacus' added value as a learning object, which seemed to be an open issue, although they had used Abacus several times in the past. M's expression that division was 'out of range' for the Abacus was indicative of her approach; the demarcation of Abacus' range in terms of mathematical content based on pedagogical and technological arguments. In the same sense, D adopted an 'empirical' pedagogical view in relation to the affordances of Abacus (line 11), realizing that, in this case, the micro-experiment would not be interesting. In lines 13 and 15, D described possible ways of addressing the challenge that, in his opinion, could be important; he exploited that rods' positions can be manipulated dynamically by giving this access to students. Later on, after adapting the micro-experiment on fish weight, in line 16, M concluded that 'addition is irrelevant' of Abacus, not just 'out of its range' and that Abacus could be exploited in a task where students could master place value through Abacus's affordances. The use of addition tasks with Abacus, like the original with the fish, even though feasible, it would not be a paradigm of Abacus' exploitation in the classroom according to D & M's agenda; Abacus affords this kind of micro-experiments technologically, and students all over Greece have addressed them several times possibly obtaining learning benefits through this engagement, since these kind of micro-experiments are available for all teachers and students in Photodentro. However, according to D and M, these micro-experiments were not fully deploying the added value of Abacus. So another criterion about task design choices that could exploit students' learning with the Abacus became apparent through the discourse. Both range and exploitation criteria that emerged through D and M's discourse did not raise and were not based on exclusively pedagogical issues in isolation to technology or mathematics and vice versa. We suggest that, with further data, a possible connection between the components of TPACK (Mishra & Koehler, 2006) and the formulation of these criteria could be apparent.

D and M contributed to the collaboration without invoking certainties; they tried to define an appropriate use of Abacus with regards to their experiences and practices, being open to experimentation and reconsideration of task design. After D's initial idea of more complicated mathematics (divisibility), he became more focused in terms of content and creativity since he came

up with a new idea. M also seemed to reconsider her view about Abacus' teaching affordances (line 16) in the conclusion of the design process. What we saw here was a 'makers' culture' approach in the collaboration and the communication between D & M that possibly led to the new version of the micro-experiment. This new version was the realization of D & M's ideas, a projection of their agenda as it became visible throughout their discourse. At the same time, although it was radically different from the original micro-experiment in terms of instructional design, it can be viewed as an extension of Photodentro's infrastructure since the conversation and communication between the teachers was fed by their experience with the use of Abacus in Photodentro as a whole. This kind of production that has the potential to change the system as a whole can be an indicator of creativity in terms of SC.

CONCLUSIONS

In the present research, we tried to shed light on designers-teachers activity and draw connections between creativity and the design for learning with digital tools. In an environment that fosters creativity (Fischer & Giaccardi, 2006), we assumed that this connection would be apparent and evident. In this case, the environment to foster creativity was the infrastructure of Photodentro and the micro-experiments that teachers can use in their classes for teaching mathematics. Stimulated by previous research findings (Kynigos, 2017), we investigated the discourse between two designers looking for elements of their professional agenda. What became apparent was that via the adaptation of micro-experiments and co-design, the emergence of creativity in design and the professional agenda of the teachers involved can be mutually understood. The adaptation and co-design of micro-experiments happened densely in this ecology where a 'makers' culture' was stimulated and─according to the SC framework─provided a fertile ground for creativity to emerge. Through the discourse in this setting, teachers' professional agenda became clearer. We observed two facets of their agenda, the one nested to the other; the designer's agenda that was apparent with the demarcation of Abacus' potential and the 'makers' culture' that emerged, and the teachers' agenda that was apparent in the argumentation about the criteria (range and exploitation) of this demarcation. What we observed was their shared agenda, shaped through their interaction in order to communicate and to achieve a common goal, something that elaborated our view about their contribution to the new version of the micro-experiment. So, in short, design with Abacus provoked the emergence of creativity in terms of SC which made teachers' professional agenda visible. Viceversa, the agenda provided a description of designers' creative contribution to the original system/infrastructure of the Abacus' micro-experiments; why it was useful for them, how it could be used with novelty in teaching, in such a way that they expressed their own teaching agendas through this adaptation, either they had already used it several times before (in the case of M), or it was rather novel for them as a tool for teaching (in the case of D). In this sense, we may investigate if the adaptation of Abacus' micro-experiments or other digital artifacts like the Abacus, could be used in the case of teachers like MSTs; as a research tool to observe teachers' professional footprints on paths of creativity.

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NATURE OF THE RELATIONS BETWEEN PROGRAMMING AND COMPUTATIONAL THINKING AND MATHEMATICS IN DANISH TEACHING RESOURCES

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This study investigates tasks in 14 didactical sequences developed to connect mathematics and the new Danish subject called Technology Comprehension, which is currently being implemented at 46 schools in a pilot project. The paper develops six categories that describe the different natures of the relations between programming and computational thinking (PCT) and mathematical competencies. These six categories are no mathematics involved, no PCT involved, mathematics as a context, PCT as a context, conceptual integration and operational integration and are defined by distinguishing between actions and concepts in both mathematics and PCT. We conclude the paper by reflecting on the usefulness of such categories beyond the national context from which they are developed.

Keywords: Computational thinking, mathematical competencies, programming.

INTRODUCTION

In recent years, there has been renewed interest in programming and computational thinking (PCT) as subjects of relevance for general education. This has led to curriculum revisions in several countries, including France, Sweden and Norway. In the case of Denmark, there is an ongoing pilot project for implementing a subject called Technology Comprehension (TC). These initiatives have led to an increased number of available mathematics educational resources that, in various ways, bring together mathematics and PCT. Shute et al. (2017) define computational thinking (CT) as 'the conceptual foundation required to solve problems effectively and efficiently (i.e., algorithmically, with or without the assistance of computers) with solutions that are reusable in different contexts' (p. 142). However, there is a debate as to whether CT is better characterised by a closed definition or its building blocks (Pérez, 2018). Moreover, given the openness of CT, not all CT frameworks include the technical skill of programming. For disambiguation's sake, we focus on PCT in this paper.

From the definition above and since Papert (1980) gave birth to the concept of CT, the similarities and potential synergies of bringing together PCT and mathematical competence have been evident. More recently, a body of research has emphasised these commonalities, indicating the many ways of establishing meaningful relations between PCT and mathematics from an educational perspective (e.g. Benton et al., 2016, 2017; Weintrop et al., 2016). As the number of mathematics educational resources involving PCT is proliferating these years, developing a vocabulary to understand the types of relations we can establish amongst these topics and how they vary across grade levels and national contexts has become increasingly important. Building on the resources designed in Denmark to support the integration of TC, this paper reports on the construction of analytical categories that articulate the nature of relations between PCT and mathematics in a Danish context.

We organise the paper as follows. We begin by describing how PCT is embedded in TC in the Danish context, in which a pilot intervention-based research project is conducted. From here, we can focus on and state our research question. Next, we introduce existing research on the relations between PCT

and mathematics and argue how our work relates and contributes to this body of knowledge. We then describe the empirical foundation and the approach to data analysis that we use to address the research question. Finally, we introduce the analytical categories describing the different relations between PCT and mathematics and provide concrete examples of how these appear in the data. We conclude the paper by discussing the possible implications of our results.

BACKGROUND

Denmark is currently implementing TC as a pilot project in 46 different schools across the country. The pilot project is designed to inform subsequent decisions on a national scale and consists of implementing and evaluating two ways of carrying out TC: (1) as a subject in its own right and (2) integrated into other subjects. Prior to the implementation, a so-called advisory expert writing group consisting of researchers in computer science education and computer science, teachers and consultants from teacher training institutions, educational consultants from municipalities, teachers from compulsory schools and consultants from the Ministry of Education developed a curriculum for the new topic of TC (Børne- og Undervisningsministeriet [UVM], 2018). This curriculum includes four competencies: *digital empowerment*, *digital design and design processes*, *technological agency* and *computational thinking* (Smith et al., 2020). Each area defines a competency goal for students after grades 3, 6 and 9. The curriculum describes the skills and knowledge that students need to acquire in four to five content areas. The content areas for CT are, for example, data, algorithms, structuring and modelling. Although programming is not amongst the four competency areas in CT, it is one of the content areas for the competency area of technological agency. In strategy (1) referred to above, this curriculum is implemented in its entirety as a new subject. In strategy (2), the elements of each competency area are added to the curriculum of the concerned subjects. No elements from these subjects' previous curricula are removed. In this paper, we focus on TC as integrated into the mathematics curriculum. The Danish mathematics curriculum is organised around the Competencies and Mathematics Learning (KOM, in Danish) framework (Niss & Højgaard, 2019) and describes competency goals for three stages in Danish K-9 schooling—goals to achieve at the end of grades 3, 6 and 9. These goals are described by a combination of competencies from the KOM framework and mathematical subject areas. TC is integrated into the mathematics curriculum by adding new competency areas, including TC subject areas described as skills and knowledge goals.

A central part of the implementation strategy for the pilot project was to develop a collection of resources aligned with the new curricula for teachers to integrate into their teaching. These materials sought to develop meaningful exercises and tasks in longer, coherent, didactical sequences that included and meaningfully called for the integration of TC and the *host subject* for each grade level (1–9, students aged 6–15). In the case of mathematics and any of the other subjects in which TC is integrated, the didactical sequences explicitly state which mathematical and TC competency areas and subject areas they include. In this paper, we will investigate the *nature* of the relations between mathematics and the competency area of CT, as well as the subject area programming, as they unfold in the tasks and exercises included in the didactical sequences. We seek to answer the following research question: *What is the nature of the relations between PCT and mathematics in Danish TC didactical sequences?*

In what follows, we summarise previous research concerning this and related matters.

RELATED WORK

Previous studies have already examined the interplay between CT and mathematics in both national curricula and textbook materials or similar resources. Amongst these studies is that of Misfeldt et al.

(2020), who compared the relations between CT and mathematics in England, Denmark and Sweden. Their work mainly focused on the *extent* of the relations at the curricular level, conceptualised as either specific, explicit, weak or non-existing (Misfeldt et al., 2019). *Specific relations* are when curricular documents state the relations between CT and a specific area of mathematics or a mathematical working process; explicit relations are when CT is related to mathematics, mathematical working processes and/or competencies; *implicit relations* are when a relation to mathematics can be inferred but is not directly mentioned; and *weak* or *non-existing relations* are when neither of the above is the case. These findings provide an important overview of differences in the prominence of mathematics in CT at a curricular level amongst different nations but do not address internal variations in *how* mathematics is specifically, explicitly or implicitly, related to CT. Whilst this study focuses on curricular documents, a recent study conducted a textbook analysis with a focus on activity types in tasks that include CT and mathematics developed for Swedish mathematics education (Bråting & Kilhamn, 2021). This study investigates the frequency of mathematics/programming *concepts* and *actions* and their co-occurrence (described as *bridging*) in Swedish mathematics textbooks. Similar to the work of Misfeldt et al. (2019), this study does not focus on variations in how programming and mathematics are bridged. Another recent study (Kilhamn et al., 2021), however, takes an explicit focus on investigating the nature of the relations between CT and mathematics. Based on interviews with Swedish mathematics teachers, this study finds different conceptions of the relations between programming and mathematics, including no relation, mathematics as a context for CT learning, programming as a tool and programming as a tool for exploration (Kilhamn et al., 2021). As noted in international overview reports (Bocconi et al., 2018), there are significant differences across countries in terms of what content programming and CT curriculum revisions include and how they are implemented in the curriculum. These differences offer the possibility of investigating the potential synergies between mathematics and CT in different countries and by drawing on different types of data. In this paper, we seek to contribute to building such an overview by developing categories of the relations between CT and mathematics in a Danish context from resources developed to integrate TC in mathematics. In what follows, we describe our approach to collecting and analysing the data.

DATA AND METHOD

As said, several activities that combine TC and mathematics for grades 1–9 have been developed and are available on https://tekforsøget.dk/forlob/. These materials are characterised by an explicit combination of mathematical competencies, mathematical subject matter areas and TC competencies. A full overview of how these competencies are combined throughout the sequences is available here.

A unique characteristic of the Danish TC curriculum is the emphasis given to design processes and critical thinking, which have been key components to ensure compatibility with the declared purpose of Danish compulsory schooling; it describes the role of schooling as to 'prepare students for participation, co-responsibility, rights and duties in a society with freedom and democracy' (§1, section 2[1]). This characteristic of both the curriculum and the resources in Denmark is difficult to compare with those of other countries. To maximise subsequent comparison with another context, we have chosen to focus on the didactical sequences that include programming (which is a component of the area of technological agency) and CT (which is an area in itself). Together, these consist of 14 of the didactical sequences.

Such sequences are broad because they consist of semester courses and their titles indicate a general guiding challenge or inquiry to address. For example, the course *'Next steps with micro:bit'* explores the capabilities and limitations of the micro:bit device as a reliable step counter. The end goal is to

construct a bicycle computer using collected and processed measurements as input for redesign. Along the way, students engage with mathematical ideas, such as bar charts, perimeter and data, and PCT ideas, such as programs, coding and algorithms. Therefore, a course too comprehensive to be a unit of analysis revealing possible fine-grained synergies between mathematics and PCT.

The sequences are structured into three phases: introduction, construction and challenges, and outro. Within each phase, activities take the form of *feedback* and *subject* (*faglige* in Danish) *loops* (UVM 2019). However, the materials signpost tasks with different names, such as *activities* and *concrete challenges*. We therefore designate the units of analysis as each task included in the 14 aforementioned didactical sequences for a total of 193 tasks.

Approach to the Analysis

We analyse these resources by taking a grounded approach in which the goal goes 'beyond creating rich descriptions *of* data to that of generating theory *from* data' (Teppo, 2015, p. 3f, emphasis in the original). Our approach to the analysis included three stages corresponding to what in grounded theory is described as *open coding*, *intermediate coding* and *theoretical saturation* (Birks & Mills, 2011). All three stages of the coding were driven by the question of investigating the relations between PCT and mathematics in the given exercises/tasks. In the first stage, we jointly coded tasks from two didactical sequences and developed an early set of categories based on negotiation and preliminary agreement. This process was thus inductive in nature, as we sought to synthesise general categories from particular instances in open coding. In the second stage, we individually analysed three didactical sequences in parallel to (1) code the tasks and (2) refine the analytical categories based on insights from the data. These individual and parallel analyses were followed up by meetings in which we again discussed and negotiated results, ultimately leading to refined versions of the categories.

This process can be characterised as abductive in nature, as we explicated and drew consequences from our preliminary categories by applying them in the coding in (1) whilst still testing them in (2) in an intermediate coding. In the third stage, we applied the categories to the remaining didactical sequences to ensure that the tasks or exercises did not include relations that could not be adequately captured by our categories. We can describe this as a deductive approach seeking to saturate and validate the categories. This process led to a total of six categories with distinct ways in which PCT and mathematics relate to each other. Below, we describe these categories and provide an empirical example from the didactical sequences for each of them.

RESULTS

In what follows, we briefly describe and exemplify each of the resulting six categories. The first two trivial categories account for those tasks that relate only to either mathematical competencies or PCT competencies.

No mathematics involved. The task exclusively includes PCT concepts or actions.

Example: 'What can a robot do?' – 3.1.9. Wrap-up. The teacher asks pupils to reflect on why not all algorithms work, and introduces the notion of debugging, asking whether they can be revised. She ought to connect these to the unplugged introductory activity of giving and following simple instructions amongst pupils. In the task, the teacher defines algorithms roughly as sequences of precise steps, such as the pupils' morning routines or the instructions given to a classmate to walk from one place to another. Debugging is the process of revising algorithms if they do not work. Overall, the task only includes PCT concepts and actions.

No PCT involved. The task exclusively includes mathematical concepts or actions.

Example: 'Statistics with bias' – 3.1.4. Subject loop 3. Students are invited to follow a tutorial on making diagrams with GeoGebra. The tutorial uses numerical variables *a*, *b* and *c,* whose values can be controlled with sliders. The template positions them on the *x*-axis as 1, 2 and 3, respectively, and three segments should be drawn from the *x*-axis to the positions given by the variables' values. The PCT learning goals invoked in the sequence refer to the use of data as representations of information from daily life situations. Therefore, the task does not include PCT concepts or actions. Making diagrams with GeoGebra activates mathematical tools and aids competency instead.

The next two categories characterise those tasks in which one of the areas—mathematics or PCT—is operationally dominant over the other.

Mathematics as a context. The mathematical concepts can be replaced with content from other contexts or subjects without changing the PCT operational task. In other words, the task is solved with PCT actions and involves mathematical concepts. This includes the case in which a mathematical concept is explored with PCT operations.

Example: 'Can you play yourself skilled in mathematics?' – 3.2.2. Concrete challenges. The core of the didactical sequence is to program a game on Scratch, in which a player is confronted with sequential arithmetic exercises to gain or lose points. The task enacts unambiguously the PCT learning goal of icon programming. However, the mathematical aspect of the task could be replaced with any other trivia-type questions and solved by the same PCT concepts and actions. Although posing arithmetic exercises can be considered a mathematical action, the sequence is oriented to sixth-grade pupils, whilst the target player should be between the fourth and fifth grades, implying no new mathematical learning by solving the task. The task is to program, mathematics is its context.

PCT as a context. The PCT concepts can be replaced with concepts from other contexts or subjects without changing the mathematical operational task. This includes the case in which a PCT concept is explored with mathematical operations.

Example: 'Concept of chance' – 3.2.4. Subject loop. Students download and use a built-in spreadsheet in GeoGebra to simulate dice rolling experiments and produce ad-hoc histograms automatically. Students do not make (program) or modify (debug) the code. However, the task enacts the stated learning goal of performing chance experiments and estimating intuitive probabilities.

The final two categories refer to the symbiotic integration of mathematical competencies and PCT in the same task.

Conceptual integration. The task in itself is not solved with mathematical or PCT actions, but it involves concepts in both math and PCT. This is the case with several *scene-setting* and *wrap-up* activities.

Example: 'Update dice' – 3.1.4. Scene setting. The teacher begins by recalling the game of Yahtzee, its rules and its mechanisms. She then asks what would happen if they modified the dice involved in the game. The process and consequences of modifying dice involve mathematical concepts (probabilities, sample space) intertwined with PCT concepts (programming). However, the task does not involve actions stated in the learning goals, i.e. *computing* probabilities, *defining* a sample space or *modifying* or *constructing* a program.

Operational integration. Mathematical and PCT competencies are interdependent. The task cannot be solved by replacing the concepts from one area or another. This is the case for tasks in which both mathematical and PCT competencies are operational in the same action.

Example: 'Polygons' geometrical characteristics' – 3.3. BeeBot and polygons. Pupils are asked to explore the figures they can make with a BeeBot. Using their knowledge of polygons, pupils explore their affordances and constraints. The suggested open questions include the following: What can it do? How does it turn? How small and big of a rectangle can they make? What does an algorithm for a square looks like? For a rectangle? Hereby, programming a robot to describe a determined polygon is, at the same time, a mathematical and PCT action.

Discussion

At this stage, we do not take a quantitative strategy, but it is safe to say that a majority of the tasks enact either mathematical or PCT competencies, thus fitting into one of the first two categories. It may be tempting to deem these tasks uninteresting or even problematic. The impression may be the same where mathematics or PCT serves as a context by idealising the cases in which mathematics and PCT integrate. However, these tasks may play a significant role in pausing and crystalising specific concepts or actions to advance the overall inquiry. Each didactical sequence is set to be enacted in several weeks (UVM, 2019), and the units of analysis are pieces of a bigger puzzle. Overemphasising integration could be counterproductive in the sense of intimidating learners who struggle with either learning school mathematics or adopting PCT (Sach, 2019).

Open and intermediate coding in grounded theory methods is based on constant comparative analysis (Birks & Mills, 2011) by asking whether different units belong to a same category. For example, in 'Polygons' geometrical characteristics', two different tasks deserve special attention. One of them asks students to program their BeeBot to *land* in determined geometric figures. Another asks them to program their BeeBot to *describe* a geometric figure with its trajectory. Available analytical frameworks (e.g. Benton et al., 2017; Bråting & Kilhamn, 2021) use the notion of *bridging* mathematics and PCT, in which both tasks may well fit. However, in the former, pupils need only to identify or recognise a figure (mathematical concept), whereas in the latter, they ought to construct the figure by enacting its characteristics (mathematical action). This subtlety gave birth to the distinct categories of *mathematics as a context* and *operational integration*. The code states that in the former, one can replace mathematical concepts with others (e.g. geometrical figures by animals or colours) and still be able to solve the PCT task. In the latter, this is not possible.

The example taken from 'Concept of chance' is classified as *PCT as a context*, despite the activity being set up in a computerised environment. The GeoGebra spreadsheet not only contains programmed code, but the teacher is also encouraged to be more or less explicit about how it works. However, the action itself is mathematical, as it activates the tools and aids competency (Niss & Højgaard, 2019). It may still enact other competency areas of TC related to digital technologies (Jankvist et al., 2018), such as *digital empowerment* and *technological agency*. That is, the use of computer software is not a direct indication of PCT actions. Moreover, the task can be solved by doing physical chance experiments and collecting data by hand, thus replacing the PCT concepts with analogue mathematical tools and aids.

CONCLUSION

In this paper, we have investigated the nature of the relations between PCT and mathematics in the tasks of 14 didactical sequences developed to connect mathematics teaching and TC in Danish compulsory schools. By conducting grounded coding and analysis of each of the tasks in this material, we have developed six categories that conceptualise the different ways in which mathematics and PCT are related, and we have provided empirical examples of each of category. The categories included no mathematics involved, no PCT involved, mathematics as a context, CT as a context, conceptual integration and operational integration. To conceptualise clear distinctions of what

immediately appeared as subtle differences amongst the tasks, we introduced a distinction between *actions* and *concepts*. This allowed us to more clearly articulate the differences between tasks in which students, for example, were to *draw* a specific geometrical figure with algorithms (actions in both PCT and mathematics) and tasks consisting of *guiding a sprite onto* a specific geometrical figure at a specific location.

We developed these categories and their definitions on the basis of data restricted to a specific context—in this case, the resources in a Danish pilot project. In that respect, the categories can be considered a *humble theory* (Cobb et al., 2003), as they apply to a rather restricted situation. We nonetheless see contributions of this work beyond a narrow national context. For one, it provides insights into a Danish case, which is relevant for comparing against other contexts in which different approaches to implementing PCT are adopted. Many countries are currently implementing or experimenting with various ways of integrating CT and mathematics. To understand the differences in these approaches and systematically evaluate and discuss their applications and potentials in different levels of mathematics education, there is a need to develop a vocabulary to articulate differences in the nature of the relations between CT and mathematics. This paper can be thought of as a contribution along this line and can serve as a starting point for similar analyses conducted in other contexts. Moreover, although developed from a Danish context, the categories provide conceptual clarity that can inform design experiments by acting as part of the hypothetical learning trajectories, which can contribute to building a solid understanding of the potentials of different PCT and mathematics relations.

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DISCOVERING THE POSSIBILITIES OF A COMPUTER-BASED LEARNING ENVIRONMENT FOR MATHEMATICAL MODELLING

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Keywords: Computer-based learning environment, digital tools, modelling, process data analysis.

INTRODUCTION

Innovative technologies can not only enrich mathematics education and competence development, but also offer new perspectives on empiricism and ways to analyse (cognitive) learning processes. Such perspectives, possibilities and enrichments should be explored as well as discussed in the workshop presented here. A digital learning environment for mathematical modelling developed within the project *Modi – **Mo**delling **di**gitally* at the University of Münster served as a focal point.

In this project, a computer-based learning environment is understood as a pre-structured but open provision of learning materials, which are delivered computer- or web-based (Baker et al., 2010; Isaacs & Senge, 1992; Jedtke & Greefrath, 2019). Here it is possible to provide various digital tools. Accordingly, such an environment also allows to stimulate independent learning processes, to enable new forms of interaction of the learners with the content, as well as to see teachers as companions (Engelbrecht et al., 2020; Greene et al., 2011; Veenman, 2007). With regard to mathematical modelling, it has already been demonstrated in video-based studies that various digital media and tools can be meaningfully integrated into all sub-processes and are also used by learners (Geiger, 2011; Greefrath et al., 2011).

DESCRIPTION OF THE PROCEDURE IN THE WORKSHOP

In this workshop, which was scheduled for a duration of 1.5 hours, a computer-based learning environment on mathematical modelling was explored from different perspectives and thus provided new incentives for participants. To this end, mathematical modelling and the potentials in connection with digital tools and media first were briefly introduced, as they served as the basis for the design of the learning environment, and thus, didactical principles could be identified. On this basis, the developed learning environment was presented, whereby the focus was on those didactical principles.

The computer-based learning environment includes GeoGebra as well as videos, pictures, a notepad and a simple calculator and could be accessed via a web browser. The technical realization was also presented shortly.

Afterwards, participants had time to explore the developed learning environment and solve the modelling tasks. Due to the quite small group, the participants worked individually. Different observation focal points were offered: how is the learning environment structured, how were digital tools integrated, how are the GeoGebra applets designed, or how is a self-directed and open learning process made possible?

The last half hour of the workshop was intended to provide a brief outlook on the possibilities of process data analysis on the one hand and to provide space for discussion on the other.

The possibilities of process data analysis and learning analytics were highlighted by taking a look at a log data file that was generated by the interaction of one randomly chosen participant and by

considering an R script using the log data generated in advance. The final discussion included technical, ethical, theoretical and practical aspects as well as an outlook concerning studies conducted with the presented digital learning environment.

DISCUSSION AND CONCLUSION

The workshop described here was able to provide a holistic view of the use of a digital learning environment for mathematical modelling. Especially the active work phase, in which the participants could use the learning environment themselves, was profitable and offered an interesting discussion. This discussion focused on the open tasks, the integration of the various tools, as well as the intuitive design. The second aspect of the discussion was related to the analysis of the data. Here, the focus was on explaining some of the analyses and making the interpretation of the results more valid. From this, approaches to further research were developed. Overall, the workshop was accordingly profitable and could set new impulses for future studies.

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HOW ABOUT THAT ALGEBRA VIEW IN GEOGEBRA? A REVIEW ON HOW TASK DESIGN MAY SUPPORT ALGEBRAIC REASONING IN LOWER SECONDARY SCHOOL

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This paper reviews the existing literature for insights on how functionalities in the Algebra View in GeoGebra can be used in task design to activate lower secondary school students' reasoning competency when working with variables. Through an extensive review, only a small number of studies were identified, indicating that this area of research so far has been neglected. Nevertheless, the studies included point towards the slider tool to be useful, as it allows students to test their conjectures about the mathematical relationship a variable represents and to experience if-and-only-if statements. More specifically, typing in algebraic expressions containing variables in an input field can orient students' reasoning towards the symbolic representations.

Keywords: Algebra, GeoGebra, reasoning competency, task design, variable as a general number.

INTRODUCTION

In this paper, I consider the potentials of GeoGebra's Algebra View to support students' reasoning with expressions and variables as a generalized number by reviewing existing literature in the field of mathematics education, with a focus on lower secondary mathematics education.

Within the last decade, the use of digital technologies in mathematics education has increased. In Denmark, this development has coexisted with the implementation of the idea of mathematical competencies (Niss & Højgaard, 2019) in the Danish mathematics programs, and both developments have been supported by the educational policies (UVM, 2019). This to such an extent that the interplay of digital tools and students' mathematical competencies have become the subject of research (Geraniou & Jankvist, 2019), as new didactical potentials and possibilities emerge when new technological tools are introduced in educational practices (Artigue, 2002).

In Denmark, Dynamic Geometry Systems (DGS), and in particular GeoGebra, have been implemented in mathematics education throughout primary and lower secondary education. GeoGebra holds the common features of a DGS but also differs by incorporating algebra, geometry, and calculus in the same dynamic software (Hohenwarter et al., 2009). What can be considered unique for GeoGebra is the so-called Algebra View (Wassie & Zergaw, 2018). All graphical objects are simultaneously expressed algebraically and numerically in this panel. Through an input field in this panel, objects can be constructed. This includes geometrical objects, but also algebraic objects such as variables (expressed by a slider), functions, groups, etc. It has functionalities such as measuring, counting objects, logical and boolean conditions, as well as functionalities similar to Computer Algebra Systems (CAS). Yet, it also holds functionalities that go far beyond, and the syntax is considerably different. That the Algebra View provides the option for students to work algebraically with mathematical objects seems in line with the developments of early algebra. Already in 1998, Kaput (1998) pleaded to algebra school mathematics across all ages, leading to an increasing number of studies and projects focusing on younger students' early algebraization (e.g., Cai & Knuth, 2011). Yet, little research has focused on the development of algebraic reasoning in lower secondary school (Knuth et al., 2011). Variable in school mathematics is used as a symbol of an 'unknown quantity',

as a 'general number' for any indeterminate quantities or, in functional relationships as 'covariation'. Generally, the research on the potentials of the use of DGS in mathematics teaching and learning has focused on conceptual development in Euclidean geometry as well as simple and complex functions (Wassie & Zergaw, 2018), including also the bridging of these two mathematical domains (Pedersen et al., 2021). Consequently, the research largely focused on either co-variance in functional relationships or invariants in geometric constructions. More specifically, the research focusing on the potential for DGS to support students' reasoning and reasoning competency has largely focused on the teaching and learning of Euclidian geometry (e.g., Højsted, 2020). How DGS and GeoGebra specifically can be used to activate students' reasoning competency when they are working with variables as a general number is less researched. In this paper, I attempt to draw out of the literature what has been researched in this matter. Hence, I ask *What existing literature for which functionalities in GeoGebra's Algebra View can be used in task design for activating lower secondary school students' mathematical reasoning competency when working with variables as a general number?*

THE KOM-FRAMEWORK AND ITS REASONING COMPETENCY

The Danish mathematics competency framework (KOM) (Niss & Højgaard, 2019) was initially developed for educational use and as such describes what mathematics as a disciple demands of cognitive processes in terms of competencies. One of the outsets for this endeavor was to overcome the understanding of school mathematics as only concerning the learning of the subject matter, but also to encompass a set of competencies that reflect what is distinctive for mathematics practice in the society. The framework constitutes eight competencies. KOM defines a mathematical competency as "someone's insightful readiness to act appropriately in response to a specific sort of mathematical challenge in given situations" (Niss & Højgaard, 2019, p. 6). In this review, I will focus on the reasoning competency. The reasoning competency is associated with situations where students analyze or produce mathematical arguments. These can be oral or written arguments and in a range of justifications form from exemplifying to deductive and formal proof. An argument is considered to be a chain of statements linked by inference to justify mathematical claims or solutions to mathematical problems. To consider what are appropriate actions in a particular situation is also restrained by the mathematical topic and the problem posed. At the lower secondary school level, the curricular goals for the competency are that students should be able to distinguish between individual cases and generalizations, as well as develop and evaluate mathematical reasoning, including when working with digital tools (UVM, 2019). The reasoning that goes on in the classroom at this level is, however, informal, and only a few—or maybe even no—deductive proofs are dealt with in class. Yet, it is at this stage that students are expected to be able to put forward justifications of mathematical relations that to a higher degree rely on theoretical knowledge and, to a lesser extent, their intuition.

METHOD

The literature search was done in five stages. In stage one, relevant texts were found by database searches in ProQuest and Web of Science, limited to the educational databases and texts in English.

In ProQuest (Hits=115) the following search string was used (10th August, 2020):

- noft(GeoGebra) AND noft(Algebra* OR vari*) AND la.exact("English" OR "Danish") AND la.exact("ENG") NOT edlevel.exact("Higher Education" OR "Postsecondary Education" OR "Adult Education") AND PEER(yes),

In Web of Science (Hits=50), two sets were created and combined (10th August, 2020):

- Set #1 (GeoGebra) AND noft(Algebra* OR vari*)

- Set #2 NOT ("Higher Education" OR "Postsecondary Education" OR "Adult Education")

20 duplicates were found using *Jabref*. All in all, 146 texts were identified. Also, the following conference proceedings were screened by searching for "GeoGebra" and then identifying any use of the Algebra View in the identified papers. This was done for CERME (N=8), ICTMT 10th- 14th (N=2), MEDA (n=0). All 156 texts were uploaded to *Covidence*, where another five duplicates were identified. The remaining 151 texts were then abstract-screened, and 48 were full text screened, following the inclusion/exclusion criteria seen in table 1. Studies that used other software in a manner that was highly similar to that of the Algebra View have also been included in the review.

	Inclusion criteria	**Exclusion criteria**
Tool use (1)	Use of GeoGebra or highly similar software, where algebra features are explicitly used.	Only use of geometric features. Not GeoGebra or software with highly similar features
Age group of participants (2)	Primary and lower secondary.	Kindergarten, adult students, in-service and preservice teachers, university or college students
Types of students (3)		Students with dyscalculia, deaf students, and students with special needs
Types of studies (4)	Empirical or theoretical.	Studies without any documentation, or any description of or reflections about students' interaction with algebra functionalities
Mathematical Content (5)	Variable as a general number.	Co-variance/functions, statistics, programming, STEM

Table 1. Inclusion and exclusion criteria

Stage 1:	156 references imported for screening
	5 duplicates removed
Stage 2:	151 studies screened against title and abstract
	103 studies excluded
Stage 3:	48 studies assessed for full-text eligibility
	44 studies excluded: 16 (criteria 1); 9 (criteria 2); 10 (criteria 4); 7(criteria 5); 2 (not English or Danish)
Stage 4:	4 studies included
	1 study included from sources identified in reference list
Stage 5:	In all 5 studies included

Table 2. Prisma of inclusion/exclusion process

By snowballing references, "Future curricular trends in school algebra and geometry: Proceedings of a conference" was identified as a source, and after the screening, one study was added.

PRESENTATION OF STUDIES AND FUNCTIONALITIES IN THE STUDIES

The five identified studies are all peer-reviewed but cannot be perceived at the same quality of a journal paper. This indicates that the research of the potentials of GeoGebra's Algebra View, and its functionalities for mathematical tasks and processes other than functions, is still in a developing phase. Two of the papers are theoretical, while three present empirical results. Four of the five studies make use of GeoGebra, and one study by Lagrange and Psycharis (2011) makes use of a programming "turtle world" software, LOGO, which makes use of very similar affordances to that of the Algebra View in GeoGebra. The software makes use of programming language, whereas the Algebra View in GeoGebra uses standard algebra notations and commands specific to the program.

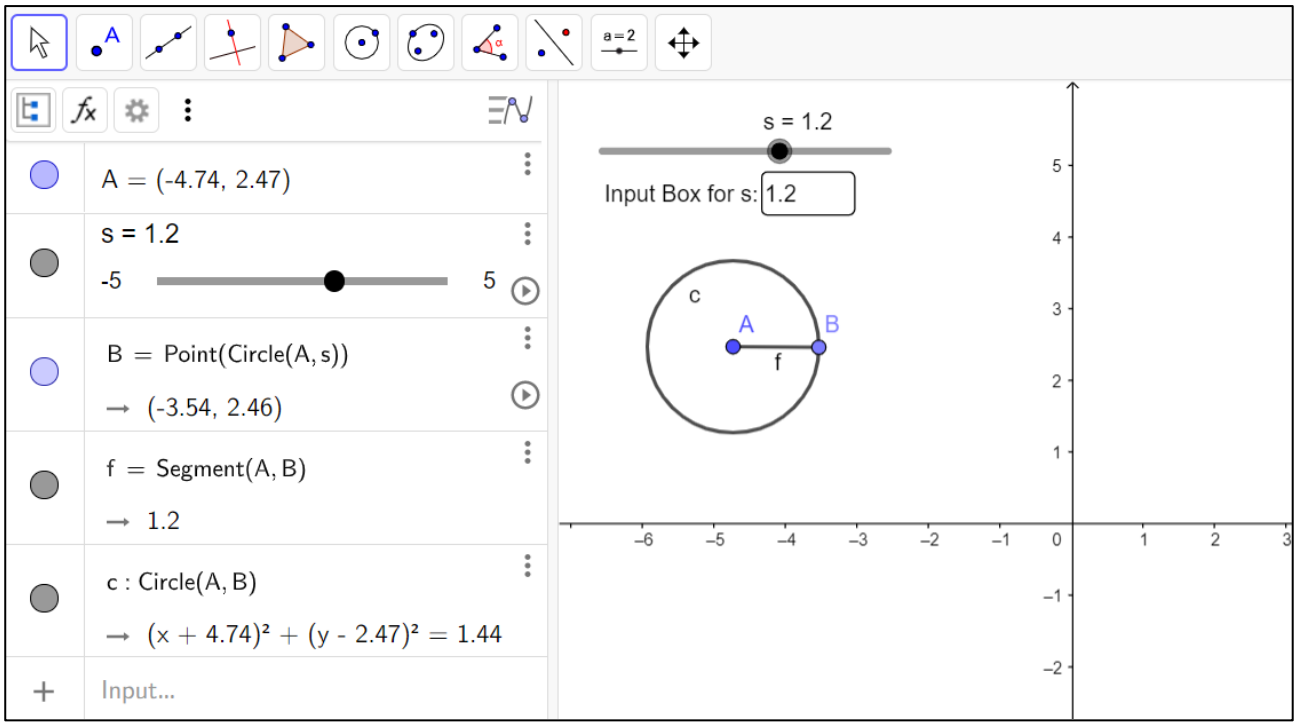

Figure 1. GeoGebra Classic online. A slider along with an input box, varying the radius of the circle *c* and line segment *f*. All visible both on the Graphics View and the Algebra View

The functionalities investigated in the five papers are the slider (in some cases also controlled by an input box) and typing expressions containing one or more variables in the input field. In the analysis of the studies, I distinguish between variables that appear explicitly and implicitly in the Algebra View. Please, refer to figure 1 for the following descriptions of functionalities. The explicit appearance is the slider tool is created by typing in the name or the letter of the variable in the input line, which produces a line segment with a dot on it (none of which are actual geometrical objects), a numerical value, and the numerical limit of the variable, which by default is -5 to +5. These limits are changeable. When dragging along the line segment, the numerical value changes accordingly. One can also adjust the increment by which the numerical value changes. A slider is visible in the Algebra View, and can also be displayed in the Graphics View. A slider can control any geometrical object in the Graphics View by defining the object by a variable. For example, in figure 1, the radius of circle *c* is defined by *s,* and hence also the length of segment *f* =*AB*, which is why both can be varied by the slider. The implicit appearance of variables happens through the construction of any dynamic geometrical object on the Graphics View, which then appears in the Algebra View with a name, definition, and value. For example, a line segment, as in figure 1, will appear with a name, e.g.,

f (a variable), defined by its endpoints, *AB*, and its value in terms of length, which in this case can be varied by dragging the slider as point *B* is restricted to the circle.

ANALYSIS OF STUDIES

The two theoretical studies, Mackrell (2011) and Jawkin (2010), are critical towards the integration of geometry and algebra in GeoGebra. Jawkin (2010) does acknowledge that analytical geometry is an obvious connection between these two domains and agrees with the intentions of the software to tackle student resistance towards algebra. One major concern, however, is that much of analytical algebra is not within the scope of elementary school mathematics. Furthermore, he argues that the unification of algebra and geometry in GeoGebra poses a pedagogical issue in the infrastructure of representation. For example, what appears to be a circle in the Graphics View is actually a plotting of a quadratic function, which loses its circular shape if coordinate axes are changed. Mackrell (2011) experiences that the construction of geometrical objects in the Graphic View that produces implicit variables can result in discrepancies in the representations in the Algebra View. For example, if a circle is dragged, the equation for the circle varies, which is an algebraic representation, whereas if a segment is dragged, then it is the measurement of its length that varies, which is not an algebraic representation (notice *f* and *c* in figure 1). Mackrell (2011) exemplifies the difficulties of using the Algebra View to calculate the relationship between the area of the circle and the radius, not only because of the discrepancy in algebraic representations but because of the large amount of information. Jawkin (2010), on the other hand, defends this discrepancy by considering the complexity of the algebraic representation that students would have to face if a line segment was represented by its equation and limits. Despite Mackrell (2011) being critical, she points out that the slider "has the potential to be an important link between geometric and algebraic representations" (p. 3), but she does not investigate the use of the slider any further. Mackrell (2011) and Jawkin (2010) both point out issues that must be considered when developing task design for GeoGebra, both in general and when designing tasks that aim at activating students' reasoning competency when working with variables. Interestingly, none of the three empirical studies seems to encounter the issues that are described here, which might be explained by the fact that they all use variables explicitly through the slider and not implicitly.

The first example of explicit use of variables is that of Lagrange and Psycharis (2011). In the task posed, they ask students to dilate an alphabet letter proportionally dependent on a single variable controlled by a slider, using LOGO. They describe how a slider provides a linkage between the algebraic and the geometrical representation by "providing a link between the graphical distortion and the symbolic aspect" (p. 199). They argue that by dragging a slider, the status of the *physical system* is connected to the status of the *symbolic system*. This allows students to conjecture about cause and effect between the numerical values and the visual variants depicted in the Graphics View. In the study by Lagrange and Psycharis (2011), the students' only possibility to change the physical system is through the symbolic system. This supports that students' reasoning about the relation between the symbolic and physical system is oriented towards the symbolic system, since students must produce conjectures about algebraic expressions and test them by dragging the slider. This explicit use of variables and students' possibility to algebraically act in the system indicates that it is possible to direct students' reasoning toward algebraic expressions.

Using sliders to validate or refute conjectures about the relations between numeric values and geometrics relationships is also elaborated by Soldano and Arzarello (2017). The task design in this study only partly uses functionalities of the Algebra View. The Algebra View is hidden, but a slider and input boxes are depicted in the Graphics View. Their task is a so-called "Hinitikka Semantical

game", where students must compete while investigating under which numerical circumstances two circles tangent. They do so by manipulating three sliders controlling the radius of each circle and the distance between the circles. The Graphics View also depicts the numeral value of three variables. The students must discover that these express the absolute value of the difference between the radii, the sum of the radii, and the distance between the centers of two circles. Soldano and Arzarello (2017) find that by dragging sliders to represent different generic states in the configurations in the Graphics View, or to consider different values of a variable, students challenge each other's claims by producing examples and counterexamples, leading students to find and even appreciate 'if-and-only-if' relationship. The task design supports students to reason about which geometrical relationships the variables resemble through two different uses of the sliders. Nevertheless, the students' argumentations are not oriented towards algebraic expression, as we saw in Lagrange and Psycharis (2011). Possibly this is because the students cannot test algebraic expressions in the task.

In Tanguay et al. (2013), the use of variables and sliders is oriented towards reasoning about arithmetical relationships. They conduct an apriori analysis of a task design that displays two sliders along with input boxes in the Graphics View, hiding the Algebra View as in Soldano and Arzarello (2017). One slider controls the *n*-number of isosceles triangles grouped around the center, and the second slider controls the angles in the center. Students are to identify cases of when an *n*-sided polygon is formed, which is when the sum of the angels in the center is 360 degrees. In the case that students calculate the angle, they can type it into the input box. The students are thus brought to examine, within a geometrical context, the list of divisors of 360. The instances (e.g., $n = 7$) that do not form an *n*-sided polygon approximation are then to be explored by increasing the number of decimals of the center angle, leading the students to experience rational numbers and the decimal limits of GeoGebra. Here the input box is utilized as the increment of the slider becomes very sensitive for a large number of decimals. We see the use of variables and sliders as a means of exploring and reasoning about arithmetical entities on geometric representations. Again, similar to Lagrange and Psycharis (2011), the students cannot test algebraic expressions.

DISCUSSION

Considering the two appearances of variables, the explicit use of variables is dominant in all three empirical studies, whereas the implicit use of variables is discussed in the two theoretical papers. Also, two out of the three empirical studies hide the Algebra View, and the study that does not hide it uses LOGO and not GeoGebra. The tool that we gain the most insight into is the slider, in two cases along with an input box and in one case along with the possibility to type in expressions. Several points can be drawn considering task design for activation of students' mathematical reasoning competency. I will synthesize these in the following.

To design tasks that support the students' activation of their reasoning competency, the slider provides a link between the graphic representations, the algebraic representations and the numeric values. Dragging the slider represents the variation of a numeric value, which allows students to test conjectures about the mathematical relationship the variable influence or represent by testing and receiving feedback from the system. This can either be for different values of the variable(s) or different states of the objects depicted in the Graphics View. In Tanguay et al. (2013) and Soldano and Arzarello (2017), we see that using the sliders only in the Graphics View can support students to activate their reasoning competency, as they can explore mathematical relationships, form conjectures, and possibly experience 'if-and-only-if" relationships. This can be related to the "cause and effect" of dragging the slider. However, leaving the variable as singular entities on the Graphics View without giving access to the Algebra View limits the students' possibilities to test their

conjectures as algebraic expressions. Task designers and mathematics educators must be aware that there is a potential in GeoGebra for students to test conjectures about algebraic expressions, thus allowing students to engage further into reasoning with variables. In Lagrange and Psycharis (2011), we see students who engage in reasoning about algebraic relationships by testing if the typed algebraic expressions result in a successful dilation of a letter and who reflect upon the results.

In the two studies, Tanguay et al. (2013) and Soldano and Arzarello (2017), input boxes are used along with the slider, allowing the students to easily test specific values of the variable, and this without struggling with positioning the slider. If there is no access to the Algebra View, task designers should be mindful of this possibility as it allows students to test exact values more easily.

Despite GeoGebra being, at least in Denmark, one of the most used DGS in primary and lower secondary school, the review reveals that there is a surprisingly small amount of studies that make use of the functionalities of GeoGebra's Algebra View in task designs that use variables as a general number. This makes one wonder if the Algebra View simply is too complex for younger students to manage. It is clear from Mackrell (2011) and Jawkin (2010) that the algebraic representations do impose challenges, not least in terms of discrepancies in the algebraic representations and the amount of information assessable in the Algebra View. Task designers and educators should keep these issues in mind when designing tasks using functionalities of the Algebra View. As in Tanguay et al. (2013) and Soldano and Arzarello (2017), designers can hide the Algebra View altogether, but there are also other possibilities to limit accessible information in the Algebra View. For example, pre-constructed objects can be hidden, or the Algebra View can be set to only show descriptions or values, making the information less complex. In addition, using the explicit appearance of variables instead of the implicit appearance of variables can ease these issues. Nevertheless, there is still much to be discovered about how functionalities in the Algebra View can be used in task design for activation of students' reasoning competency when it comes to variables as a general number. In Lagrange and Psycharis (2011), we do get indications of how typing in expressions that contain a variable that is simultaneously influenced by graphic representation in LOGO can do exactly this. Will a similar design bring similar results if tried out in GeoGebra? And how can we design tasks drawing on these functionalities to engage lower secondary students in mathematical reasoning on core concepts and structures in algebra, such as generality, equality, additive, and multiplicative structures?

CONCLUSION

What can be concluded from this review is that, in general, very little has been researched about which functionalities in GeoGebra's Algebra View for working with variables as a general number, as well as how to use the functionalities in task design for activating lower secondary students' mathematical reasoning competency. Still, the review does indicate that using the slider for explicit variables can be used for this aim, and typing in expressions containing variables should be further explored in the context of GeoGebra.

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GO ONLINE TO GO OUTDOORS – A MOOC ON MATHCITYMAP

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During a math trail, students discover authentic mathematics problems in their environment. Studies have shown that this method has several advantages for learning outcomes. Nevertheless, it is a rarely used way to teach mathematics. By means of a Massive Open Online Course (MOOC), it is the aim of the MaSCE³ project to educate teachers in designing outdoor activities for their students and implement them into their regular teaching. A MOOC has several benefits and provides new possibilities in the design of a training session, with respect to "traditional" trainings. In this paper, we will focus on the outcomes of the MOOC with respect of the participating teachers. Moreover, in an interview-based comparison with our beforehand made training experiences, we highlight the benefits and challenges of online professional development via MOOC, taking also into account the coronavirus situation.

Keywords: MathCityMap, MOOC, outdoor mathematics, teacher training.

INTRODUCTION

Teaching mathematics often seems to be artificial or embedded into quasi-realistic contexts (Vos, 2015). It is questionable whether students can be motivated to understand mathematics in this way. Outdoor mathematics is an approach to let students discover mathematics in their environment by leaving the classroom. The environment is rich in objects and situations that have mathematics in it. One idea of doing mathematics outdoors is mathematics trails (also math trails). Along such a trail, students are led to solve math problems (or tasks) on real objects by measuring or counting (Shoaf et al., 2004). With the MathCityMap (MCM) system, the math trail idea is manifested into the educational context and supported by a digital component. This system was founded at Goethe University Frankfurt in 2012 and is currently available in 12 languages with more than 20.000 tasks. The system takes advantage of the following benefits: (i) The mathematical tasks are authentic and taken from the students' everyday life; (ii) The smartphone is a relevant technical tool for the students and supports them in the solution process; (iii) Studies on the use of MCM show that students tend to remember the mathematical contents longer in comparison to concepts solely learnt inside the classroom if the method is used regularly (Gurjanow et al., 2019a). In addition, math trails have the potential to increase students' motivation (Gurjanow et al., 2019a). Despite these advantages, our experiences show that math trails are often solely used for excursion days and that some teachers have concerns about the increased time exposure, supervision and outdoor task design while using outdoor mathematics in their teaching. Apart from the technical development of the system, it is therefore of high relevance to educating teachers in using outdoor mathematics regularly.

In order to introduce teachers to MCM and its potential for the use in mathematics teaching/learning, we have held regular face-to-face training courses during the last years. Recently, the aim of educating an increasing number of teachers has been pursued within the Erasmus+ project MaSCE³ (Math Trails in School Curriculum and Educational Environments of Europe; 2019-2022; www.masce.eu), especially in the Intellectual Output "MOOC (Massive Open Online Course)". Seven partners from five different countries, i.e. Estonia, France, Germany, Italy, Portugal and Spain, work together

hereby. This international collaboration and the partners' experiences in teacher education and professional development allow the hypotheses that a MOOC might convince a huge amount of teachers in using math trails more regularly. To underline this potential, we focus on the following research question: What are the benefits of the MOOC in comparison to previous face-to-face mathematics teacher training experiences in MCM?

THEORETICAL BACKGROUND

Mathematics Teacher Education and Professional Development

In professional development initiatives, teachers are engaged in activities that aimed at increasing their own knowledge of mathematics, either by looking more closely at particular topics or by learning about a new mathematical development (Matos et al., 2009). Teachers can choose from a big selection of professional development activities offered by universities, their own schools or educational systems, which often reflect current interests or trends within the system. These activities do not attend exclusively to mathematical content knowledge, but engage teachers in the process of acquiring new techniques for mathematical instruction (Liljedahl et al., 2009). Courses can focus, for example, on issues such as implementing new guidelines or standards, collaborative groups, or the use of particular resources to teach specific content topics. These professional development experiences are offered in formats such as workshops or seminars, providing teachers with "opportunities to connect with outside sources of knowledge in a focused, direct and intense way" (Loucks-Horsley et al., 1998, p. 87). Although the organization and duration of these initiatives vary, participating teachers often add something new to their knowledge or pedagogical stock, which they could take back to their classrooms.

MOOCs for Mathematics Teacher Education

MOOCs are courses offered openly to learners through the web, and appear as dynamic and diversified learning spaces with varying factors, such as flexible time frames, a massive number of participants from different geographic areas, motivation to continue learning, and opportunities for designers to implement novel pedagogies (Manathunga et al., 2017). The activities provided in these courses range from watching certain videos, posting on forums or blogs, sharing experiences on social media, responding to quizzes, doing learning tasks for individuals or workgroups, and/or conducting peer review activities (Taranto, 2020). Learners are involved to various degrees: many just want to check out the resources and the new educational model, while others are really motivated and follow every aspect of the course, often interacting with other MOOC participants. Likewise, educators' involvement varies substantially: in some courses, the educators disappear when the course starts; in others, they are intensively involved, injecting dynamism to the proposed activities and providing their students with feedback (Daza et al., 2013). It is worth noting that the emergence and use of MOOCs for professional teacher development are still uncommon, especially in mathematics. In fact, although there is a wide choice of many different topics, when looking specifically for a MOOC aimed at mathematics teacher education, the range is limited (Aldon et al., 2017). Nevertheless, there is a growing interest in MOOCs involving mathematics teachers as participants. Therefore, MOOCs for teacher education are on the verge of gaining a foothold. In this stream of ideas, the University of Catania has designed, delivered and monitored a MOOC for teacher education on the MaSCE³ project issues, as an element of innovation. We will present the MOOC in "Results" section. First, we will give an overview of MCM as the main focus of the MOOC and the teacher training experiences gained with it so far.

THE MATHCITYMAP PROJECT

MCM consists of two components (see Figure 1): a web portal for teachers to view, create and store outdoor tasks and trails and a mobile app to solve the tasks that belong to an outdoor trail. The mobile app guides the students on a map, provides hints, validates the solution and displays a sample solution (see also Ludwig and Jablonski (2019) for a detailed description).

Figure 1. A math trail in the MCM web portal (left) and an example task the MCM app (right)

During the last years, the MCM system has been developed and improved through new technical features. In the interface of web portal and app, the *Digital Classroom* helps teachers to organize a math trail with MCM. It is a temporary digital environment to see where the students are, which task they work on and how they perform. With the chat function, the teacher can communicate with the students in case of questions. All the inputs and events, such as opened tasks and used hints, are saved in an e-portfolio.

METHODOLOGY

In order to answer the research question, we used a mixed-methods approach. On a quantitative level, we have the answers of teachers participating in the MaSCE³ MOOC to a pre- and a post-questionnaire. From the first, we deduce from which countries the teachers come from; from the second, limiting ourselves only to the answers of the teachers who have completed the MOOC, we analyse a question concerning the degree of agreement of the teachers on asynchronous communication. On a qualitative level, we analyse the benefits and disadvantages from the perspective of a trainer. For this purpose, we interviewed one of the MOOC instructors who has gained in addition experiences in face-to-face teacher training during the last years.

RESULTS

Face-to-face Teacher training on MCM

The MaSCE³ MOOC is based on the experiences made from more than 50 teacher pieces of training on MCM all over the world. For that, a Short-Term Curriculum (STC) has been developed. It consists of five consecutive modules which are described in detail in Milicic, Jablonski and Ludwig (2020).

As one example of how the STC can be used for teacher training, we will present a training course conducted by the University of Catania. The University of Catania organizes yearly training sessions for teachers of all school grades. For mathematics teacher training, both in 2018 and 2019, the training offer included a course on MCM, addressed to 20 teachers (for each year). In both editions, the course consisted of 3 meetings of 4 hours each, delivered at a certain time distance (every 2–3 weeks) from

each other (Figure 2), in order to allow the teachers to assimilate the topics explained and to have time to find measures and data on site, so as to create their own tasks. The aim of the course was to introduce the idea of mathematical trails for doing outdoor mathematics with MCM.

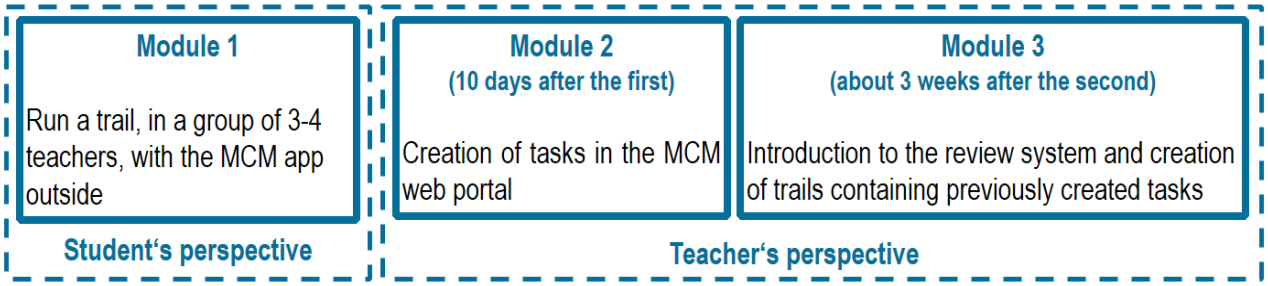

Figure 2. MCM teacher training courses in Catania

In the first meeting (Figure 2), teachers take the perspective of school students and run an outdoor trail using the MCM app. In the following meetings, teachers take the perspective of task and trail designers. Hereby, the teachers had to design 4 tasks (in the first course) and 6 tasks (in the second course) that would form a mathematics trail. The tasks could be located in their own city or, in 2019, also in their own school[1]. This choice made it easy for the teachers to find the necessary data on-site, but since not everyone was from the same city, in the first year this led to the choice for some to work individually. In the second year, however, we preferred a methodology of working in groups, even though the members were not all from the same town. All the tasks were revised by the course teacher educator. All the teachers completed the teacher educator's requests: in the first course, 7 math trails were implemented, and in the second course, 6 math trails were implemented.

The MaSCE³ MOOC

As mentioned in the introduction, one of the Intellectual Outputs of the MaSCE^3 project is an open and freely accessible MOOC to educate a wide range of teachers to use MCM and its new tools in their teaching. The MOOC title is "*Task Design for Math Trails*" and it is aimed at mathematics teachers of all school grades from all over Europe and the world. It is delivered from 8 March to 30 May 2021 through the DI.MA. platform (http://dimamooc.unict.it/) managed by the University of Catania. 513 teachers enrolled in the MOOC, from 36 countries, including 18 European countries. The presence of qualified instructors from different countries (Estonia, France, Germany, Italy, Portugal and Spain) guarantees fruitful monitoring of the participants' learning pace and prompt support in case of necessity. The aim of the MOOC is to raise the awareness of participants with regards to the potentials of the math trails tasks, and make them autonomous by showing them how to create tasks and use them with their own students. The MOOC is organized in 12 weeks in total as follows: the first week to enter the world of outdoor mathematics learning; four thematic modules of two weeks each, in which teachers will be guided in the creation of 8 math tasks of different types, located in their own environment, which will form a trail; three weeks in which the teachers will be able to accomplish the final homework, i.e. run their own math trail with their students and report back on this experience. Table 1 also gives an overview of the topics that will be covered during the MOOC. The teachers that completed the MOOC were 93 from 12 countries. They have created 911 tasks and 105 trails.

Name of Module	Topics
Introduction	Overview: outdoor mathematics; why use MCM; getting to know the instructors
Module 1: Outdoor learning and Task Design	- Organisational aspect for outdoor learning - Creating an account on the web portal - Overview of the web portal and how to create a task on it - Format and Criteria for Task Design (Exact Value, Multiple Choice)
Module 2: Subtasks and Task Wizard	-Use of subtasks -Functionality of the Wizard
Module 3: Interval	-Another format Task Design: the Interval -Choose an appropriate interval, errors and examples
Module 4: Trail and app	-Create trails -Introduction to the idea and the use of the Digital Classroom -How the app works and the narrative feature
Final	Experimentation: run the trail with your students using the Digital Classroom

Table 1. MaSCE³ MOOC: structure and topics

Comparison of the Different Training Formats

After an exemplary description of the previously hold face-to-face trainings and the MaSCE³ MOOC, we compare the benefits of these training formats. The presentation is grounded on the questionnaire data from the MOOC participants as well as an interview with a MCM trainer that was involved in both, face-to-face trainings and the MOOC instructions. The first point we want to take into account is the *target group for presenting innovations*. The teachers who take part in the MOOC are a massive community spread all over the world. Our trainer summarizes in the interview that

> "a MOOC makes it much easier to train more participants and, of course, time flexibility is one of the best advantages.".

The availability of the MOOC materials in the digital format is therefore a strong benefit in reaching a worldwide community and spreading technical innovations of MCM produced within the MaSCE³ project. For example, the Digital Classroom (mentioned in the section "The MCM project") meets many of the requirements that teachers made in face-to-face training, especially the possibility to see where students are geolocated and how they move, what tasks they are doing, and if they need some help; to be able to interact with them even if you are not physically close thanks to the chat function. By means of the MOOC, the innovation could be spread rapidly all over the world. In addition, the MOOC could be conducted despite the COVID-19 pandemic without travelling. Still, it is clear that the

> "materials [of the MOOC] must be prepared in advance, mainly videos because they take time to be prepared and as MCM goes very fast in its development, it is difficult to prepare the lastest novelties in the training. In this sense, face-to-face modality is more flexible to change the schedule or to adapt it to introduce something new."

The second point of our comparison focuses on the *intensity of training*. As the previous presentation shows, the MOOC training is spread over a larger number of weeks than face-to-face experiences. This allows teachers to gradually internalize the content and become more aware and autonomous in practicing mathematics outdoors. In face-to-face training, it has often been necessary to repeat

instructions on how to operate the web portal several times in different meetings. The MOOC videos overcome this obstacle [2]: they can be watched and reviewed. In addition, teachers have more time to design and create their own tasks. Still, the face-to-face training allows a common introduction to the MCM system which is especially highlighted in the expert interview as one benefit of face-to-face trainings:

> "Regarding math trails, for me, the main difference between face-to-face training and the MOOC is the possibility of running a trail with the participants in the first case […] From my experience, running a trail with teachers as training is very beneficial as a starting point."

So we identify a gap between a common and intensive starting point in face-to-face trainings on the one hand and an intense work on task design over a longer period and in the teachers' locations on the other hand. In a third aspect, we focus on *language and communication*. A main contrast between face-to-face trainings and the MOOC is that the first is normally in the native language of the participants and the second is in English. Still, the MOOC is supported with subtitled materials and the opportunity to design tasks in a chosen language, so our expert states that "language was not a barrier for participants."

Another main difference is the format of working. It is true that the task creation work in the MOOC is done individually, but teachers work in an environment where peer interaction is encouraged. In fact, there are specific spaces in the MOOC (forum and padlet) to encourage communication and sharing among participants in this learning experience. From the final questionnaire (here we only consider the answers of the 93 finalists), with a Likert scale question (from 1 = strongly disagree to 4 = strongly agree), we asked the teachers to express how much they agreed with this statement: *Communication message boards encouraged interactions with other participants*. It turned out that 82% (considering together items 3 and 4 of the Likert scale) agreed. This testifies to the fact that most of the finalists not only used the message boards, but also appreciated the opportunity to exchange views with their peers. Therefore, interaction in the MOOC is possible even though in a different form than in face-to-face trainings.

CONCLUSION

In the comparison of face-to-face teacher trainings and the MOOC, we identified several benefits of both formats which lead to the answer of our research question and future challenges in designing MOOCs. Through its online and long-term format, the MOOC has many benefits in comparison to "traditional" teacher pieces of training, e.g. the possibility to participate independently from the location, and even in times of a pandemic. Also, the number of participants in the MOOC is massive compared to the numbers hosted by previous face-to-face MCM training courses. Furthermore, it is true that the MOOC lasts a certain number of weeks and there is a lot of flexibility in both attendance and deliverables. The teachers that completed the MOOC were 93 from 12 countries. This gives an idea that the MOOC reaches more teachers than the previously conducted teacher events on MCM. On the same side, this results in a significantly higher number of produced material, i.e. tasks and trails, from which teachers not enrolled in the MOOC can also benefit, as these productions are public on the web portal. Through the international working space, the MOOC underlines the idea of MCM as an international tool for mathematics education immensely. Still in this setting, it remains a challenge to allow collaboration as in face-to-face training. The MOOC provides several possibilities for interaction among the participants, i.e. in forums and through the MCM community website. In addition, the review team consists of instructors from five different countries (Estonia, France, Germany, Italy, Portugal and Spain), fluent in English. Hereby, it can be guaranteed the teachers receive the necessary support and monitoring concerning their learning progress. The MOOC is therefore a merge of both: individual professional development, on the one hand, and the integration

into an online community, on the other hand. Still, or especially through the online format combined with individual outdoor activities, it seems to be an adequate approach to educate mathematics teachers in outdoor mathematics.

Future research should investigate additional developments in MOOC designs. MOOCs generally encounter a high drop-out rate, which has never been the case in MCM face-to-face courses. Although the literature reports that the high drop-out rate is a physiological factor of MOOCs (Daza et al., 2013; Taranto, 2020), for future research it may be interesting to investigate how we can try to reduce its being so high. For example, we should take into consideration the benefits of face-to-face trainings (e.g. flexible in presenting innovations, common activities and direct communication) and try to adapt them to the digital training format.

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NOTES

1. During the first year, especially teachers of students in lower grades pointed out that it was not always easy to take the children out during curricular hours (both for safety and responsibility reasons). The following year, in agreement with the team in Frankfurt, the opportunity was given to also create trails within one's own school (i.e. in places that are actually not accessible to everyone, but to the student population of that particular school).

2. We specify that the videos are accompanied by subtitles in the instructors' languages (this makes the training easy even for teachers who are not fluent in English).

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DEVELOPING SILENT VIDEO TASKS' INSTRUCTIONAL SEQUENCE IN COLLABORATION WITH TEACHERS

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Previous research indicated that silent video tasks might be useful for formative assessment practices. Thus, in a second data collection phase of a doctoral study on the development and design of silent video tasks, participants were selected to have some previous experience with formative assessment practices. Three teachers in two Icelandic upper secondary schools collaborated with the first author of this paper to develop silent video tasks' instructional sequence. One teacher implemented three different silent video tasks with his low-achieving students in 11th grade and two teachers implemented one silent video task each with their groups of low-achieving 11th-grade students. The collaboration with teachers influenced the way feedback was given to students (immediate instead of delayed) and instead of the former practice of selecting a few student responses to discuss with the whole group, teachers went through all student responses with the group and reacted thereto.

Keywords: Encouraging discussion, formative assessment, silent video task, teaching with new technologies.

INTRODUCTION

In a silent film or video, no commentary directs the viewer's attention where to look. It requires a considerable amount of work from viewers to internalise and describe for others what captured their attention (Pimm, 1995). According to Pimm, spoken language is an important learning medium in mathematics, and we often grasp concepts by talking about them in our own words (Pimm, 1987). When learners get asked to record their voice over to an animated video clip, as is done in silent video tasks, they are restricted by not being able to point or touch. This restriction can help encourage learners to move from predominantly informal spoken language (with which students are more or less fluent) to the formal written language of mathematics (Pimm, 1989). When learners report to their peers, they must choose what to say, consider what they know, and what they believe their audience knows. Such practice can place sophisticated linguistic demands on learners' communicative competence and help teachers gain access to learners' proficiency (Pimm, 1989). Despite its importance, the discursive practice of explaining (asking students to give oral explanations) in mathematics was identified by Erath to be not only too seldom practised but also too seldom set as an explicit learning goal (Erath, 2017).

Silent video tasks require teachers to set the discursive practice of explaining as an explicit learning goal. They involve teachers asking students to work in pairs to record a voice over with explanation, description, or narration of what they see in a short (less than 2 minutes long) silent animated video that focuses on one mathematical concept or phenomenon. Results from the first phase of a doctoral research project on the design and development of silent video tasks (Kristinsdóttir et al., 2020a) indicated that these tasks could be used as part of teachers' formative assessment practices, making classroom discussion based on students' responses around the issue of differences in conceptual understanding possible. This paper describes some initial results regarding teachers' preferred ways of implementing silent video tasks in their classrooms with formative assessment practices in mind.

FORMATIVE ASSESSMENT

For formative assessment practices, Wiliam and Thompson (2008) described the following five key strategies: i) Clarifying and sharing learning intentions and criteria for success, ii) Engineering effective classroom discussions and learning tasks that elicit evidence of student understanding, iii) Providing feedback that moves learners forward, iv) Activating students as instructional resources for one another, and v) Activating students as owners of their learning. Behind these five key strategies lies a definition of formative assessment:

> An assessment functions formatively to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have made in the absence of that evidence (Wiliam, 2011, p. 43).

For technology-based formative assessment, Wright et al. (2018) defined six potentials that support teaching and learning: a) Provide immediate feedback, b) Encourage discussion and develop cooperation, c) Provide an objective and meaningful way to represent problems and misunderstandings, d) Provide opportunities for using preferred strategies in new ways, e) Help raising issues that were previously implicit and not transparent for teachers, and f) Provide different feedback outcomes (Wright et al., 2018, p. 219). After the first implementation phase, findings suggested that silent video tasks supported all but the first of these potentials since feedback was delayed and took place in a follow-up lesson day(s) after students handed in their task responses (Kristinsdóttir et al., 2020a). This paper, which focuses on a second implementation phase that took place in fall 2019, reports on new findings where the instructional sequence of silent video tasks was changed (upon suggestion from participating teachers) in order to make immediate feedback possible.

RESEARCH DESIGN

Because results from the first implementation phase of the research project had indicated that silent video tasks might be useful for formative assessment practices, for the second implementation phase, I (first person refers to the first author) searched for upper secondary schools to work with that officially mentioned the use of formative assessment in their school policy. Mathematics teachers at these schools were invited to join the project regardless of what mathematics courses they taught. I intended to create silent videos based on their needs for the courses that they taught. Three mathematics teachers in two upper secondary schools in Iceland accepted participation. It so happened that all three teachers taught remedial classes for low-achieving 11th-grade students in courses focusing on algebra and coordinate geometry. Since the course schedules for these courses allowed for time for experiments, the teachers suggested that I would make silent video tasks on topics from this course.

In collaboration with the teachers, I prepared three silent videos with a focus on characteristics of the coordinate system, line slope, and linear functions. They were all made with GeoGebra and screen recording software. The first silent video (SVT1) highlights different parts and certain characteristics of the coordinate system (see https://youtu.be/8cLrbJM4F-I). The second silent video (SVT2) shows the graph of a line that rotates around a point making pauses at different slopes (see https://youtu.be/-snC4JLe63g). The third silent video (SVT3) shows discrete and continuous graphs of two lines with positive slopes that intersect (see https://youtu.be/aBtlIVTcs8M). The original plan was that all teachers would implement all three videos over the course of one semester, but in the end, only Orri at Blackbird High School (teacher and school names are pseudonyms) did so, whereas Andri and Edda at Mallard High school implemented one silent video in their classes.

Data that was collected in this phase of the research project included interviews with teachers, students' responses to peer evaluation and self-reflection worksheets that teachers prepared, students' responses to the task, classroom observation notes, communication with teachers via email and WhatsApp, and notes from my research journal. Semi-structured interviews (Brinkmann & Kvale, 2009) were conducted at the start of the semester, and before and after each silent video task implementation. The interview guides were based on earlier interview guides from the first implementation phase. They were updated with a few new questions, two of which were inspired by Schoenfeld (2007). These questions addressed whether the task sufficed in on the one hand making students' voices be heard in class and on the other hand in providing teachers with information regarding common misunderstandings.

With Andri and Edda, I conducted two preparation interviews with both of them together and one interview with each of them separately after they tried out a silent video task in their classrooms. I transferred ideas from the first interview with Andri and Edda to Orri (see Figure 1) and conducted five interviews with Orri, before and after using the first two silent video tasks, and after using the third silent video task. Teachers were informed that the focus of the interviews would be on their experiences with the implementation of silent video tasks. As compensation for their participation, I offered participants support meetings in the case that they were working on changes in their practice. Orri accepted this offer and we met six times to discuss ways to build a thinking classroom (e.g. Liljedahl, 2018) at meetings that were recorded but not transcribed or analysed.

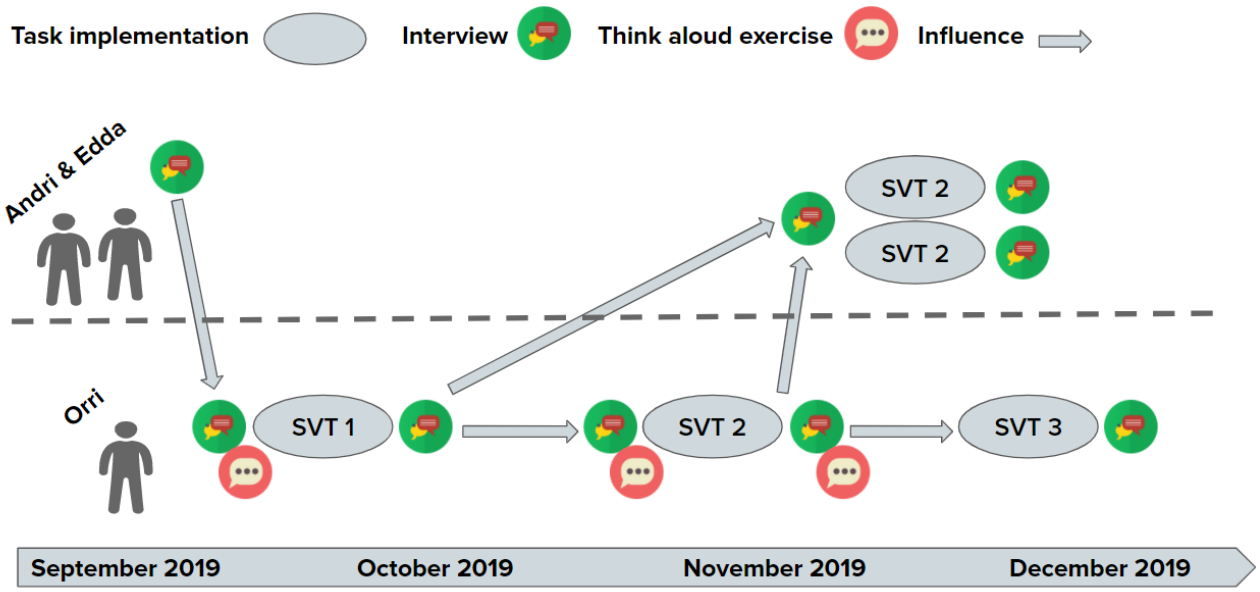

Figure 1. Approximate timings of silent video task implementation and interviews with Andri and Edda at Mallard High School and Orri at Blackbird High School during fall semester 2019.

The Icelandic Data Protection Authority was informed about the research project. Participating teachers and their school leaders received information and signed an informed consent explicitly stating the research aims, the intended data collection, treatment of data and in what ways results would be communicated. It is important to note that in fall 2019, the General Data Protection Regulation (GDPR) had recently been introduced to school leaders in Iceland. They were very cautious regarding data collection in classrooms. In order to better grasp speech, gestures and mimes that often happen simultaneously in classrooms, I had planned to video record the task implementation, but school leaders preferred field notes over video data. To build trust and positive

correspondence needed for research that is done in collaboration with teachers, I, upon their recommendation, decided to refrain from collecting video data and take field notes instead. No personal or personally identifiable information was collected from students and they all received information sheets about the research project that their teachers were taking part in.

Freudenthal (1973) discusses the use of some kind of think-aloud exercises that he calls *thought experiments* and, according to Gravemeijer (1994, p. 448),

> […] the elaboration of an educational design is, in practice, constituted via a thought experiment.

In addition to setting a mathematical learning goal before starting developmental research, the thought experiment is intended for developers to envision how the teaching-learning process might proceed in the classroom. This free-flow and in-the-moment exercise is a tool that I used in three of the five interviews with Orri (see Figure 1). I asked him to imagine how he would implement a silent video task and think aloud what he would do and why. He could change his mind on the go when needed, in the midst of wondering about different variations of instructional sequences that he might use.

In the first interview, I used the thought experiment to hear Orri's initial ideas and in later interviews as a means to reflect on his experiences and keep a record of his ideas for the next implementation round. He not only thought aloud but also reflected on his in-the-moment thoughts and related them to planned actions for the next round of implementation. Normally, one would expect it to require training to remember and reconstruct one's own interpretations, moments of thought, sensemaking of any kind. It might be that since we sat down for the interviews immediately after implementation it was easier to apply this method despite no training.

DATA ANALYSIS

For the purpose of this paper, the data that was analysed included transcribed interview data, research journal notes and notes from classroom observations during the implementation of silent video tasks. Interviews were transcribed verbatim in Icelandic and when possible, I transcribed them immediately after the interview took place. As I asked teachers to implement silent video tasks in their classrooms and reflect on their expectations and experiences, I took a hermeneutic (interpretive) phenomenological stance (Van Manen, 2016) to answer the question of how and why these tasks could be used for teaching and learning in the mathematics classroom. Especially in the case of Orri, I studied teachers' actions and reasons given for their actions with the aim to make the instructional sequence generalizable. In other words, to describe their instructional sequence in such a way that could be helpful for other teachers aiming to use such tasks.

It was important that this happened in the busy setting of participating teachers' own classrooms where I could observe their work and make notes preparing for the next interview. In the interviews, I referred to teachers' actions and when possible, they would state the reasons for their actions. Moreover, teachers gave their personal insight as to whether and how they could use this tool for the teaching and learning of mathematics. Directly after our meetings, I reflected on teachers' insights in writing and then repeated that process when I analysed the transcripts from our interviews. Thus, I used iterative cycles of writing notes and reflections, considering how excerpts from the data contributed to evolving understanding of the way silent video tasks could be used in the mathematics classroom.

In the first data familiarization phase, which started right after the first interview and lasted until the last interview had been transcribed, I focused on the instructional sequence design, making sure that ideas would be transferred between participating teachers and paying attention to how these ideas developed over the semester. In a second familiarization phase, I used open coding to mark anything

I found interesting in the data. To summarize and deepen my thoughts, I wrote detailed notes in English in the third round of reading through the transcripts. On the basis of these detailed notes, I created a condensed overview of how Orri's ideas, experiences and expectations developed in time on a large sheet of paper (630 × 891 mm). To document the way that the instructional sequence developed, I drew comics and flow charts.

FINDINGS AND DISCUSSION

To enrich the discussion, especially in the case of students' responses being similar, I had received a suggestion to prepare pseudo-responses for teachers to use in a mixture with their students' responses to the task. This idea of "the use of pre-designed student responses to unstructured mathematics problems as a possible resource for teachers to develop their capacity of acting contingently in the mathematics classroom in a productive way, whilst teaching" (Evans & Ayalon, 2016) also seemed to me to be a possible workaround in case of tensions coming up regarding the use of student task responses as a basis for discussion–a tension that teachers had experienced in the first implementation phase of the research project. Being willing to create some pseudo-responses for teachers was therefore one of the ideas that I introduced to teachers who participated in the second implementation phase of the research project. Upon receiving this suggestion, teachers were not fond of the idea. They explained that pseudo-responses might affect the trust between teacher and students as students might recognize that none of them had created that response. They also pointed out that such responses might address conceptual conflicts that their own students had not yet encountered. Therefore, pseudo-responses were neither created nor used in this implementation phase.

Initial ideas had suggested that teachers would listen to all students' responses and select and sequence some of them to be played at the start of the follow-up lesson where a whole group discussion would take place. For example, responses could be sequenced such that they would range from everyday language to formal mathematical language. However, similar to what Wright et al. (2018) suggest, Andri and Edda wanted feedback to be immediate and to take place directly after students handed in their responses. They underlined, based on previous experiences with other types of tasks, the importance of immediate feedback for formative assessment practices. As for the sequencing of students' responses, they pointed out that students might interpret any sequencing to be "from the worst to the best", affecting their attention and shifting the focus away from listening, reflecting, and learning. Andri and Edda also mentioned that it might be interesting to have students give each other feedback, for example by each pair of students listening and reacting to two or three of the other student pair's responses as a warm-up for the whole group discussion.

Andri and Edda's suggestions and ideas were transferred to Orri. He decided to prepare and try out peer assessment and self-assessment in his implementation of SVT1. Each pair of students would be assigned two other pairs' responses that they were asked to reflect on along with reflecting on their own task response. It so turned out that little time was left for group discussion based on SVT1 and Orri felt like students got away little effort, giving and receiving too meagre of a feedback. He rather wanted to play all students' responses in a random order and react to them "on the go" in a whole-group discussion. Furthermore, Orri decided to show the next silent video, SVT2, twice to the student group: first at the start of the learning sequence, before they would learn about the slope of a line, to collect students' ideas in a word cloud, and then again at the end of the learning sequence, asking students to record their voice-over task responses. This way, Orri was able to collect students' initial ideas (some of which surprised him) and receive the important information that even though his students according to course schedules were supposed to have learnt about the concept of a line before, none of them came up with the word "line" when watching the video. This information helped

him plan the lesson sequence accordingly. All three teachers experienced surprise when it came to listening to students' responses. They discovered new obstacles and misunderstandings that had been previously hidden to them, similar to previous results from the first implementation phase (Kristinsdóttir et al., 2020b).

By deciding to lead a group discussion without having heard any of the students' task responses beforehand, Orri put quite some strain on himself. Despite that strain, he felt that it was a more genuine and interesting way of giving feedback and that this way it was easier to involve students in giving feedback to each other as compared to asking them to write reflections about only a part of their peers' task responses. After the lesson, as Orri and I listened to students' responses to SVT2 during our interview, Orri sometimes noticed something that he had not paid attention to in the action of leading the group discussion. He mentioned that in such cases he could give written feedback afterwards on top of the feedback already given during the group discussion.

Information about Orri's experiences was transferred to Andri and Edda and implemented SVT2 in the same way as Orri. Similar to what teachers had predicted, students seemed to find it important that their task response was played during the whole-group discussion. During the discussion, for example, teachers asked students to pay attention to if they could notice any differences and similarities between responses and asked students to repeat what they had understood from a given response. No student-to-student debate was observed during classroom observations of SVT2 and the effort made by teachers to involve students in the discussions resulted only in short reflections and surface discussion. At the end of the lesson, after the group discussion, Andri and Edda asked students to answer in writing what (if anything) they would have liked to change in their voice-over, if there was anything (then what) that surprised them or made them wonder, and if they would like to add any comments or questions regarding the silent video task.

Since Orri felt that students had not participated actively in the group discussion of SVT2, he decided for the implementation of SVT3, after the whole-group discussion, to ask students to record a new task response. However, during our interview when listening to students' initial and new responses to SVT3 side-by-side, Orri decided that in future he would either skip this re-recording part completely or maybe–similar to what Andri and Edda had done–he would let students reflect in writing on what (if anything) they would like to change and why. The reason for this was that he found the original versions more informative than the new ones, and he was not sure why students had changed what they changed. All three teachers mentioned that leading group discussions was a challenge, similar to previous results (Kristinsdóttir et al., 2020b). During the implementation of SVT3, Orri, however, managed to engage students more effectively as participants in the whole group discussion.

To summarize the ways in which participating teachers decided to implement silent video tasks, the following describes the instructional sequence developed:

1. (optional) Teachers might show the silent video at the start of a learning sequence and collect words that come into learners' minds (e.g., in a word cloud).
2. Teachers show the video to the whole group of learners toward the end of a learning sequence.
3. (optional) Teachers might guide students through closing their eyes and making gestures to indicate what happened in the video.
4. Teachers make learners aware of that different approaches can be taken to create a voice-over, that their task response might help other learners to gain access to the mathematics shown in the video, and that it is theirs to decide what to focus on in their response.
5. Teachers assign learners into groups of two using a visibly random method.

6. Pairs of learners watch the video as often as they like on their own device while they prepare their voice-over recording.
7. Learners get an opportunity to acknowledge, express and discuss what they know, understand, or think about what they see in the video.
8. Teachers refrain from answering proximity questions (asked only because the teacher is nearby) and stop-thinking questions (such as "Is this correct?").
9. Learners record their voice-over and send it to their teacher.
10. All learners' responses get listened to (randomly) and discussed in a whole group discussion.
11. Teachers encourage learners to reflect on and reason about their own and their peers' responses to the task.
12. Teachers keep their ears open to possible conceptual obstacles that can be addressed in the group discussion.
13. Teachers facilitate awareness of the importance of precision in language use during group discussions.
14. (optional) After group discussion learners can write a reflection in their journals ("notes to my future self").
15. Teachers re-listen to students' responses for reflection and to plan further teaching activities.
16. (optional) Teachers might send individual feedback to students.

CONCLUSION

New ideas regarding task implementation, that might seem obvious in retrospect, might not have come up without the collaboration with the participating teachers. What we have discussed in the findings of this paper is rather descriptive, but we believe that it might give interesting insights into decisions that teachers make regarding task implementation. For example, observing teachers make decisions that lead to extra workload in connection with orchestrating classroom discussion based on student responses without any previous preparation, only for the sake of importance of immediate feedback seems to us to be important. In retrospect, if the teacher had taken time to reflect and prepare the discussion, one could imagine that the discussion would have become more teacher-centred as only the teacher would have had time to sit back, think and ponder over the different student responses. Instead, by reflecting "on the go", teachers aimed to get students to think along with them. Even though students were observed to take little part in the discussion at first, during the second time round when Orri initiated whole-group discussion students were observed to take a more active part. Since orchestrating a meaningful classroom discussion is a tough task (Stein & Smith, 2011) that requires practice, it might be that it gets easier with time to activate students in the discussion.

Previous results regarding fulfilment of five out of six potentials listed by Wright et al. (2018) were confirmed during the second implementation phase reported on in this paper. In addition, the sixth potential about providing immediate feedback was also fulfilled, as it was identified by teachers to be of great importance for formative assessment practices.

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SYNCHRONOUS DISTANCE LEARNING WITH MCM@HOME: A CASE STUDY ON DIGITAL LEARNING ENVIRONMENTS

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Teaching and learning in Europe have undergone massive changes due to the Covid-19 pandemic. Based on challenges of distance education, we derive the following criteria for the design of distance learning environments: enabling synchronous lessons, facilitating communication despite spatial separation, fostering individual support and assessment, as well as following a low-tech barrier approach. The MathCityMap@home (MCM@home) concept is one example that addresses these requirements for mathematics teaching and learning. With this concept, the students work self-regulated on digital learning environments, i.e., pre-structured task sequences. By means of a case study with three German teachers, MCM@home is evaluated as a suitable tool for mathematics distance education. In line with theoretical considerations, the interplay of assessment and interaction is seen as a condition of success for the development of distance learning environments.

Keywords: Digital learning environment, distance education, MCM@home, mobile learning, synchronous online teaching.

Because of the Covid-19 pandemic and the subsequent reorganization of teaching in the virtual space, new ways of instruction have occurred – and new tools for distance education have been developed (Flores & Swennen, 2020). In this paper, we firstly identify the requirements for the design of distance learning environments in mathematics education. Secondly, the MathCityMap@home concept (MCM@home) is introduced as an approach for the conduct of mathematics distance education that has been developed since spring 2020. Thirdly, it is investigated to what extent the MCM@home concept meets these requirements from a theoretical perspective. Finally, we present a case study to evaluate the use of MCM@home in school practice and derive further needed developments of the MCM@home system. Since this case study was conducted in Germany, the situation of Covid-19 distance education is sketched for Germany in the following.

DISTANCE EDUCATION IN GERMANY DUE TO THE COVID-19 PANDEMIC

As in most European countries, teaching and learning in Germany has changed massively since spring 2020 due to the Covid 19 pandemic. Instead of learning in a common place, namely the classroom, the place of learning has been shifted to the children's rooms. In the first period of spring 2020, lessons were mostly not held at a common time – a predominance of asynchronous learning settings is reported for spring 2020 (Drijvers et al., 2021; Wößmann et al., 2021). Since the average learning time decreased from 7.4 to 3.6 hours per day in the first school lockdown (Wößmann et al., 2021), it can be assumed that not only the place but also the time of learning varied in contrast to the familiar classroom setting.

This major reorganisation of learning at home goes hand in hand with a decrease in student-teacher interaction (Aldon et al., 2021) and student-to-student communication (Drijvers et al., 2021) resulting in a perceived lack of personal contact (Barlovits et al., 2021). During distance education, students take on greater responsibility for structuring and organising their learning progress (Barlovits et al., 2021; Wacker et al., 2021) as both the teacher and classmates are not immediately available. From the teacher's perspective, this more independent learning impairs the implementation of formative

and summative assessment (Aldon et al., 2021). Regarding formative assessment, especially the diagnosis of learning progress and individual support are perceived as challenges in distance education by both teachers (Aldon et al., 2021; Barlovits et al., 2021) and students (Wacker et al., 2021).

Since lower-performing students and students from households with a lower social status report more problems with distance education, an increase in social inequality is feared (Wößmann et al., 2021). From a technical perspective, the so-called digital divide is exacerbated because students with a lower social status tend to have less access to digital tools (DESTATIS, 2020). But not only the availability but also the handling of the technology is reported as an issue for teachers and learners (Barlovits et al., 2021).

From the outline of the educational situation in Germany in spring 2020, the following design requirements for distance learning tools are derived: Firstly, it can be assumed that (i) synchronous and strongly pre-structured distance learning environments counteract the loss of familiar school structures (Barlovits et al., 2021) and self-organization (Wacker et al., 2021), as students - despite spatial separation - learn at a common time in a prepared setting. Secondly, (ii) a possibility for direct interaction is needed to address the decrease of content-related (Aldon et al., 2021; Drijvers et al., 2021) and personal communication (Barlovits et al., 2021). Thirdly, distance education tools should enable teachers to provide (iii) appropriate individual support (Barlovits et al., 2021) respectively formative and summative assessment (Aldon et al., 2021). Finally, from a technical perspective (Barlovits et al., 2021.; DESTATIS, 2020), (iv) a low-barrier and user-friendly approach is required.

MCM@HOME: THE CONCEPT

Based on these four assumptions on distance education, the MCM@home concept for the conduct of mathematics distance learning has been developed since spring 2020. Here, the freely available system MathCityMap, which was originally developed for outdoor mathematics, has been adapted to the needs of distance education. The MathCityMap system provides two working spaces, namely a web portal for teachers and a smartphone app for students. In the MathCityMap web portal, teachers can create own or select publicly available task sequences. These task sequences can be characterized as internet-based, structured, and guided learning environments for students' autonomous and self-regulated work on a particular topic (Greene et al., 2011; Lichti & Roth, 2018). They can aim for a broad repetition of previously learnt topics or focus on a single topic as a theme-based learning environment. The workspace for learners is the free and add-free MathCityMap app. The students can access the digital learning environments through the smartphone app which is presented in the following.

The MathCityMap App: Asynchronous Support for Students

To participate in a lesson conducted with MCM@home, students simply need to enter the code of the digital learning environment into the MathCityMap app to access the tasks. To encourage the task solving process, students can call up to three hints on demand. After entering, the app validates the answer, i.e., students receive an immediate systemic feedback on their calculated solution. Further, students receive up to 100 points depending on the quality of the entered solution. Thus, the app provides a shallow gamification of the digital learning environment (Gurjanow et al., 2019). Additionally, students can compare the calculated solution with a sample solution. These asynchronous features of the MathCityMap app are shown in Figure 1.

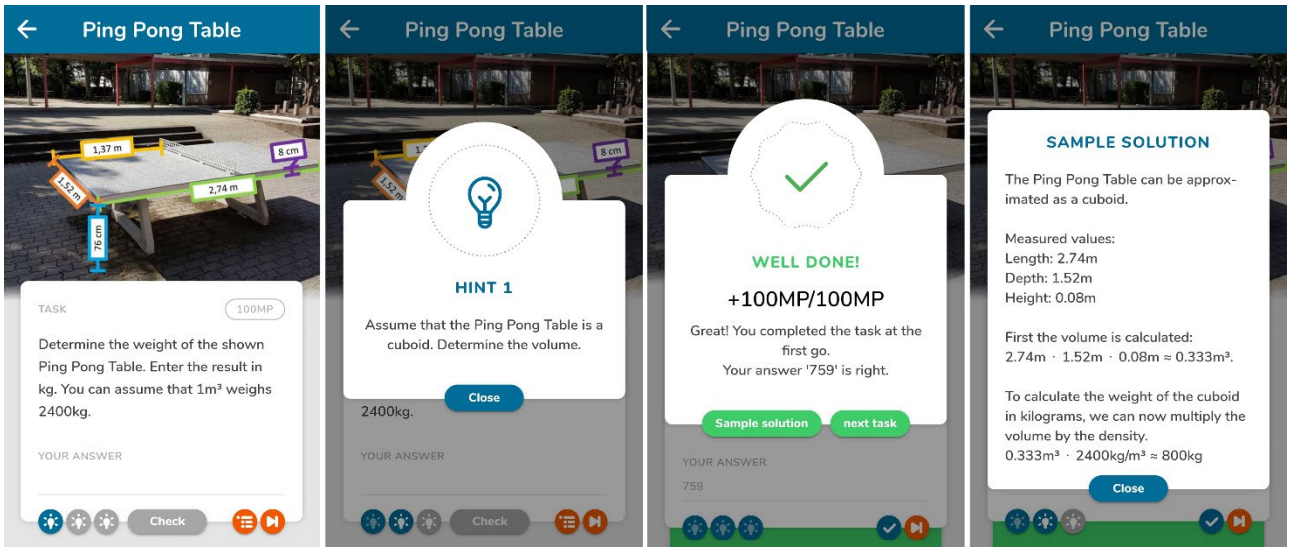

Figure 1. Working on the task "Ping Pong Table" supported by the MathCityMap app: task formulation, hints, answer validation incl. gamification and sample solution (from left to right).

The "Digital Classroom" of the Web Portal: Synchronous Monitoring of the Students

The MCM@home concept aims at the simultaneous teaching and learning of mathematics in distance education settings. Additionally to the described asynchronous functions of the app, a feature for the synchronous conduct of math lessons is provided: the so-called "Digital Classroom".

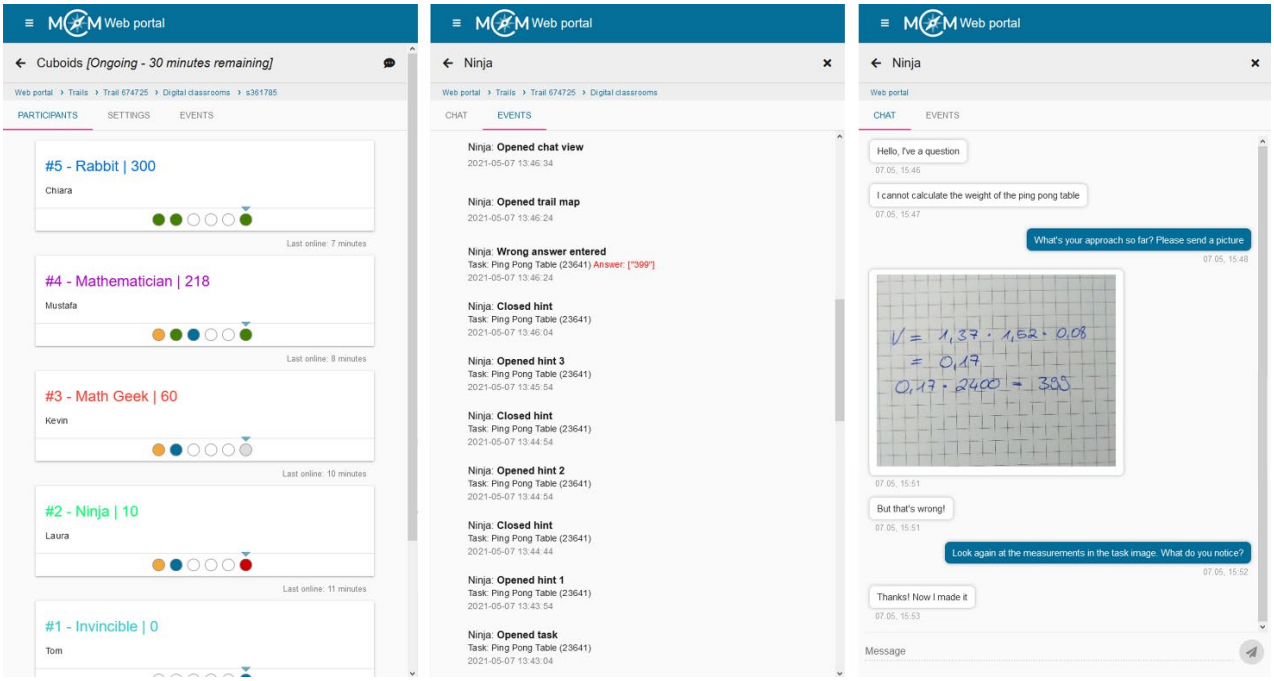

Figure 2. The feature "Digital Classroom" in the web portal: monitoring on class level and individual level (e-portfolio) as well as the chat function (from left to right)

The "Digital Classroom" is a tool for teachers to monitor and support students on an individual level (Baumann-Wehner et al., 2020). It consists of two main components, namely a chat and a monitoring tool. The chat enables a direct student-teacher interaction. For example, the learners can ask their teacher for help by sketching their solution process via text and audio message or by sending a picture

of their notes (Figure 2, right). Besides, the "Digital Classroom" provides a monitoring view for both, class and individual work progress. Teachers can observe the learning progress of all students in real time through the number of tasks accessed, the quality of task solution and the achieved scores (Figure 2, left). In the e-portfolio, the teacher can retrace the individual work process of a single student. Here, all entries of a student into the MathCityMap app, e.g., the entered solutions or the use of hints, are stored (Figure 2, middle).

MCM@home in View of the Pandemic Situation

With regard to the identified requirements for distance learning (i)-(iv), the paper aims to examine the extent to which they are met by MCM@home.

(i) Synchronicity and guided structure: The demand for synchronicity due to the loss of familiar school structures can be fulfilled using the "Digital Classroom", in which all students work on the same topic at the same time. Further, learning environments provide guided and pre-structured task sequences (cf. Lichti & Roth, 2018) that allow autonomous and self-regulated learning (cf. Greene et al., 2011). Therefore, MCM@home seems to pre-structure the individual student's work process in distance education.

(ii) Communication and interaction: The requirement of online discussions (cf. Drijvers et al., 2021) and personal communication in distance education (cf. Barlovits et al., 2021) is taken up by the availability of the chat function. It enables teachers to communicate directly with their students and to support them individually. By implementing the chat, a feature is provided which at least partly counteracts the lack of personal contact in distance education settings (cf. Barlovits et al., 2021). However, it seems questionable whether this form of interaction is completely sufficient for distance learning purposes.

(iii) Individual support and assessment: The demand for formative assessment in distance learning situations (cf. Aldon et al., 2021) is addressed within the feature "Digital Classroom": Teachers can monitor the student's learning progress and analyse their individual work process in the e-portfolio. However, the need for summative assessment (cf. Aldon et al., 2021) is not currently met by MCM@home. Therefore, requirement (iii) is only partly fulfilled by MCM@home.

(iv) Technical availability and handling: From a technical perspective, only a smartphone with an active internet connection and the installed, cost- and add-free MathCityMap app is required on students' side to use MCM@home. As smartphones are a worldwide used device (Deloitte, 2017) and moreover widely used for educational purposes in Europe (European Commission, 2019), the MCM@home concept can be seen as a low-barrier approach to distance learning.

To conclude, MCM@home meets most of the theoretically identified requirements for distance education. In the following, we will focus on the questions of whether this hypothesis can be empirically confirmed and whether the chat can fulfil the need for interaction in distance education. This should be analysed within a case study which is presented in the following.

RESEARCH QUESTIONS AND METHODOLOGY

To answer the research question, we conducted a case study in Germany in summer 2020. It aims to evaluate MCM@home from the teacher's perspective (RQ 1) and to investigate the use of the chat. (1) Concerning the design requirements (i), (iii) and (iv), how do teachers evaluate the use of MCM@home for mathematical distance education during Covid-19 pandemic? (2) Concerning design requirement (ii), how do teachers and students use the chat channel provided within MCM@home?

To answer these research questions, three classes of a German Gymnasium (grammar school; grades 6, 9 and 10) were accompanied while solving mathematics tasks with the system. During the school closure period in June and July 2020, the three classes worked on MCM@home learning environments for one double lesson each. The different class levels were chosen in order to observe learning groups that were as heterogeneous as possible and to answer the research questions.

Due to the different grade levels, the learning environments cover different topics: The tasks for grade 6 deal with the calculation of decimal numbers, while the tasks for grade 9 aim at an overall review of the school year. The learning environment for grade 10 deals with the topic of data and statistics. Unlike the actual use of MCM@home, which is organised by the teachers themselves, these three digital learning environments were created by the researchers. This approach should ensure a certain comparability of the task design in order to analyse the user behaviour of teachers and students in digital learning environments.

After conducting the double lesson with MCM@home, the three teachers received an online questionnaire with fourteen open questions (free-text answers) focussing on the design requirements (i) synchronicity and guided structure, (ii) communication and interaction, (iii) individual support and assessment as well as (iv) technical issues and handling. If at least two out of three teachers report similar or the same experiences in the questionnaire, their statements are considered to be essential in this paper. To analyse the communication and interaction via chat (RQ 2), the number of chat messages and their purpose are further taken into consideration.

RESULTS: MCM@HOME IN TEACHING PRACTICE

RQ 1: The teachers' evaluations of the tool are subsequently presented on behalf of the identified design requirements (i), (iii) and (iv).

(i) Synchronicity & guided structure: The synchronous conduct of distance education within the "Digital Classroom" is highlighted by the teachers since it allows the real-time monitoring of the student's individual work progress. On the other hand, the synchronous conduct demands for an immediate support of the students: "It is difficult to help students with problems, […], at least when many problems arise at once." For structuring the students' work process in distance lessons, the asynchronous support functions of the MathCityMap app are emphasised, i.e., hints, immediate answer validation, gamification and sample solution. Moreover, the possibility to work at one's own pace is stressed: The use of MCM@home "has given students who are usually quieter in class the opportunity to engage intensively with the tasks".

(iii) Individual support & assessment: The monitoring function of the "Digital Classroom" is seen as a well-structured tool for tracking learners' work processes in distance learning: "The display showing which student is working on which tasks or has already completed them is great!" For monitoring students' work process, the teachers reported both, class overview and e-portfolio, to be suitable for analysing and retracing students' progress. The e-portfolio is, in particular, analysed to provide individual feedback for the students via chat (see RQ 2). However, to immediately support the students after entering a wrong answer, further development of the "Digital Classroom" is desired: "I think it is important that e.g., a pop-up message is displayed directly when a student has entered a wrong solution".

(iv) Technical availability & handling: On the teachers' side, no problems are reported regarding the handling of the "Digital Classroom". The three teachers could create a "Digital Classroom" session the MathCityMap website without any difficulties. For navigating in the "Digital Classroom", the clear structure is highlighted: "I like the clear presentation. […] Clear icons, concise naming of the

areas, clear arrangement, clear differentiation of the areas from each other." On the other hand, the map-based design of MCM@home is criticised. Since the original system MathCityMap has been developed for teaching and learning mathematics outdoors, all tasks are related to a specific object and thus marked on a map. This, of course, is for the purpose of distance education no longer needed. Thus, teachers consistently expressed the wish for "utilizing the space of the map" for distance learning purposes, e.g., to see the work progress of several students at a glance or to show teachers the tasks and the sample solutions while simultaneously observing students' work progress in the "Digital Classroom". In addition, the use of the smartphone on side of the students is highlighted. Due to the high availability of smartphones and the students' experience in using them, "it is an advantage to work well on own mobile devices". Only a few difficulties were encountered in dealing with the MathCityMap app, all of which were due to an unstable internet connection. Otherwise, no problems were reported in dealing with the app, regardless of the age of the students.

RQ 2: To analyse the use of the chat, the teachers' answers to questions related to the requirement (ii) communication and interaction are considered. Further, the number of organisational and content-related conversations as well as communications aiming at personal contact are taken into account (Table 1). Hereby, one conversation contains all messages related to one reason for communication.

Initiation by	Contact person	Number	Number per cause of communication		
			O	C	P
Teacher (N = 3)	All students	8	7	1	0
	Single student	27	4	23	0
Student (N = 43)	Teacher	68	9	48	11

Table 1. Use of the MCM@home chat: number of conversations for organisational (O), content-related (C) causes as well as for personal communication (P).

Based on the number of communications, it can be assumed that the MCM@home chat function can be used properly by both, students and teachers. The communication initiated by the three teachers is mainly content-related: In 23 out of 27 one-to-one messages, the teachers gave individual support to a student – often after retracing the student's work progress in the e-portfolio. The possibility to send messages to all students at once is mainly used for organisational reasons (7 of 8 class messages). In addition, the students also requested advice from the teacher in 48 communications, which is almost equivalent to one request for help per student (N = 43). Besides this two-way content-related use, the chat – in student's view – can be used for personal communication, e.g., for questions about the well-being of their teacher.

In the questionnaire, teachers emphasise the possibility to send messages to all learners to give general instructions or to clarify open questions. To support an individual learner, teachers stressed the possibility to support learners individually after analysing their entered answers and invoked hints in the e-portfolio. As a further development, a teamwork mode is requested, where learners can use the chat to collaborate with other learners on the tasks.

CONCLUSION: CONSEQUENCES FOR THE LEARNING PLATFORM

Teaching and learning in distance due to the Covid-19 pandemic is perceived a major challenge by teachers and students. In this paper, we identified design requirements for distance learning environments with regard to the challenges that arose in Germany in spring 2020: (i) synchronicity and guided structure, (ii) enabling communication and interaction, (iii) fostering individual support and formative assessment as well as (iv) low-tech approach and user-friendly handling.

Hereinafter, the MCM@home approach was presented as a promising approach to address the described challenges of teaching and learning at home. After evaluating MCM@home in view of the design requirements from a theoretical perspective, a case study was conducted to evaluate the system's use in school practice. Based on the case study, we draw the following conclusions about MCM@home – being aware of the small sample of three teachers.

RQ 1 aims to evaluate MCM@home from a teachers' perspective in terms of design requirements (i), (iii) and (iv). The three teachers consistently describe the "Digital Classroom" as a suitable tool to manage distance education in real time. This finding is in line with our assumptions on the value of synchronous lessons in order to overcome the loss of familiar structures in distance education (Barlovits et al., 2021). Also, the guided structure (cf. Lichti & Roth, 2018) and the self-regulated learning (cf. Greene et al., 2011) in digital learning environments created within MCM@home, e.g., by the availability of hints and immediate answer validation, is highlighted.

Regarding the monitoring function of the "Digital Classroom", the retracing of students' work progress is seen as a major advantage of MCM@home. Hereby, teachers used both, the class view and the e-portfolio for an overall respectively individual monitoring. Desired features are automatic pop-up messages about the student's individual progress and the replacement of the map-based view (due to the original outdoor purpose of MathCityMap). Overall, it can be assumed that MCM@home meets the requirement of formative assessment (cf. Aldon et al., 2021). On the other hand, currently, no functionality for summative assessment (cf. Aldon et al., 2021) is available.

Finally, from a technical perspective, no problems with handling can be reported on the side of teachers (web portal) or students (app) in this sample. The approach of using smartphones is highlighted by the teachers in terms of ease of use.

RQ 2 focuses on the use of the chat within MCM@home. The teachers emphasize the chat channel to support students on an individual level after retracing their work progress in the "Digital Classroom" Thus, the interplay of the design requirements (ii) assessment and (iv) communication is highlighted. Furthermore, it is remarkable that not only the teachers – based on their observation in the e-portfolio – contacted their students, but the learners themselves ask for help via chat. Thus, the chat is described by teachers as a suitable and convenient way to fulfil the demand of teacher-student communication in distance education settings (cf. Aldon et al., 2021). Since learners also contacted their teachers for personal communication, e.g., asking about the teacher's well-being, it can be assumed that the chat can also help to compensate for the perceived lack of personal contact during distance learning (Barlovits et al., 2021).

However, the chat tool does not provide an opportunity for communication between students. In line with the need of interaction among students (cf. Drijvers et al., 2021), the teachers requested a teamwork mode for online collaboration. Being aware that also a decrease in content-related discussions is reported (cf. Drijvers et al., 2021), we recommend supplementing lessons conducted with MCM@home with synchronous video conferencing.

Overall, the case study shows that MCM@home can help teachers to monitor the work progress on both, class and individual level within the "Digital Classroom". Via chat, the teachers were able to individual support their students. Therefore, from both theoretical and empirical perspectives, MCM@home fulfils the demand of (ii) formative assessment and (iv) communication. However, the features requested by the teachers focus precisely on these two design requirements. Thus, the successful interplay of assessment and direct communication can be assumed – in a broader sense – to be a condition of success for the development of distance learning environments.

OUTLOOK: DEVELOPING THE ASYMPTOTE SYSTEM

With a focus on this interplay, the MCM@home concept is further developed to the stand-alone system ASYMPTOTE. From spring 2022, the web portal (www.asymptote-project.eu) and app of ASYMPTOTE is publicly available. Here, special attention is given to the integration of a teamwork mode and an improvement of the "Digital Classroom" evaluation. In addition, a broad open database of prepared digital learning environments on different topics is created during the project lifetime until February 2023. Finally, systemic adaptivity and automated assessment of student work is integrated to fully exploit the potential of digital learning environments (cf. Greene et al., 2011).

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WRITING ATOMIC, REUSABLE FEEDBACK TO ASSESS HANDWRITTEN MATH TASKS SEMI-AUTOMATEDLY

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Written feedback is powerful in math teaching yet highly labour-intensive. Hence, digital assessment with fully-automated feedback has received much attention. Unfortunately, learners solve higher-order questions more naturally when using paper and pencil. Therefore, we investigate semi-automated assessment: a method in which a teacher works with a computer system to assess handwritten tasks. When a teacher writes feedback for a student, the computer saves it, so the text can be reused when following students make the same or similar mistakes. To make feedback more reusable, we devised atomic feedback. During the workshop, participants learned how to write atomic feedback, experimented with providing feedback using the semi-automated system implemented in Moodle, and gained a thorough insight into the research project.

Keywords: Atomic feedback, handwritten task, reusable feedback, semi-automated assessment.

RESEARCH CONTEXT

In this research project, we investigate how we can give feedback to handwritten math assignments more efficiently. After all, handwritten tasks remain important to train higher-order thinking skills and genuine problem-solving in mathematics education. Therefore, we propose a semi-automated approach: teachers write feedback items, the computer saves these items so they can easily be reused when other students make similar mistakes (Moons & Vandervieren, 2020).

How to write feedback that can easily be reused for other students? Long pieces of classic feedback are often too targeted to a specific student. Hence we came up with atomic feedback: a collection of form requirements for written feedback that is hypothesized to be more reusable. To write an atomic feedback item, teachers must: identify the independent error occurring and; write short feedback sentences for each error, independently of each other. A comparison between classic and atomic feedback can be found in Figure 1.

In a crossover study with 45 math teachers in Belgium (Moons et al., in press), we could already prove that atomic feedback was significantly more reused than classic feedback (odds ratio: 2.6). Furthermore, results showed no significant time differences between paper-based feedback versus semi-automated feedback, but the teachers in our sample wrote significantly more feedback using the semi-automated system with atomic feedback compared to giving feedback with paper-and-pencil (d = 0.41). The semi-automated system with atomic feedback has been implemented in Moodle. A final version is planned to become available as open-source software when the research project is finished. We currently investigate ways to combine it with a marking system and suggest already provided feedback more intelligently.

WORKSHOP REPORT

The workshop wanted participants to experience the possibilities of semi-automated assessment of math tasks with atomic feedback and give them a deep inside of the current research project. First, participants learned how to write atomic feedback. Second, they tried to provide atomic feedback

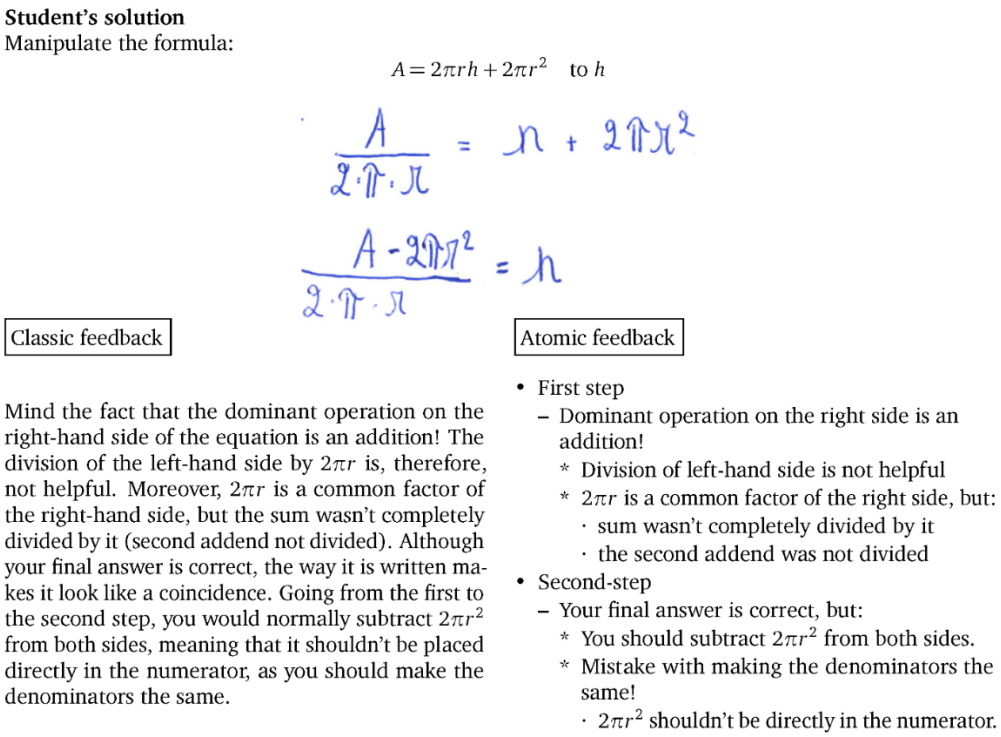

Figure 1. Example of classic versus atomic feedback

using the semi-automated system in Moodle on some students' tasks on linear equations. Sufficient time was provided to experience the effect of reusing feedback. Next, certain feedback examples of the participants were scored on their atomicness by the attendees using the codebook from Moons et al. (in press). This phase served as a reflective moment for the participants to deeply understand the atomic feedback concept. In the last half an hour, the experimental design of the study and the results were presented. The workshop ended with a lively, inspiring discussion on the following steps in the research project, the participants' experiences during the workshop, and their envisioned potential of semi-automatically assessing math tasks with atomic feedback. Several attendees taught mathematics to large groups themselves and were immediately interested in when they could implement the semi-automated approach with atomic feedback in their lesson series.

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RELATIONAL THINKING SUPPORTED BY AN ALGEBRAIC MODELING TOOL ON THE WEB

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A new web-based implementation of the FeliX system that combines algebra and geometry in a way that supports relational thinking is introduced. It allows to explore the role of algebra for the modelling of geometrical relations. The rationale behind the system is described, its design principles based on mathematical logic are explained and some use-cases are described. Especially, the didactical implications of a new feature that incorporates minimization is elaborated. Examples range from elementary explorations of geometric and algebraic relations to advanced applications.

Keywords: Algebra, geometry, educational software, relational thinking.

INTRODUCTION

Algebra offers different thinking tools; among the most important are variables, expressions, functions and equations. Working with and thinking about functions is coined together into the famous concept of functional thinking, which dates back to the days of Felix Klein more than 100 years ago (Weigand et al., 2017). Despite its age, the concept is still of great importance, and the use of spreadsheets and dynamic geometry systems such as Geogebra put even more weight on this. Both types of software realize a functional view of computations: There are certain input elements (cells with numbers respectively basic objects) from which other objects are functionally dependent (cells with formula respectively constructed objects). Dependent objects may be the input of further calculations or constructions so that a directed graph of dependencies is constructed. When an input cell is changed, or a basic object is moved then following the route of this graph allows propagating recalculation through all dependent objects. Thus, existent technology uses and fosters functional thinking.

Functions have a direction: $x \mapsto f(x)$. This is the reason why they are so often useful in modelling causal chains or channels of information transport. However, in the real world, there are also many relations that are undirected. Consider economics: Are the wages a function of the prices or the prices a function of the wages? Or consider physics: Is the pressure of a gas a function of its volume, or is its volume a function of the pressure? In such examples, there is no clear direction and functional thinking falls short as a mental tool to model such situations. What is needed is relational thinking. This concept is not as widespread as functional thinking. Stephens (2006) has reviewed some of the literature with a focus on primary education where relational thinking shows up in being able to see the equal sign not only in an operational sense. In secondary education, relational understanding also links to understanding the meaning of the equal sign (Bardini et al., 2013). For the present paper, relational thinking is understood as the thinking that relates several quantities. The aspect that equations are relations or restrictions on values of variables is also elaborated in Drijvers (2011).

Given this background, the project described here addresses the following research questions:

1. Is it possible to design a consistent software environment that supports relational activities?
2. Is it possible to integrate relational and functional modelling tools?
3. Do examples of tasks exist that exhibit the power of relational thinking for modelling?

The methodology to answer all three questions is constructive existence proof. It is thus a theoretical and empirical paper where the empirical part consists of computer implementation experiments. Therefore, data collection is reduced to observing the behaviour of the implementation. Based on old ideas (Oldenburg, 2007), a new system (Oldenburg, 2021) called FeliX has been developed. The present paper describes the implementation and the possibilities it opens up to integrate relational and functional thinking. The paper first gives a short overview of the system from a user's point of view, then shortly explains the design choices. A short technical part explains central ideas of the implementation. The paper concludes with a small kaleidoscope of applications.

FELIX FROM THE USER'S PERSPECTIVE

The user interface of FeliX consists of three main components: A geometry view that shows part of the Euclidean plane with a Cartesian coordinate system (cf. Fig. 1). Points and other elementary geometric objects can be created, and points can be dragged. The second component is a table that shows all points and their current coordinates. Points may be moved with the mouse or by entering new coordinates in the coordinate table, i.e., these representations are bi-directionally linked. The last and most important component is an equation table that may in fact take equations, inequalities and expressions that are built up from the coordinate variables of the points.

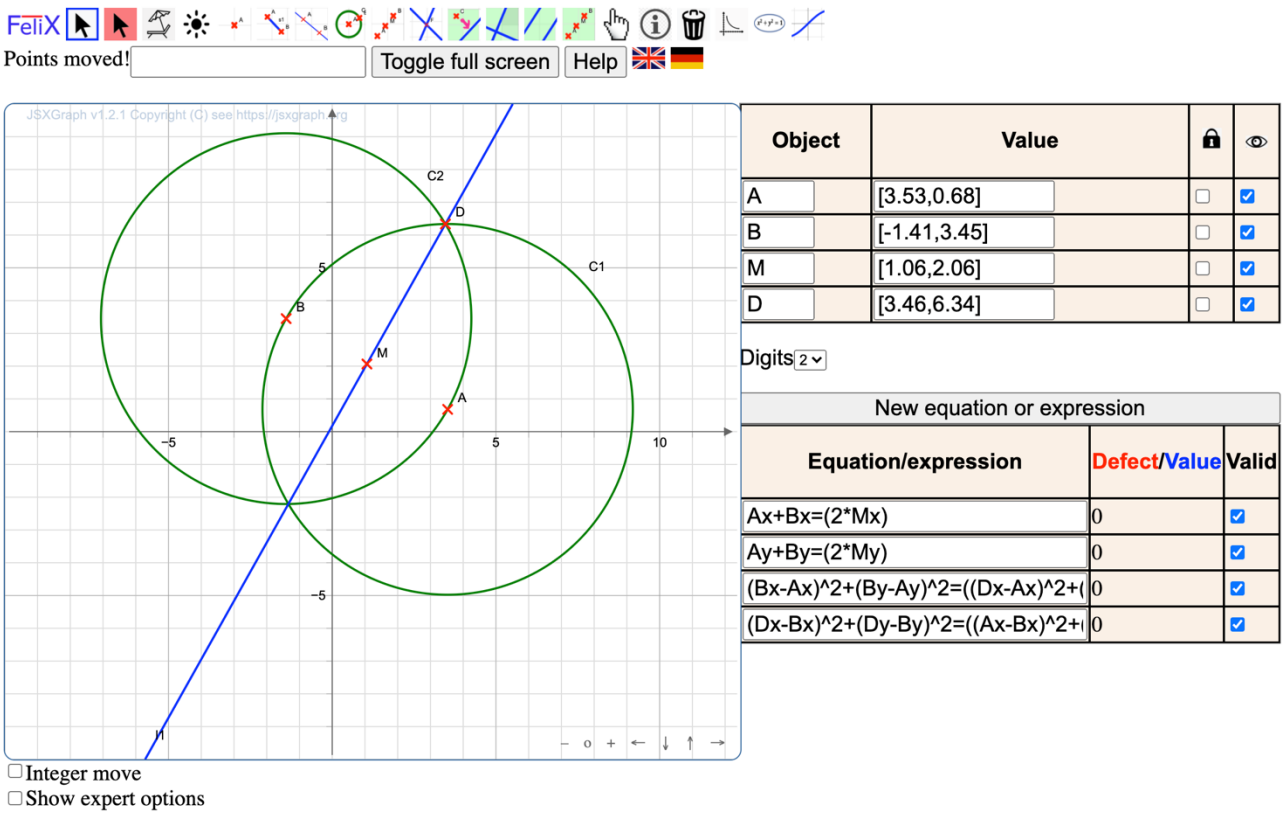
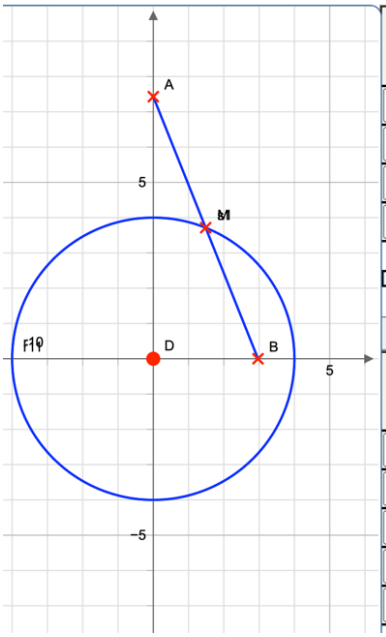


Figure 1. The main components of the FeliX window are a Euclidean plane, a table of points and a table of equations/inequalities and expressions

If there is a point P, then its Cartesian coordinates are Px and Py. The equations and inequalities are respected while dragging. Assume one has three points A, B, C and enters the equations 2*Bx=Ax+Cx and 2*By=Ay+Cy then B will be the midpoint of A and C. Still, all three points can be dragged with the mouse. In fact, if one of the points is dragged the system has some freedom to adjust the other two points so that the equations are fulfilled. To make the behaviour more

deterministic, one may set any of the points (temporarily) fixed using a checkbox in the object table. Such fixed points can still be dragged, but they will not move when the system tries to fulfil the equations. If, e.g., B is fixed, then A or C can be moved, and the other one follows as a reflection of the moved point. Equations and inequalities may be nonlinear and involve all usual mathematical functions such as the absolute value abs, trigonometric functions sin, cos, tan, exponential and logarithms and much more. There are helper functions to calculate distances, angles and lengths of segments. Moreover, some common equations (e.g., orthogonality or parallelism) may be set by using a button for convenience: This results in the appropriate equations being inserted just if they were entered by the user directly. Of course, all equations can be modified to explore the meaning of these algebraic relations. They can be set valid or invalid to explore their meaning. If the user enters contradictory equations, such as Ax=1 and Ax=3 then the system will show red defect values for those equations that cannot be satisfied. In this particular case, Ax would take the value 2 with defect 1 for both equations.

Figure 2 illustrates how FeliX can be used to model the classic sliding ladder problem. A "ladder" of length 8 with endpoints A and B is modelled by the equations Ax=0 (i.e., endpoint A is on the y-axis as a wall), By=0 (i.e., the other endpoint is on the x-axis as the ground). The formula for the length of the ladder was entered as Len(s1)=8 which FeliX automatically expands to the Pythagorean form given in Figure 2. Next, the midpoint M of the ladder was constructed. It can be moved with the mouse only on a curve, and, using Groebner basic methods, FeliX can calculate the equation of this curve and plot it. It is a circle which is optically highlighted by moving D to its centre (but D is not necessary in this example to create the circle!). Such graphs of curves that result from constraints on the freedom of a point are called "relation graphs" in FeliX language.



Object	Value
A	[0.0,7.43]
B	[2.97,0.0]
M	[1.48,3.71]
D	[0.0,0.0]

Digits 2

New equation or expre

Equation/expression
Ax=0
By=0
√((Ax-Bx)^2+(Ay-By)^2)=8
Ax+Bx=(2*Mx)
Ay+By=(2*My)

Figure 2. The sliding ladder problem

One could further construct an orthogonal line to the segment that passes through D. To do this, one has to construct a further point F, construct the line through D and F and enter the equations Dx=0, Dy=0, (Fx-Dx)*(Bx-Ax)+(Fy-Dy)*(By-Ay)=0. The last equations can also be set by using the green "declare orthogonal" button. Then the intersection of the line and the segment follows an algebraic curve of degree 6, which can be calculated but not yet plotted.

Figure 3 presents two other simple problems where the calculation of curves is interesting. On the left-hand side is an ellipse constructed from the defining equation that point C is on the ellipse if the distances to focal points A and B sum up to a fixed number. This fixed number is Dx in this case, so moving D deforms the ellipse.

The second example in Figure 3 starts from two segments that are set orthogonal. FeliX can calculate the curve that C can move on easily. As in Figure 2, the midpoint D of A and B is obsolete but was included in the figure to optically underpin that it is the centre of the Thales circle.

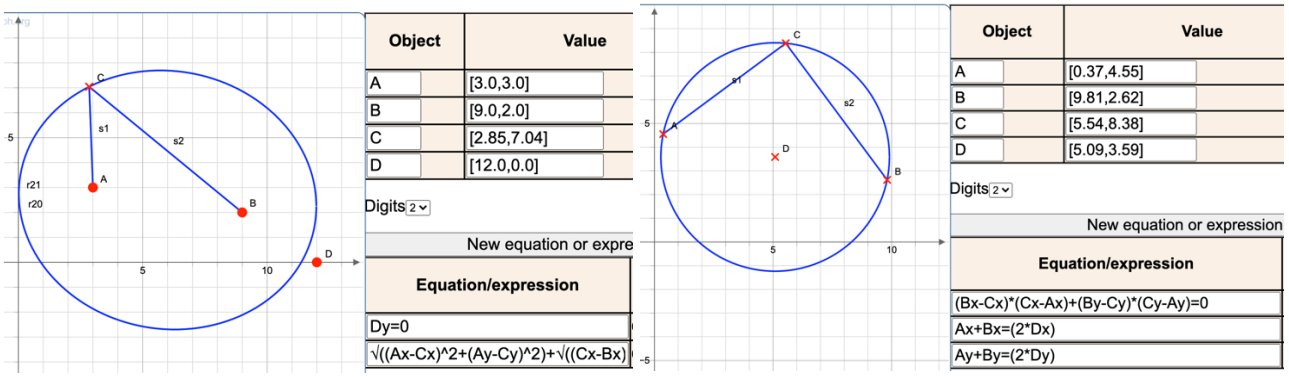


Figure 3. An ellipse from Len(s1)+Len(s2)=Dx and a Thales circle constructed from setting s1 and s2 orthogonal.

A feature not touched upon in the examples so far is that FeliX can handle inequalities. For example, one may enforce that two points always have at least a distance of 1 by entering (Ax-Bx)^2+(Ay-By)^2>1. This results in a construction where one of these points may be used to push the other around if they would come too close to each other.

Yet another feature is the ability to draw function graphs from expressions that are allowed to involve all other objects' variables. Using the "bind to" tool, points can be bound to lines, segments, circles, function graphs or relation graphs.

All these functionalities add up to a system that is very flexible in modelling geometric configurations and exploring the meaning of a wide range of algebraic relations.

DESIGN CHOICES

The design of FeliX follows from some very basic principles that have motivations both in mathematics and in didactics:

- Full Information: All information that governs the behaviour is fully visible on the screen. This is in contrast, e.g., with spreadsheets where one usually sees only the values in the cells, not at the same time the formulas (or, in formula view, vice versa).
- Based on concepts from mathematical logic: Object tables with the values of coordinates are essentially interpretations in the sense of mathematical logic (e.g., Hamilton, 1988): At each time, they assign numerical variables to all variables in such a way that, if possible, this is a model of the equations, i.e., that they are all fulfilled.
- Object creation and imposing relations between objects are independent operations. This allows a step-by-step specification so that the user can symbolize its knowledge about the situation dynamically as it evolves.
- Everything can be changed at all times. The order in which objects or equations are created has no impact on the behaviour. For example, the equations in the equation table are logically

connected by the "and" operator, which is commutative—any order of creation or order of imposing relations thus leads to equivalent configurations. Thus, no user will ever be caught in a deadlock because (s)he took a wrong decision at some time of the "construction" process.
- There is no restriction on the type of operations and functions that can be used.

It is interesting to compare these characteristics to those of dynamic geometry systems such as GeoGebra which differ much in all these points. Focusing on geometrical aspects, one may realize that the functional dynamic geometry is very different from the relational geometry described here. For example, to construct a triangle with sides 3,4,5 in a dynamic geometry system, one has to come up with a construction (which is a real problem for students that encounter it the first time), while, in relational geometry, one simply specifies the parts, removing *this* occasion for problem solving. From a pedagogical point of view, one may wonder what kind of geometry is more important to master. As compass and ruler constructions dominate in schools, functional geometry is more important there. However, in the professional world of computer-aided design systems such as AutoCAD and FreeCAD, the relational approach is dominant.

IMPLEMENTATION

The goal of FeliX's design is to have a system that has a clear and mathematically-defined semantics. Thus, it is basically an interface to a numerical constraint optimization algorithm. If, e.g., a point P shall be dragged to coordinates (x_0, y_0) then the expression $(P_x - x_0)^2 + \left(P_y - y_0\right)^2$ is minimized subject to the constraints given by the equations and inequalities. Coordinate values of fixed objects are inserted in the constraints, of course. At each point, the current configuration is used as the starting point for the search for a new solution. This approach is rather simple, but some caveats are in place: The solution process is done by a numerical algorithm (Powell, 1998) and hence may fail to find a solution, or there may be some noticeable inaccuracy. Moreover, some artefacts may result from this design. If a point is bound to a line that is fixed because some fixed points lie on it, then dragging the point is restricted in the sense that it will stay at the line and move to that point on the line that is closest to the mouse position. It may be the only sensible behaviour in this case.

Often there are many solutions, but only one will be found and realized. The equation that states that the lines through A,B and C,D are orthogonal leads to a scalar product equation that is not only fulfilled if the lines are orthogonal but also if A=B or C=D. Hence, such a degenerate solution may be found and realized by FeliX. There are two ways out: One may add an inequation that says that the points shall have some minimal distance. The other way is to use the "shake button" that randomly moves points and will often get one out of degenerate solutions.

Another issue is that of points going to infinity: Consider the lines through A,B and through C,D and construct the intersection point F. Set A,B,C fix and move D around C. The intersection F should move to infinity and come back from the opposite side. This may bring the solver into trouble. There is a more powerful move tool that sets the coordinates of the moved points and searches then for a solution. With this move tool, one may move through such degenerate situations.

The approach further implies a kind of existential quantification. If one has two circles and an intersection point, one cannot move the circles so far apart that they no longer intersect. Dragging mode will stop when they are tangent at their intersection point. Moreover, an equation like Ax=sqrt(Ay) will, as a side-effect, constrain A to the first quadrant of the coordinate system.

In contrast to the dynamic of moving objects, the calculation of relation curves for a restricted point P is done symbolically by calculating lexicographic Groebner bases (Cox et al., 2005) and eliminating non-fix auxiliary variables. In the resulting equation, only variables of fixed points and of the

generating point P occur. The method of Groebner bases is suited only for polynomial equations. Thus, FeliX eliminates in a preparation step subexpression $a^{\frac{n}{m}}, n, m \in \mathbb{N}$ and replaces them with v^n where v is a new variable, and it adds the equations $v \geq 0 \wedge v^m = a$.

Taking these elements together, research question 1 is answered positively by this constructive proof of existence.

EXAMPLES OF OPTIMIZATIONS

FeliX has a powerful feature that is almost invisible from its user interface: The checkbox to set equations either valid or invalid (an invalid equation will be ignored while dragging) is also in place for expressions. In contrast to equations, expressions are set to invalid by default. Setting them to valid means that they will be minimized! This gives some more modelling possibilities and combines relational and functional thinking, as the following examples will show.

Figure 4 illustrates a possibility to explore the optimality of certain figures. Five points are created and connected by segments s1,s2,s3,s4,s5 to form a polygon. The user entered the equation abs(polyArea([A,B,C,D,E]))=50 which is expanded by FeliX using the Gauss formula for the area of a polygon. It is then interesting to move around the points and see how flexible a polygon with fixed area of 50 is. Entering Len(s1)+Len(s2)+Len(s3)+Len(s4)+Len(s5) as an expression displays the circumference, e.g., the polygon on the left in Figure 4 has area 50 and length 32.48. Setting the expression for the circumference valid, i.e., minimizing its value, immediately moves the points to form the shape on the right-hand side of Figure 4. Similar investigations can be undertaken, e.g., to find shapes that have maximal area under fixed circumference or to solve other optimization problems.

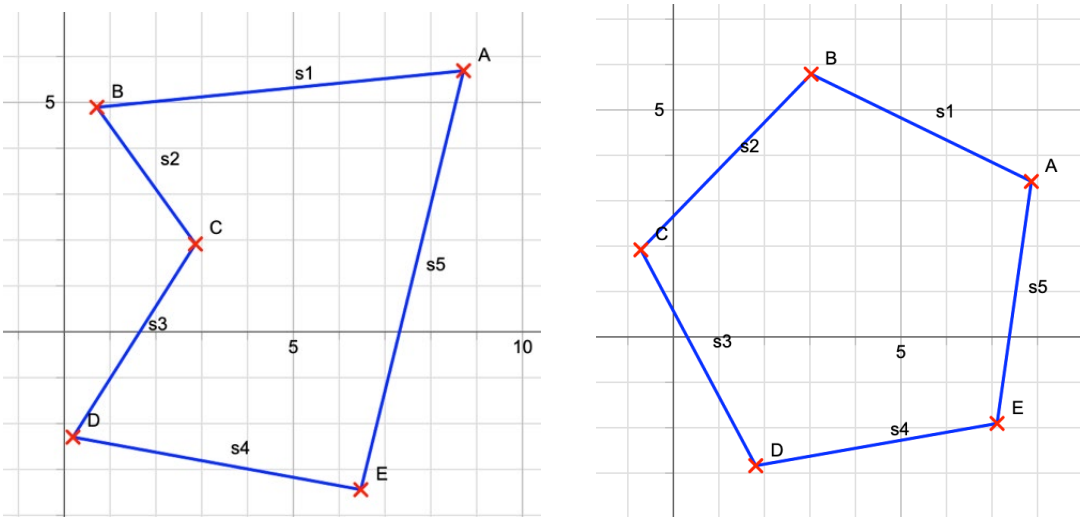


Figure 4. One click transforms the polygon of area 50 to one with the same area but minimal circumference

Figure 5 shows a discrete hanging chain. Points A and E are fixed (recall that they still can be moved explicitly with the mouse), and B,C,D are unrestricted points. The segments s1,s2,s3,s4 between AB, BC, CD, DE are all set to have length 3. Their midpoints F,G,H,I are constructed, and the expression Fy+Gy+Hy+Iy that corresponds to the potential energy of the chain is minimized. It is interesting how natural the chain behaves when, e.g., E is raised further or moved horizontally.

Another example shown in Figure 5 is a shortest-path problem which is also classic: The Fermat point is that point P inside a tringle ABC that minimizes the sum of the distances AP+BP+CP. FeliX allows students to observe the fact that at the optimal point, the three segments to A,B,C form angles of 120°. Of course, this is not yet the solution, but it may give hints to come up with a theoretical solution.

Both problems support a combination of functional and relational thinking. They combine the variability of a function value (which is minimized) with the invariance of a relation, which is preserved. The existence of these examples answers research questions 2 and 3 positively.

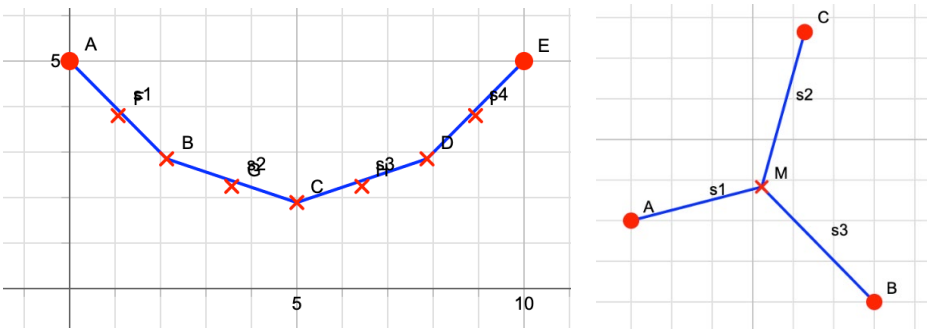


Figure 5. Discrete hanging chain and Fermat Point of a triangle

DISCUSSION AND OUTLOOK

The use of the old FeliX system 15 years ago was very limited because it was extremely difficult to install, but with the new web-based approach presented here accessibility of FeliX shall no longer be a problem. There are quite a number of areas were the work with FeliX might open promising perspectives. This final section will discuss some of these. Implementing the use of FeliX in schools is a complex task. First, teachers need to get an idea about what FeliX is and why its use could be rewarding. Next, one needs concrete ideas of how to introduce the system and what topics to use it for.

As a first contact with FeliX the problem of finding the midpoint of two points is very rewarding, because it exhibits most of the semantics of FeliX. Moreover, the equations that one needs are very easy and can either be entered by the students or created using the convenient tools. To explore und understand the midpoint relation it is useful to use FeliX's option "integer move". When this is activated, the dragged point moves discontinuously jumping only to points with integer coordinates. This eases mental calculations to check and understand what the system does. Moreover, modifying the midpoint equations is an interesting problem: How to get the 1:2 point that divides AB in this ratio? From there, one can, in principle, go on to find a form of the equation of a line.

A next activity may be to investigate one-dimensional equations. For example, one sets two points, A,B on the x-axis by Ay=0, By=0, and then relates A and B, e.g. by Ax+2*Bx=12 or by Ax*Bx=100. The same strategy can also be applied to equations that relate more than two variables, e.g., the "lens equation": When a lens forms a sharp image of an object, then there holds the relation $\frac{1}{f} = \frac{1}{a} + \frac{1}{b}$ between the focal length of the lens f and the distances a, b between then lens and object and between the lens and image. This interesting example has been used by Drijvers (2006) to illustrate the many roles variables can take when a computer algebra system is used, but the same consideration applies here: Setting a point fixed, e.g., turns it from a variable into a parameter.

Another field that can be explored with FeliX consists of the many relations between the various forms of quadrilinears. One may start with a general quadrilinear, possibly with its diagonals, and then may impose more and more relations and explore how rigid it becomes. As a last area of applications, the large field of mechanical linkages shall be mentioned. FeliX both provides easy ways to model and simulate them and to calculate relation curves of the movement of certain objects.

Of course, plenty of research lies ahead. One may ask if the experience with equations in this relational sense enhances students' performance on reversal error tasks (Rosnick & Clement, 1980).

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"… THEN IT LOOKS BEAUTIFUL" – PREFORMAL PROVING IN PRIMARY SCHOOL

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With a focus on the primary level, the project "Prim-E-Proof" pursues the goal of developing learning environments to support proving skills in mathematics education at primary school. The aim is to support teaching and learning processes with substantial learning environments in which – if useful – digital media (free applets on tablet PCs) are applied. In this paper, the state of the art of a learning environment for proving the theorem "the sum of two odd numbers is always even" is described and evaluated. A single case study explores the question of which epistemic actions are recognizable in a primary school child's activities according to Abstraction in Context to provide indications for the further development of the learning environment and the applet.

Keywords: Preformal proving, arithmetics, primary level, digital media.

THEORETICAL FRAMEWORK AND OBJECTIVES

Addressing the topic of proof already at the primary level can contribute to learners' perception of mathematical proof at secondary school or university as a natural extension of their earlier mathematical experiences (Stylianides, 2016)─and thus of mathematics education as a coherent whole (Wittmann, 2014). Digital media can have the potential to help children learn to prove through self-activity, which is a crucial aspect in learning mathematical proving (Freudenthal, 1979).

Blum & Kirsch (1991) distinguish (1) experimental 'proof', (2) action proof, (3) 'inhaltlich-anschaulich' proof, and (4) formal proof, whereby they summarize the action proof and 'inhaltlich-anschaulich' proof under the generic term *preformal proof*. The border between proofs that are none and real proofs runs between (1) and (2). In an *"inhaltlich-anschaulich" proof* (Blum & Kirsch 1991; Wittmann & Müller, 1988), something general is proven on a concrete, visually perceptible object usually presented in an iconic way. By trained observation, a learner can understand it as an object of a more general kind, whereby she/he has to mentally see the more general in the special of this example, to be able to produce an 'inhaltlich-anschaulich' proof. In 'operative proving' (Wittmann, 1985), this object represents a simple, generally executable operation that can be applied to an entire class, e.g., the shifting of tiles or the swapping of summands. Before these operations can be used in proof and have their effect, they must first be recognized as generally executable (Krumsdorf, 2015). A learner has to verbalize an 'inhaltlich-anschaulich' or operative proof so that what is initially subjectively found to be universally valid can be socially shared and recognized by others (Wittmann & Ziegenbalg, 2007).

The objective of *Prim-E-Proof* is to develop *substantial learning environments* for primary school mathematics lessons to support proving skills. Wittmann (1998) describes substantial learning environments with the following criteria: They must (1) represent central goals, contents, and principles of mathematics teaching, (2) provide rich opportunities for the mathematical activities of students, (3) be flexible and easily adaptable to the specific conditions of a given class, (4) integrate mathematical, psychological, and pedagogical aspects of teaching and learning holistically and therefore offer a broad potential for empirical research.

THE LEARNING ENVIRONMENT "DAMON & PHINTIAS"

In a study on the first version of the learning environment with 23 fourth graders of a German primary school, a need for proof could not be awakened for each learner (Platz, 2019). That is why a reference to the Pythagoreans is established in the sense of a historical digression (Krauthausen, 2018) to support the arousal of a need for proof. A *researcher's booklet* (www.melanie-platz.com/Forscherheft.pdf) deepens the historical digression using an introductory story about the Pythagoreans Damon and Phintias (cf. Dinger, 2014) as an introduction to the learning environment, because: "For primary school children, there are different experiences of relevance than for mathematicians. On the detour via a story, mathematically necessary relevancies can also implant themselves in the child's mind." (Kothe, 1979, p. 280). The structure of the researcher's booklet is based on the didactic model for learning and teaching proof concerning the process model of school proving (Brunner, 2014) with a particular focus on supporting the generalization process. A discursive framework initiates a need for proof, the oracle of Delphi poses a riddle to the children: *If I add two even numbers, is the result even or odd? If I add two odd numbers, is the result even or odd? Can you convince me that your assertion is always true?* (Platz, 2020b). These two proofs are developed based on tasks or task sequences (i.e., preliminary exercises). The second proof is partially analogous to the first proof (Fischer & Malle, 2004). Central tasks are for each of the two proofs: (1) What were even and odd numbers again? (2) Create tasks (even plus even or odd plus odd). What did you discover? At this point, the *Steinchen-Applet* (www.melanie-platz.com/Steinchen-Applet/Steinchen.html) is introduced to the learner. It allows to create, move and delete square (single) tiles, two-, five-, or ten bars on the screen. The objects can be rotated, grouped, or broken up into units. Furthermore, rectangles (or rectangles with a 'nose', see (3)) can be formed by touch actions. A grid in the background functions as an additional structuring aid. Different colors can be used, and a pen function allows to write on the screen. The mentioned functionalities allow a more robust fit between action and mental operation (Walter, 2017) than it would be possible with actual tiles. (3) The children then work on a task format based on Akinwunmi (2012, p. 128f), in which the learners deal with a geometric-visualized sequence of square tiles (arithmetic tiles; figured numbers) to provide a meaningful approach to algebra (see Figure 1).

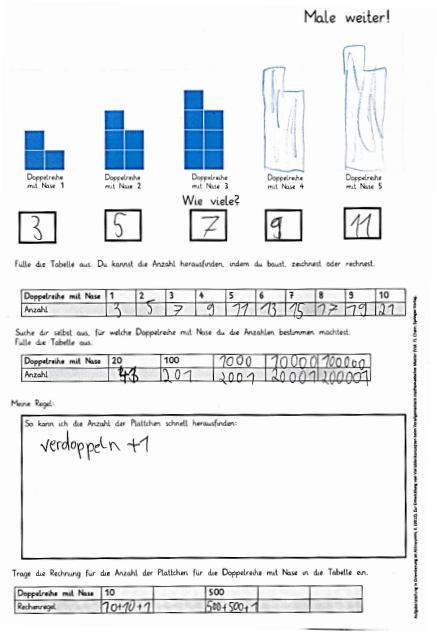


Figure 2: task in the researcher's booklet edited by Bob

The sequences of even and odd numbers, represented by tiles, are connected with an arithmetic form given in a table. The geometric figure shown in the task is called a 'double row' or 'double row with a nose' (Wittmann, 2014; Krauthausen, 2018; the 'nose' is the single tile at the top of the double row). (4) What happens if I place 2, 4, 6, or 8/ 1, 3, 5, 7, or 9 tiles on an even/odd number? Is the new number even or odd? (5) How do you know that the sum of two odd/even numbers represented by tiles will result in an even/odd number? (6) We explained it only for an example. Why does this apply to all odd/even numbers?

This work on preliminary exercises for proving and decomposing the proof intends to help learners structure the proof. Besides, the teacher can more easily recognize and name the respective degree of generalization of the assertion or the conclusion of a partial argument instead of remaining with the question 'Why does it always apply?', because: "The student may misjudge the limits of the generalizability of assertion and proof, especially if he is at the forefront of his knowledge when discovering, and not only when checking, an assertion." (Krumsdorf, 2015, p. 354).

RESEARCH QUESTIONS

This paper targets the following research questions: *RQ(1)* Which epistemic actions are recognizable in a child's activities working on the learning environment "Damon & Phintias"?; *RQ(2)* Is the developed learning environment substantial, and which consequences can be drawn for the further development of the learning environment and the applet?

METHODS AND PROCEDURES FOR COLLECTING AND ANALYZING DATA

Qualitative data collection and analysis methods were used within a single case study (Yin, 2018). A clinical interview (only key questions are defined and the interviewer follows children's thinking) was conducted for data collection. A clinical interview is, in principle, analogous to classroom management in implementing a substantial learning environment (Wittmann, 1998). This way, the collected data can provide information "[...] about teaching/learning processes, thinking processes and learning progress of students [...]. On the other hand, they help evaluate and revise the learning environment to design teaching/learning processes even more effectively." (Wittmann, 1998, p. 339).

The author of this paper conducted a clinical one-on-one interview in June 2020 with a typical fourth-grader (Bob) from a Hessian (Germany) primary school. Bob was chosen because he is not excited about mathematics but also not averse, and his usual performance in mathematics is average. Two sessions of 45 minutes each took place. The interview was videotaped and transcribed.

For data analysis, Abstraction in Context (AiC) is used, which allows studying, at a microanalytic level, learning processes that lead (for the learner) to new constructs (concepts, strategies, etc.). The central element of AiC is a theoretical-methodological model, the dynamically nested epistemic action model, according to which the emergence of a new construct is described and analyzed based on three observable epistemic actions (RBC model):

> […] recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner seeing the relevance of a specific previous construct to the situation or problem at hand. Building-with comprises the use and combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem. […] Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. (Dreyfus et al., 2015, p. 188)

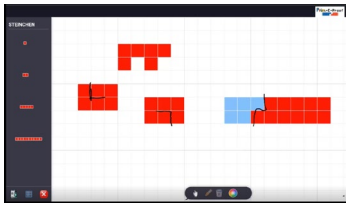
An a-priori-analysis according to AiC for the given proving task was done in Platz (2020a).

RESULTS AND DISCUSSION

RQ(1)

In the first interview session, Bob worked on pages 1–9 of the researcher's booklet (the proof of the theorem *the sum of two even numbers is always even*). In this session, Bob explains the term "beautiful" with the expressions "always two tiles on top of each other" or "without gaps", meaning a double row (even number). He explains, "if something does not look nice, then it is not even" meaning a double row with a nose (odd number). One week later, Bob worked on pages 10–18 of the researcher's booklet, and *scene 1* follows, where Bob tries to prove the theorem *the sum of two odd numbers is always even*. He initially stuck to the example 3+3 and explained the found pattern according to the interviewer's advice using the example 5+11.

Scene 1.



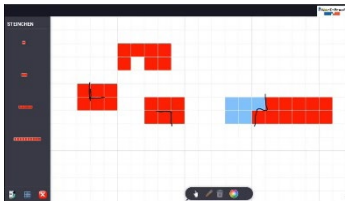
1	I:	hm *(affirmative) (...)* Exactly, which means that you always represent an odd number with a nose, right?
2	B:	hm *(affirmative)*
3	I:	exactly
4	B:	but you just have to rotate it correctly.
5	I:	hm *(affirmative)* and how to rotate?
6	B:	once around its axis. So half-way. […]
7	I:	and then what happens after you've rotated it? //*(not understandable)*
8	B:	Then it is/// then it looks beautiful.
9	I:	hm *(affirmative)*
10	B:	Then it goes like this. *(indicates rotation of the triple-figure on the right side of the tile pattern in the upper left corner, which stands for the task 3+3, i.e., a 'double row with a nose')*
11	I:	hm *(affirmative)* ok exactly, because then the two of them// *(not understandable) (points to the 'noses' of the double rows)*
12	B:	are united.//
13	I:	exactly, and then it will be
14	B:	without gaps.

Bob represents a number as a set of tiles (recognizing: cardinal number aspect). The applet facilitates the use of structured number representation. Bob understands an actual situation with similar objects (tiles) as a natural number. He also recognizes even and odd numbers and builds with the constructs: To distinguish them, he develops arithmetic and geometric patterns, modifies them systematically, and describes them. This process and the used constructs are strongly instructed in the researcher's booklet (see Figure 1). Bob uses presumably the basic idea of dividing (pairing of tiles), wherein an odd number one tile is added (Figure 1: "verdoppeln +1" – in English: "double+1") or missing and in an even number (*scene 1, turn 14*: we do not have "gaps"). Actions on sets of objects (sets of tiles)

and the actions themselves (pairing) are related, and Bob recognizes mathematically structural properties of even and odd numbers. He forms the sum and uses these structural properties in terms of constructing. Bob discovers that the two individual tiles can be joined together and that an even number can be produced when forming the sum by cleverly joining the double rows (*scene 1, turn 4*). By actions with (addition) and at sets of objects (pairing) as well as by relating the actions (to the previous construct), there is a mathematical transfer of the structural properties of even and odd numbers to the structural properties and the result of a basic arithmetical operation (addition). The reasoning of Bob is still tied to the presented examples. He verifies the statement by showing its validity in a typical case. It is not entirely clear if he manages to appeal to the structural properties of mathematics with reference to a generic example (Balacheff, 1991). Bob would have to recognize and describe the regularities in geometric and arithmetic patterns and continue them for generalization. Actions have to be put in relation to each other, a construction of actions has to take place, to which the action structure (previous construct) can be applied meaningfully, and the conditions of their use have to be worked out (Peschek, 1989). Mathematically, Bob has to recognize the structural properties of the sum of any two uneven natural numbers. What becomes evident in the presented scene is that the interviewer assists a lot. With help, Bob makes a representative selection of examples, and he tries to find possibilities for generalization. He seems to have a suitable justification idea that has not yet been taken to its conclusion in the sense of a conclusive argumentation. The resulting construct is still fragile and context-dependent and can only be used freely and flexibly by the learner through consolidation (Dreyfus & Kidron, 2014). Such a flexible use can be interpreted in the following scene:

Scene 2. After *scene 1,* Bob mentions that *the sum of an even and an odd number is always odd* (this was not a task, and Bob concluded this on his own).

1	I:	How would you/ or how can you see that this must always apply?
2	B:	Well then remains/ it remains a nose.
3	I:	hm *(affirmative)*
4	B:	So here like before, if I would put another one on here, that would be even *(puts a new single tile to the right side of the tile pattern in the upper left corner, which stands for the task 3+3, and creates a square by putting the single tile into the 'gap' to create the task 3+4)*



5	I:	hm *(affirmative)*
6	B:	so (..) then something remains, a nose.
7	I:	hm *(affirmative)*
8	B:	or a/ or a gap, no matter how (.) depends on how you do it.

In conclusion, the joint development of a kind of word or sentence memory (Selter & Sundermann, 2012) might have been useful to support Bob in verbalizing the generalization so that less assistance from the interviewer might be necessary. This word or sentence memory should be supplemented when the learner defines terms (e.g., *scene 1, turn 10*: "then it looks beautiful"). This way, a minimal professional consensus in terms of shared mathematical knowledge (Brunner, 2014) between

interviewer and learner can be established, "[...] by first clarifying the central subject-specific concepts and terms that are needed in a joint conversation and thus actually ensuring the shared meaning to the problem context." (Brunner, 2014, p. 89f).

RQ(2)

Especially criteria (2) and (3) of Wittmann (1998) are not entirely fulfilled because of strong instruction, which could lead to a restriction of individual and creative solutions. According to Wittmann (1998), "[...], a substantial learning environment is open in principle, and only the key information that the teacher gives at the beginning of each stage is fixed. Further interaction with the students and among the students remains open" (p. 339). Such an open interaction is not given in the current version of the learning environment. To create an openness of solutions, consideration could be given to providing learners with different approaches by offering various tasks and means of representation:

> (School) mathematics is a language that uses different regulative and regulated symbol systems: formal-algebraic, constructive-geometric, and verbal-conceptual or networks of these pure forms; these can be expressed predicatively or functionally […]. Thus, different approaches to mathematics […] can be accommodated. (Lambert, 2012, p. 19)

In this version of the learning environment, the presentation and structuring of the tiles are strongly guided. However, it can be helpful if a learner can experience the same situation in various representations and explicitly compare them. "A fixation on the same tool should be avoided." (Krauthausen, 2001, p. 107). In this sense, selection options could be implemented in the applet to allow working with other means of representation and tasks or examples that would enable different approaches and structuring. In addition to the presentation, the choice of examples also plays a role. A number example that is not too trivial in the sense of big numbers (Martin & Harel, 1989), i.e., dealing with large or unhandy numbers, can help to prevent the learner from perceiving the task as too easy, which can prevent the development of a need for proof. This could also be extended through a visualization where counting the tiles is not possible anymore. Thereby, it can be prevented that the learner only inductively tests the assertion on the example so that the learner's gaze might fall more on the structural aspects of the presented example (Krumsdorf, 2015). Not the quantity of given examples is essential: "[…] it seems to make more sense […] to offer the student selected examples according to his situational level of knowledge. Not quantity, but the quality of examples is crucial." (Krumsdorf, 2015, p. 353).

CONCLUSION

Bob was supported by the applet primarily through the possibility of rotating the grouped tiles and the structuring aids (grid) and the visualization options (different colors, pen tool) offered by the applet, which enabled Bob to take aesthetic actions (to make something look "beautiful"). As shown in *scene 2*, the developed learning environment could be used as a kind of preliminary exercise to enable the pupils to build constructs that can be used in other, similar proving tasks (e.g., inspired by the doctrine of even and odd in the *Book IX of Euclid's Elements*). However, the means of representation and arithmetic and geometric patterns are strongly instructed in the learning environment, which does not enable individual creative solutions. Therefore, a new learning environment, allowing discovering the assertion, building on previous knowledge, and promoting individual, creative solutions will be developed. In clinical teaching experiments (Wittmann, 1998), different support options will be tested depending on the proof phase, e.g., the variation of appropriate materials or representations (Krauthausen, 2001), the variation of appropriate example-based proofs (Krumsdorf, 2015), the use of Big Numbers or merely imagined examples (Martin & Harel, 1989),

or the use of pre-exercises for proving (Brunner, 2014). Nevertheless, a one-time implementation of a learning environment into mathematics lessons does not support proving competencies in a sustainable way. A culture of proving must be established in the classroom.

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FORMS OF EPISTEMIC FEEDBACK

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Feedback in a digital learning system has two functions, a pragmatic and an epistemic one. In previous papers, we were concerned with the pragmatic function, showing how digital feedback initiates acting with the system in groups of students. This paper addresses the epistemic function of this feedback by describing how feedback supports understanding algebraic principles. We report on an investigation of learning algebraic rules conducted with a virtual manipulative learning system, distinguishing, identifying and describing different forms of epistemic feedback, that is, how feedback provided by the learning system fosters disclosing and using algebraic principles.

Keywords: Algebra, epistemic feedback, integer arithmetic, representations, virtual manipulatives.

INTRODUCTION

Manipulatives have long been used in mathematics education to provide concrete representations of mathematical concepts. Symbols, however, are the primary representations used in mathematics, and students must become familiar with their use. We have been researching how to support this transition using digital feedback in a multimodal algebra learning system. In previous papers, we have focussed on the pragmatic function of feedback (Bikner-Ahsbahs et al., 2020, 2021), showing that digital feedback evokes students' actions by a feedback loop constituted in the learning groups. This paper reports on a complementary study in which we focus on the epistemic function of feedback in the same system. That is, we are investigating if our conceptually designed forms of feedback actually support the understanding of algebraic principles. Our report illustrates several kinds of epistemic feedback in the system, focussing on a virtual manipulative environment for integer addition. In this case study, we show particularly how this feedback allowed two boys to connect actions on the manipulatives to symbolic representations, in the process identifying and using several algebraic principles. We answer the research question of how the different kinds of feedback may foster disclosing and using algebraic principles.

THEORETICAL FRAME

As Duval (2006) notes, "mathematical processing always involves substituting some semiotic representation for another" (p. 107). Here we are interested in the substitution of symbolic representations for action on (virtual) physical objects. We make use of the ideas from research on representations by Kaput and Duval.

Kaput (1998) notes that already twenty years ago, "bi-directional links between pairs of the traditional … (numerical, graphical and character-string) notations have dominated the attention of educators and researchers" (p. 272) and that this focus informed software design. One example is the SimCalc project (see, e.g., Roschelle et al., 2000), in which actions on graphs affected other graphs and motions of cartoon figures. A danger of such environments, Kaput notes, is that, if the

representations only refer to one another and not to the students' experiences in the world, any fluency in translation between them is empty.

Duval (2006) uses the concept of registers to clarify the kinds of transformations of representations that occur. When two different representations are linked, for example, symbols and manipulatives, two kinds of transformations can occur. Duval calls transformations that occur within a single register, that is, transformations of the manipulatives or of the symbols, 'treatments'. Transformations that occur between registers, he calls 'conversions'.

These transformations are central to learning using models. As Yopp (2018) puts it:

> The concept is conceived by individual learners through:
>
> • Working with objects that fall under the concept
>
> • Working with representations of those objects, and
>
> • Working with relationships between these objects and their representations. (p. 45)

THE MAL-SYSTEM

We are involved in a research project, the Multimodal Algebra Learning (MAL) project, with the goal of developing an "intelligent" system of algebra tiles with the ability to give feedback to its users (see, e.g., Janßen et al. 2017, 2019, 2020; Janßen & Döring, 2017; Janßen, 2017; Reinschlüssel et al. 2018). We developed an app with virtual algebra tiles for testing various features. The app provides the same visual and symbolic feedback as is planned for the physical tiles. It is based on a balance model, which associates the physical act of placing or removing objects on each side of the balance with the mathematical operations of adding to and subtracting from each side of an equation. Like a physical balance, the app provides feedback if the quantities on the two sides differ, but unlike a physical balance, the MAL-system can handle negative and unknown quantities.

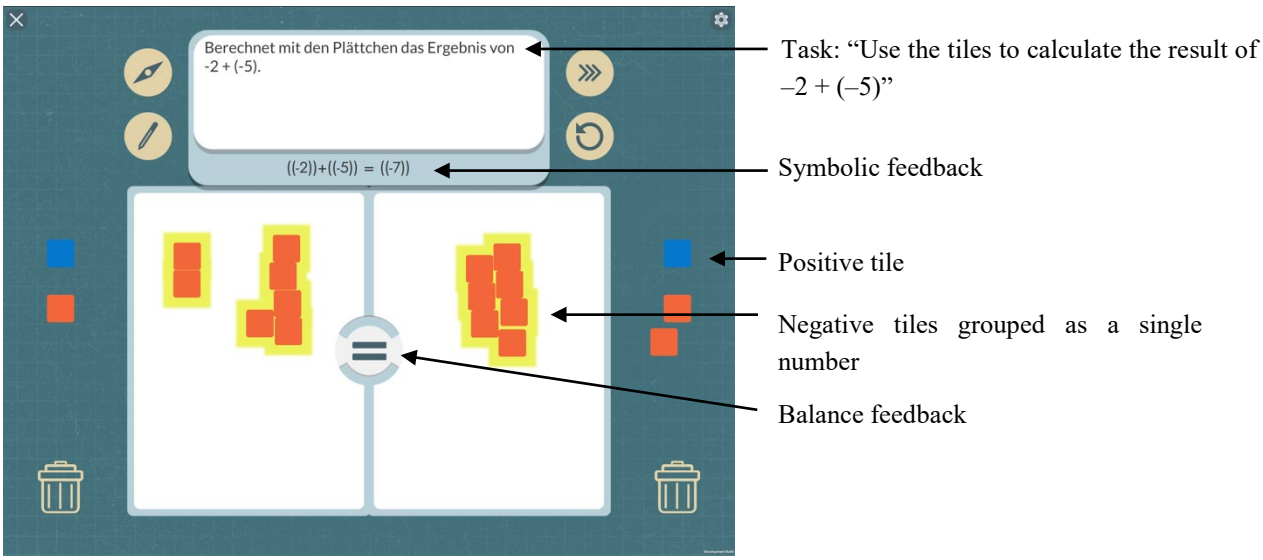

Figure 1. Some key features of the MAL-system

Some key features of the MAL-system are visible in Figure 1. Red square tiles, representing negative numbers, have been placed on both sides of the 'mat'. On the left side, they are grouped into a group of 5 tiles representing (–5) and a group of two tiles representing (–2). Because they are both on the same side, they are considered to be added, but because they are in two groups, the addends are represented, not the sum. On the right side, the tiles represent the sum (–7). The "balance feedback",

the equal sign between the two sides, provides feedback that the two sides are equal. If they are not, it becomes a not-equal sign. Above the mat is the statement of the task and the symbolic feedback: "((–2))+((–5))=((–7))". The double parentheses are due to a bug in the programming. This symbolic feedback updates as tiles are regrouped, added or removed, but it cannot be changed directly. Both the balance feedback and the symbolic feedback are always visible.

To represent an equation like 3 – (3) = 0, a special zone is used, the subtraction zone (see Figure 2). Tiles in this zone are considered to be subtracted from the tiles outside it, on that side. In the symbolic feedback, the symbolic form of the tiles in a subtraction zone are always enclosed in parentheses to indicate that the subtraction sign is not a negative sign. The zone outside the subtraction zone is referred to as the 'addition zone'. Red tiles represent negative numbers that are added, while blue tiles in a subtraction zone represent positive numbers that are subtracted, making the distinction between these two situations visually evident.

In Duval's (2006) terms, the MAL-system operates in two registers, the tiles register and the symbolic register. The symbolic feedback provides a 'conversion' from the tiles register to the symbolic register. Manipulations of the tiles involve 'treatments' in the tiles register, and the balance feedback operates in that register as an indication that the manipulations are legitimate ones in that register.

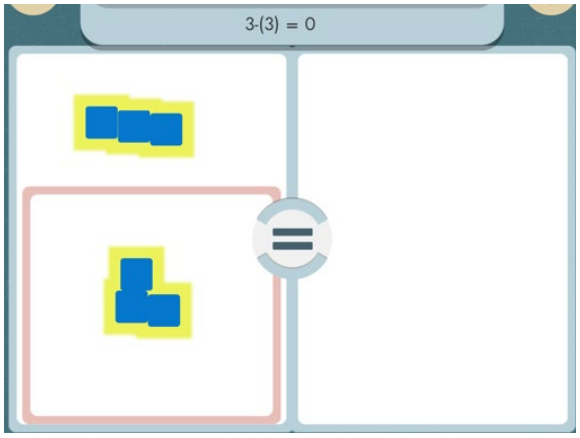

Figure 2. A Subtraction Zone used to represent 3 – 3 = 0

METHODS

The data analysed here come from a larger study exploring students' reactions to features in the MAL-system. Four pairs of Grade 5 students (aged about ten years old) from a German Gymnasium (upper stream secondary school) were given tasks about addition and subtractions of integers in a clinical interview setting outside the classroom (Hunting, 1997). The pairs were videotaped. We focus here on two boys, Simon and Timo, who were more vocal than the other pairs.

RESULTS AND ANALYSIS

In this section, we will describe results related to each kind of feedback and implications of the boys' reactions to that feedback. The boys' comments quoted below have been translated from the original German, and their names are pseudonyms.

The Tiles as Objects

One form of feedback that is not immediately obvious is the identification of the virtual tiles as behaving like physical objects that can be moved, and that have important properties like being countable and preserving quantity when rearranged. In the first task, only blue tiles were available,

and the boys were asked to place a single tile and observe what happened. The boys placed three tiles, predicting that the symbolic feedback would show "3", and were surprised when it showed "1+1+1". We interpreted their surprise as an indication that they expected the symbolic feedback to count the tiles in the same way they would, which in turn indicates that they expected the tiles to be countable in the same way that physical tiles are. As the boys placed more tiles and rearranged them, the virtual tiles continued to behave as if they were physical objects. This meant that the boys could *make use of their prior experiences with physical objects* when working with the MAL-system.

Grouping Feedback

As they rearranged the tiles in Task 1, the boys accidentally brought two of them close enough that the yellow outline marking a group was activated, and the symbolic feedback changed. They then moved all the tiles together. Simon commented, "When you move them together, then they are counted together as one number", indicating an understanding of how grouping works in the app. This suggests that the grouping feedback, in conjunction with the symbolic feedback, had allowed the *grouping convention to become transparent* (Meira, 1998) to the boys.

Balance Feedback

The first part of Task 2 states: "Represent the number 5 on the left side of the mat." As the boys did so, Timo commented, "At the beginning, there was an equal sign in the middle, that means it's equal." We believe he had observed that when no tiles are present, the balance feedback is "=" and that as soon as they started adding tiles to the left side, it changed to "≠". The next part of the task reads: "So far, you see an unequal sign in the middle. Move tiles from the stock to the right side and observe what happens. What do you have to do to make the unequal sign an equal sign?" Timo conjectured that the sign would again be an equal sign "when there is equally many in there" and confirmed his conjecture by moving two tiles from the left side to the right side and adding one tile to the right side from the stock, making three tiles on each side. Here we see the linking of the balance feedback to the state of the tiles on the two sides. Here Timo seems to understand the use of *the equal sign to represent the equality of two quantities*, which is fundamental to algebraic thinking. This understanding of the equal sign was not the only one they used, however. Later, in Tasks 4 and 5, Simon seemed to interpret the equal sign as an "instruction to calculate" (Behr et al., 1980).

In Task 3, which asked the boys to create a subtraction zone and explore how it worked, Timo again referred to the balance feedback. They had placed a tile on one side and made a subtraction zone. Timo placed a tile in it and said, "and when I put that in here, then it's equal. It's exactly equal and it shows an equal sign." The interviewer asked him why, and he explained, "Because one minus one is zero and zero is the same as zero, and there, there's zero." It is interesting that he does not simply say "Because one minus one is zero", which uses the equal sign to separate a calculation and its result, but continues "zero is the same as zero" *using the equal sign to mark equality,* which according to Knuth et al. (2008) is a prerequisite for further algebraic understanding. Here Timo's treatment in the tiles register supported his interpretation of the balance feedback within that register; no conversion to the symbolic register was involved.

Tile Colours for Representing the Sign

In the first five tasks, only blue tiles were available. In Task 6, red tiles were added to the store, and Timo immediately noticed them. The task asked them to find the result of 2 – 3. Timo remarked, "I think it could still work because it could be possible that the red tile is negative", and Simon agreed, "I think so, too". The boys followed the procedure they had used in previous tasks to find the positive difference between two numbers. On each side, they placed three blue tiles in a subtraction zone and two blue tiles outside it. This gave them a starting position in which the two sides were visually equal,

and the balance feedback indicated "=". They had learned that removing one tile from a subtraction zone and one from outside it preserves the equality. They did this twice on the right side, leaving one tile in the subtraction zone and none outside it.

The instruction in the task then told them to place a red tile and a blue tile together outside the subtraction zone. As he placed the tiles, Timo commented, "I think it stays equal," adding, "yes!" when the balance feedback remained "=". The balance feedback, operating in the symbolic register, provides evidence that a red-blue pair has a value of 0 in the tiles register. The boys made a direct connection between the tiles and integers. When the interviewer summarised "a red and a blue together always gives zero", Timo explained, "because a red one is negative", and Simon commented, "that [refers to a blue tile] is 1 and that [refers to a red tile] is, I think, minus 1."

Zero Pairs Feedback

In Task 8, a new kind of feedback was introduced. When a red and a blue tile are grouped as a pair, the pair vanishes. They seemed to be aware of this new feedback and its mathematical meaning. Simon observed, "it disappears," and Timo explained, "The result is 0 and therefore they are only additional tiles which are not needed." The task then instructed them to place more tiles on the left side, so that the balance feedback stayed "=". Timo immediate stated "simply zero pairs." At first, they had difficulty placing zero pairs because the new feedback immediately removed them, but Timo realised that if they were not placed together, they would remain, thus identifying the principle of *adding a number and its additive inverse cancel each other***.**

Symbolic Feedback

The most important feedback in terms of making a connection between the boys' actions on the manipulative and symbolic representations is the symbolic feedback, but this connection is made with the support of the other kinds of feedback we have already discussed.

The boys had noticed the symbolic feedback already in Task 1, when they saw that it showed a sum when tiles were placed apart on the mat, and a single number when they were grouped. However, in Task 2, in which they were to make as many representations of 5 as possible, the interviewer had to explain that "representations" referred to what was written in the symbolic feedback.

In Task 3, Timo was clearly attending to the symbolic feedback as he commented on two different symbolic expressions. They had placed the tiles as shown in Figure 2, when Timo remarked, "before there was a brackets calculation" and separated the group of tiles in the subtraction zone into a pair and a single tile. This changed the symbolic feedback to "3 – (2+1) = 0" and Timo noted, "with brackets." We believe he had noticed the symbolic feedback earlier, when they were first placing the tiles in the subtraction zone and had not yet grouped them into a group of three.

The boys were also aware of the relationship between the symbolic feedback and the tiles when they began Task 4, which told them to begin by placing tiles to show "3 – 2" on both sides. They placed the correct tiles, but due to a bug in the software, the symbolic feedback did not show "3 – 2 = 3 – 2" as expected. They restarted the task and proceeded once the symbolic feedback was what they expected. They also referred directly to the symbolic feedback at the end of the task, when they were trying to rearrange tiles to change the symbolic feedback from "2 + (–2) – (1)" to a single number.

In Task 8, Timo explained why the symbolic feedback displays "3 + ((–3))" when they have a group of three blue tiles together with a group of three red tiles. He said, "So there appears a double bracket because, otherwise, it would calculate plus minus three, and that would not make sense, and therefore there appears the bracket because it is another calculation." We believe he is saying that writing "plus minus three," that is, "3 + –3" "would not make sense." His exact meaning is not clear, but he is

certainly relating the symbolic feedback to the tiles and to the arithmetic operations on natural numbers that he is familiar with. The brackets are doubled due to a bug in the software, but the boys did not remark on the doubling.

USE OF FEEDBACK IN ESTABLISHING NEW KNOWLEDGE

In Task 9, the boys were to represent the number –5 on the left side and then on the right side, find different possibilities for representing the same number. Simon began placing five red tiles on the left side to represent (–5) while at the same time, Timo created a subtraction zone on the right side and started placing blue tiles in it to represent 0 – 5. They next placed another blue tile in the subtraction zone and a matching tile in the addition zone on the right side to represent 1 – 6. Timo then wondered, "What happens then if I …" and he placed a red tile in the addition zone. The balance feedback showed that the two sides were no longer equal. Timo removed one blue tile from the subtraction zone, which restored the equality. He said, "That works, too. Because they are negative, they don't need to be in the subtraction zone". Simon agreed, "That would also work," and he removed another blue tile and added another red one to produce (–5) = (–2) – (3). Timo explained, "Yes, that would work too because it results in zero." This is an interesting remark. Unlike a zero pair, which gives the feedback that it is equal to zero by disappearing, the zero here is indicated more abstractly, by the fact that the balance feedback remains equal when a blue tile in the subtraction zone is exchanged for a red tile in the addition zone. Here the boys used the interaction of feedback from the tiles, the symbol feedback and the balance feedback to conclude that, as Simon put it, "We always place one more red tile in the addition zone, then we take one blue away. That stays always the same." He also noted, "The same works the other way around" and removed a red tile, placing a blue tile in the subtraction zone, observing that the balance feedback continued to indicate equality. The boys had discovered an important principle in integer arithmetic, that *subtraction of a number and addition of the opposite number are equivalent.*

CONCLUSIONS

The different kinds of feedback offered in the MAL-system allowed the boys to make links between their experiences of physical objects, the virtual tiles and symbolic expressions.

Conversions from the symbolic register to the tiles register and back again occurred throughout. Because the tasks presented symbolic representations, the boys converted to the tiles register when setting up the tasks. They were guided in this by their connecting the virtual tiles to real objects. The symbolic feedback provided a check on this process, converting the arrangements of tiles the boys produced back into the symbolic register. This feedback was combined with the grouping feedback to establish what representations in the tiles register correspond to numbers, and how addition is represented in the tiles register.

The subtraction zone does not correspond to a real-world object, and so establishing its meaning involved comparison of the symbolic feedback and the tiles representation. In Task 4, the boys said that tiles representing 3 – 2 on one side of the mat are equal (according to the balance feedback) to a single tile on the other side. This fits with the boys' real-world experience of subtraction as removing two objects from a set of three. The balance feedback in the tiles register gave them immediate confirmation that the subtraction zone representation corresponded to subtraction as taking away.

The balance feedback was also important, in Task 6, in establishing that a red tile is the opposite of a blue tile. The boys saw that a red-blue pair is equal to an empty side representing zero, and hence that the red tile represents a negative number. When zero pairs feedback was introduced in Task 8, this identification was reinforced.

In addition to using the feedback to establish how to convert between representations, there were also occasions where the boys used the feedback to reach conclusions. One example of this is in Task 9. They could explain why exchanging subtraction of a positive number corresponds to the addition of a negative number, but they also relied on the balance feedback to confirm that the action in the tiles register of exchanging a blue tile in the subtraction zone for a red tile in the addition zone, produces an equivalent symbolic expression.

The feedback in the MAL-system offers support for mathematics learning in a number of ways. It helps to establish the meanings of representations (through conversions), it supports the definition of new objects and operations (through conversions and treatments), and it helps to confirm hypotheses, primarily through conversions; hence, the feedback functions in various epistemic directions.

Our study shows that the combination of manipulative material, and feedback offered by software, makes it possible to overcome the danger pointed to by Kaput, of conversions between registers becoming a meaningless game. We also believe that the feedback offered by the MAL-system allowed Simon and Timo to discover for themselves an important principle in integer arithmetic, that *subtracting a number is equivalent to adding its opposite*. In contrast, using physical manipulatives alone require interventions by a teacher to say how representations should be interpreted. That is, physical manipulatives provide pragmatic feedback, while the digital feedback in the MAL-system provides both pragmatic and epistemic feedback.

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CHALLENGES OF PROCEDURE-ORIENTED COGNITIVE CONFLICT STRATEGIES FOR UNDERGRADUATE STUDENTS

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Keywords: Cognitive conflict, misconceptions, overgeneralization, interactive videos, intrinsic cognitive load.

INTRODUCTION

At German universities, systematic errors caused by overgeneralizations can frequently be found in first-year STEM students' solutions of tasks related to differential calculus (Schirmer et al., 2020). These errors often remain stable and hinder students from successfully completing advanced mathematics courses (Kersten, 2015). Hence, there is a need to facilitate students to recognize their own errors so that they can adequately expand on their previous knowledge.

Students often make use of online videos to consolidate their knowledge. Despite a high number of freely available online videos intended to support students in consolidating their school knowledge in preparation for university mathematics, the quality of these videos is sometimes questionable or these videos do not explicitly address systematic errors (Oldenburg et al., 2020).

CONFLICT-INDUCING INTERACTIVE VIDEOS

The purpose of the present study is to design interactive videos that can induce a cognitive conflict. For that purpose, videos use the ECRR (elicit – confront – resolve – reflect) instructional sequence (Engelman, 2016) in order to reduce a typical student error. The study focused on a typical procedural error that first-year students show when deriving products of functions, and that can be described as overgeneralization of the differentiation rules for sums of functions to products of functions. This overgeneralization is also referred to as "illusion of linearity". As the concept of linearity is intuitive, students tend to rely on it without reflecting on its limitations and, in turn, inappropriately expand the domain of applicability (Verschaffel & Vosniadou, 2004).

To induce a cognitive conflict, we used a function that can be represented as the product of two polynomials and differentiated it in two ways. We designed a video that contrasted the incorrect solution of determining the derivative caused by overgeneralization and the correct solution based on elementary derivation rules. The idea was that the comparison of the two unequal results coupled with the reliance on elementary derivation rules for polynomials would lead (1) to the awareness of the presented inconsistency, (2) to a dissatisfaction with the existing unsustainable concept—a condition for a conceptional change (Hewson, 1992)—and (3) to a rejection of the incorrect rule. To reinforce this rejection, we presented the solution obtained with the help of the product rule and highlighted the equality of the result with the result we got when using elementary derivation rules for polynomials in order to convince students of the necessity of the product rule.

EXPLORATORY STUDIES

In our exploratory studies, we investigated the effects of the conflict students experienced. Particularly, we examined whether students recognized the contradiction presented in the video and whether they considered it to be useful for their learning. 16 second-semester engineering students worked with the interactive video and were interviewed afterwards in videotaped sessions. In a pretest, no student made the error based on overgeneralization. One-third were able to recognize the error as the primary learning objective of the video. The most frequent classification of the learning objective was the deepening of the product rule. Furthermore, five students expressed surprise or uncertainty when they watched the presentation of the wrong solution. Four out of these five students initially doubted their own solution based on the product rule, which they had previously solved in a test task. We concluded that the function we used to contrast two different solutions was not suitable to trigger a cognitive conflict.

In a follow-up interview study, engineering students were presented with different contradictions and two means to make sense of these contradictions: the first example was more procedurally oriented, the second more conceptually oriented. The students were asked to give reasons for the contradiction. Results showed that students struggled to compare ways of calculating a derivative which we, based on our analysis of the data, attribute to high cognitive load. It was noticeable that some students recited concepts about calculating extrema, even though the interview question referred to the verification of solutions of different calculations. For the conceptually oriented example, students used arguments based on geometric properties of functions, such as the concept of slope, to explain the contradiction. We conclude that a procedurally oriented example is less suitable to reveal a contradiction and hence is not likely to trigger cognitive conflicts.

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Theme 2: Making Sense of 'Classroom' Practice

with and through technology

SHIFTS FROM TEACHING MATHEMATICS *WITH* TECHNOLOGY TO TEACHING MATHEMATICS *THROUGH* TECHNOLOGY: A FOCUS ON MATHEMATICAL DISCUSSION

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Mathematical discussion plays a key role in teaching-learning processes, and it has been mostly studied when implemented in physical classroom contexts. However, during the lockdowns imposed by the Italian government (and many other governments worldwide) because of the COVID-19 pandemic, teaching had to be performed exclusively online. Therefore, also mathematical discussions, when implemented, had to be carried out through digital technology. In this paper, using the perspective offered by the Theory of Semiotic Mediation on collective mathematical discussions, I analyze an online mathematical discussion on division algorithms in a 6th-grade class. The analyses point out significant differences in how the actions were implemented online and in their effect in the discussion, especially on the emergence and elaboration of signs.

Keywords: Online mathematical discussion, signs, teacher actions, Theory of Semiotic Mediation.

HOW MATHEMATICS CAME TO BE TAUGHT THROUGH TECHNOLOGY IN ITALIAN CLASSROOMS: THE COVID-19 CRISIS

To contextualize this study, I start by citing a very representative quote from a recent paper by Ramploud et al. (2021) describing the general feeling of most (if not all) Italian teachers during the COVID-19 crisis:

> […] it was a violent and uncontrolled cultural change that teachers had to face at a moment of extreme isolation, since communication with and among students was to be necessarily combined with technological tools, changing the nature of communication […] (Ramploud et al., 2021, p. 5)

The authors describe the situation as a *crisis*, referring to Yerushalmi (2007):

> Crisis situations, as I am defining them, occur during ordinary life occurrences when one's personal mechanisms for organizing experience cease to function. […] Often the primary experience is one of internal imbalance and lack of control over one's life. (Yerushalmi, 2007, pp. 359–360)

Although, unfortunately, this was the situation within which most Italian teachers felt trapped, they (at least the teachers from our research-action groups [1]) continued doing their best to support the emotional well-being and mathematical learning of their students. In this crisis situation, many students did not dispose of adequate devices and internet connections, so most of the teachers decided not to propose activities with digital artefacts during the regular online lessons. Such a decision was in contrast with what I and other researchers involved in the research-action projects tried to propose and had hoped to be able to study. However, many of the teachers in our groups did try to promote mathematical discussions online. This is the situation that I use in this paper to exemplify teaching *through* technology. In particular, I will highlight shifts in the practice of mathematical discussion when it is carried out online as opposed to in physical classrooms.

Conceptualizing the use of digital technology in the "crisis" context and my objective.

The didactical tetrahedron (Ruthven, 2012) foresees two main roles of technology: one describes digital technologies in their mediational role with respect to specific mathematical content, while the other is described by Ruthven (2012) as follows:

> Another line of development in the educational use of digital technologies has sought to update and enhance the basic infrastructure that supports classroom communication between teacher and students, and assists their use of content-related resources within and beyond the classroom. (Ruthven, 2012, p. 628).

Because of the data that I was able to collect and analyze prior to this talk, I will focus on lessons in which digital technologies are framed within this second line of development. Moreover, this will allow me to focus specifically on shifts that occur due to the different form of communication (online as opposed to "in presence") between teachers and students, avoiding overlap between the two lines of development. Within this second line, we can talk about way in which teachers use digital technology in terms of "schemes", as discussed in the Theory of Instrumental Genesis (Monaghan et al., 2016).

Within this perspective, our teachers were faced with the task of communicating with their students, to teach them mathematics, and to accomplish such a task they made use of the (digital) tools of Google Meet and Jamboard. So, studying how mathematical discussions are carried out through technology becomes a matter of inferring teachers' schemes, by looking at techniques (the visible parts of such schemes) used to communicate with digital technology during such online discussions.

Objective of this Paper

Specifically, I will focus on the online implementation in Meet and Jamboard of the typical actions (see the following section) used by the teacher in a mathematical discussion, which I expect to be qualitatively the same online and in presence. I will pay particular attention to how the teachers use this digital technology to foster the emergence and elaboration of *signs* in such a discussion. An underlying hypothesis is that there are differences in how the actions are implemented online and in their effect in the discussion, especially on the emergence and elaboration of signs.

SIGNS AND MATHEMATICAL DISCUSSION FROM THE PERSPECTIVE OF THE THEORY OF SEMIOTIC MEDIATION

The Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) was developed to analyze the relationship between students' accomplishment of a task through an artefact and their mathematics learning, precisely addressing the issue of how students can become aware of the meanings stemming from the activity with an artefact to accomplish a task, and of how such meanings can evolve towards target mathematical ones. In this section, I introduce *collective mathematical discussions*, as they are conceptualized within the Theory of Semiotic Mediation, focusing specifically on the theorized teacher actions within such discussions according to the goal of promoting the evolution of signs. I will use the notion of *sign* in a broad sense, including any kind of perceivable spatio-temporal entities which might be uttered, spoken, written, drawn, encoded electronically, or in general used by someone to express some meaning.

Collective Mathematical Discussion

Classroom discussion to promote mathematical learning has been conceptualized in different ways (e.g., Michaels & O'Connor, 2015; Stein et al., 2008). Here I refer to *collective mathematical discussion* as introduced by Bartolini Bussi and Mariotti (Bartolini Bussi, 1998; Bartolini Bussi & Mariotti, 2008); it is part of each didactic cycle, and it is a kind of activity involving the whole class:

various solutions are discussed collectively, students' situated signs (with personal meanings) are collectively analyzed, commented, and elaborated. Students' interventions are coordinated by the teacher with the goal of generalizing the situated meanings, emerging from the activities with the artefact, and moving them towards mathematical meanings. (e.g., Bartolini Bussi & Mariotti, 2008; Mariotti & Maffia, 2018).

Teachers' Actions in Collective Mathematical Discussions

The study of teachers' actions aimed at fostering the production and development of signs during mathematical discussions has led to a classification of such actions (Mariotti, 2009) that Mariotti and Maracci (2010) have also described in terms of schemes. Below are the four types of actions conceptualized in these terms.

Back to the task: the class of situations characterized by the need of promoting the students' production of signs related to the actual use of the artefact for accomplishing a given task.

Focalize on certain aspects of the use of the artefact: the class of situations when the discussion has led to the emergence and sharing of a rich net of signs related to the use of the artefact and there is the need of selecting the pertinent aspects of their shared meanings in respect to the development of the mathematical signs that constitute the final education goal.

Ask for a synthesis: the class of situations when the discussion has led to the emergence of shared and stable signs condensing the key aspects of the common experience with the artefact, and there is the need of generalizing and decontextualizing the meanings that emerged.

Provide a synthesis: the class of situations, when the discussion has led to the de-contextualization and generalization of meanings form the context of use of the artefact towards the context of mathematics, and there is the need of ratifying the acceptability and the status of a sign within the mathematical context.

Application of this Theoretical Perspective to the Study in Focus

The teachers in this study were aware of the four types of actions described above, and they were used to enacting them during collective mathematical discussions conducted in presence in their classrooms. However, the teachers had not conducted discussions completely online before.

A slightly delicate issue is that of the artefact around which the discussion is centered, according to the didactic cycle. In many studies in the literature, including most of my own (e.g., Antonini et al., 2020; Baccaglini-Frank & Mariotti, 2010; Baccaglini-Frank, 2019, 2021), the artefact is a digital one. Such studies would, therefore, fall into the first role of technology described by Ruthven, that I introduced above. However, in this paper, the artefacts around which the discussion is promoted are two division algorithms [2] (Lisarelli et al., 2021). Therefore, the signs I will be analyzing refer to these algorithms and mathematical meanings behind them, such as division, place value of digits in numbers written in base ten, or powers of ten.

However, here I will not be looking at how digital technology mediates mathematics, but at how it mediates the communication *about* mathematics between the teachers and the students, which is consistent with Ruthven's second role of technology.

PARTICIPANTS, MATHEMATICAL CONTEXT AND DATA COLLECTION

The mathematical discussion in this paper is part of data collected during the Italian lockdown in March-April, 2020 from classes of the teachers of our research-action group [1]. Specifically, the discussion occurred online in one of the two subgroups that a 6th-grade class was divided into for

some of the online activities (as described in Lisarelli et al., 2021). The online discussion was conducted by F., the main teacher, together with G., a researcher, who had been involved in the design and implementation of the teaching sequence on division algorithms.

The class was part of an English school in Northern Italy, in which mathematics had been taught in English until the end of grade 5, while in grade 6 it was taught in Italian. In primary school the students had been taught an algorithm they called "DMSB". The acronym stands for the steps: divide, multiply, subtract, bring down (Figure 1a). Prior to the lesson in focus, the class had been introduced to what was referred to as the "Canadian" algorithm (also known as "Nuffield", see Figure 1b) [3], through the escamotage of a letter from an imaginary Canadian student Nadège. Prior to this lesson, the students had used both algorithms to calculate quotient and remainder of various divisions, but they had never been asked to compare these two algorithms.

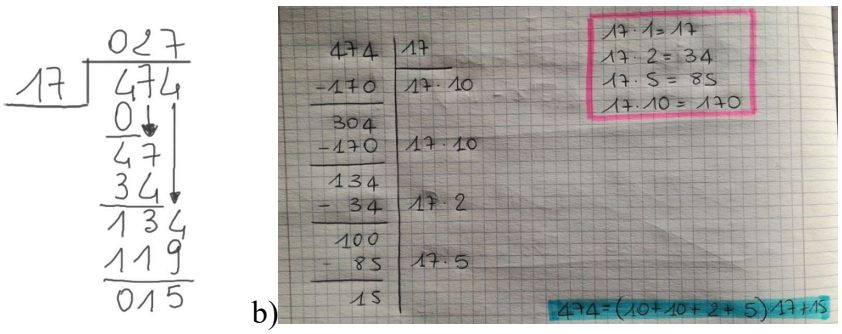


Figure 1. a) Divide, Multiply, Subtract, Bring down (DMSB) algorithm; b) "Canadian" (non-optimized) or "Nuffield" division algorithm

During the online lessons (including the one in focus), the students were asked by their teacher F. to keep their webcams and microphones off unless she asked them to activate the microphone, while the webcams had to stay off. The students had free access to the online chat in Meet. The mathematical discussion occurred in Italian, and it was transcribed in Italian and then translated into English.

Setting the Scene of the Lesson with Respect to the Didactic Cycle

F. and G. had assigned the division $395 : 16$ to be carried out with the DMSB and with the Canadian algorithms as homework. They had collected the students' work and transcribed some of the solutions to use at the beginning of the lesson, before launching the collective discussion. G. was in control of the Jamboard and, together with the class teacher, F., guided the discussion. F. had announced, at the beginning of the lesson, the objective in the following terms: "Let's be mathematicians and try to compare these two algorithms and discover why they produce the same quotient and remainder."

SPECIFIC FINDINGS FROM THE ANALYSES OF EXCERPTS FROM THE ONLINE COLLECTIVE MATHEMATICAL DISCUSSION IN A 6TH GRADE CLASS

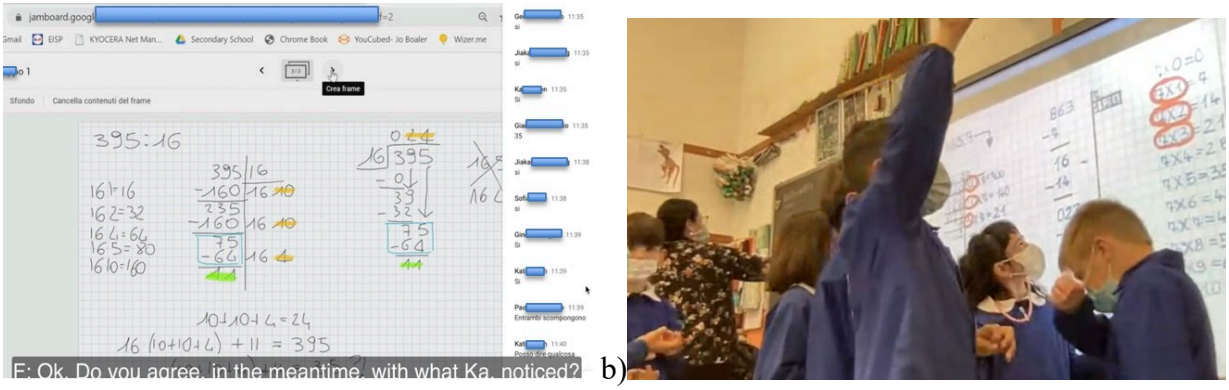
For space limitations, I will not include here the complete transcriptions of the excerpts I report on, some of which are included as subtitles of the video clips shown during my talk (visible here: https://youtu.be/WqEQHtP5_fs). Here I will only list some of the findings from such analyses, organizing them around the main tools used, and trying to follow the discussion's temporal evolution.

Using the Chat in Google Meet to See Students' Signs and as a Window onto the Class

Both F. and G. frequently open the chat and scroll down, in search for students' signs to pick up and use in their actions. Especially at the beginning of the discussion, they use the students' signs written in words for focalization actions. We note that signs in chat are only expressed in a single modality,

as written words, unlike in "in presence" classrooms, in which F. would usually capture gestures and oral expressions together with written signs on the students' worksheets or notebooks. So, this online action appears to dramatically narrow down students' initial modes of sharing signs they have produced. When promoting online focalization actions, F. and G. tend to call on students who have produced certain signs to turn on their microphone and explain, adding another mode of production and potentially new signs to the semiotic bundles (Arzarello, 2006) produced.

The teachers also use the online chat to get an overview of how the class is following. For example, after providing a synthesis of what Ka. has noticed, Figure 2a shows F. asking whether the class agrees with Ka. and immediately after, she and G. scroll through the chat. Many students write "yes" or words suggesting approval. However, F. seems to use that chat also to see who remains silent, that is, "invisible", to later call on them and ask them to participate.



a) b)

Figure 2. a) The teachers using the chat as a window onto their online class; b) visual glimpse at student participation in a physical classroom, in response to a similar question by the teacher

This way of using the chat seems to be an online version of what visual and acoustic signs convey in physical classrooms. Figure 2b shows students in a physical classroom responding to a similar intervention of their teacher in a mathematical discussion. However, unlike in a physical classroom, the chat allows to "see behind" students' raised hands, because frequently students write brief answers to questions posed by other classmates or by the teacher. In the discussion analyzed here, a student, Al., referring to the "75" that appears in both algorithms, disagrees with a classmate who says: "they are the same" and asks to speak, writing in the chat: "[but] coming from two different sums".

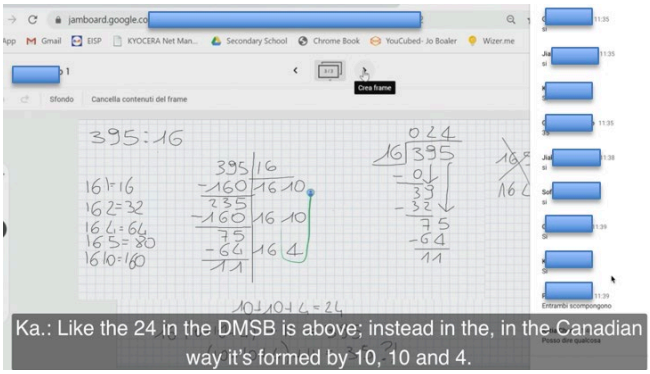
Using Jamboard for "Online Gesturing" in Coordination with Students' Speech through the Mic

Initially, G., who is in control of the shared board, tries to coordinate her production of signs Ka.'s words as she speaks (without being seen or being able to act on the Jamboard). For example, in Figure 3a, G. highlights with the laser tool the numbers Ka. is referring to. This technique, a sort of "online gesturing", serves to bring attention to certain graphical signs, part of the bundle with the oral signs, as well as to confirm to Ka. that she is being understood. Notice that, in a physical classroom, these or similar gestures accompanying verbal utterances would all be produced by the student.

F. suggests a modification of G.'s technique to produce more permanent signs on the board, as shown in Figure 3b.

Ka.: Like the 24 in the DMSB is above; instead, in the, in the Canadian way it's formed by 10, 10 and 4. (Figure 2a)

F.: Uhm, can you circle them, G., with the same color using the highlight tool, maybe? ... Leave them colored.

a) 

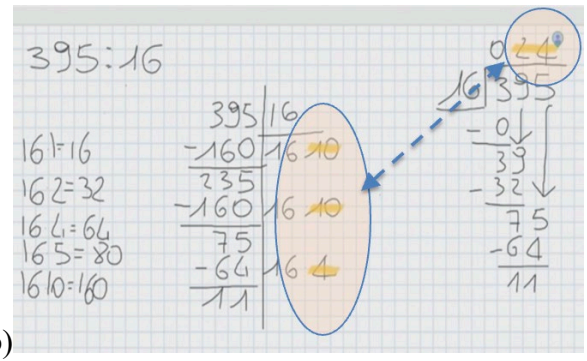
b) 

Figure 3. a) G. uses the laser tool as Ka. speaks; b) correspondence between signs described by Ka. and highlighted graphically by G.

Now Ka.'s verbal utterances are completed with corresponding graphical signs, that are highlighted to make the correspondence explicit. The correspondence between the 10, 10 and 4 and the 24 shown in Figure 2b is expressed orally, graphically, and through G.'s online gesturing. So the set of signs in focus is produced jointly through a collaboration between Ka. and G.. In this joint effort, the actors seem to have different, though harmonious intentions: Ka. wants to explain the correspondence between signs she has perceived but that she can express only verbally, while G. works hard to interpret Ka.'s words, in order to "see through her eyes" and show Ka. she is being understood, while at the same time enriching the set of signs with graphical components in the hope that other students will see the same connections that Ka. is describing. This coordination effort can be seen as a focalization action.

Indeed, G. notices Al.'s comment, finds it relevant in the orchestration of the discussion, and immediately uses it in a focalization action.

G.: What does "broken into two different sums" mean?

Al.: The two...I mean in the first one there is 235 minus 160... and, eh, and in the one on the right there is 39 minus 32.

F. & G.: Uhm! Uhm!

G: What do you see as similar? What does "broken un into two different sums" mean?

Al.: No, in the sense that they derive from two different sums, the same subtraction.

F: It means that the 75 comes out from two different processes, Al. is saying.

G: So Al., are you suggesting that this package—I don't know if I'm going too far beyond or if it's what you meant—should somehow correspond to this little package here? (Figure 4)

In the excerpt above, notice how Al. uses a sort of verbal enaction of a pointing gesture ("on the left/right"), elaborating on the sign "different sums", which he then refers to as "the same subtraction". F. elaborates on Al.'s sign, restating it as "different processes". This builds up to a key moment in the discussion, at which G. expresses the sign in terms of "packages" that she interprets, as shown in Figure 4. Such an interpretation does not seem to be completely coherent with Al.'s meaning.

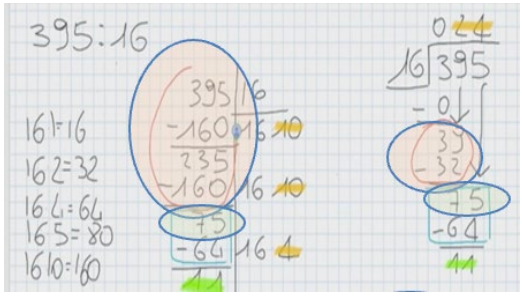


Figure 4. G. highlights the "packages" that she interprets Al. sees as corresponding in the production of the 75s in the two algorithms

G.'s focalization action includes a strong interpretation of Al.'s sign: while the student's intervention refers to the two 75s "coming from different sums/subtractions", G. uses oral and graphical marks to bring the class' attention to "39-32" in the DMSB and to longer sequence in the Canadian division, adding the word "little package(s)" in correspondence to her signs (Figure 4). Moreover, Figure 4, together with Al. and G.'s verbal utterances, illustrates how different components of the semiotic bundle are produced and constantly re-elaborated by the actors, in this case, the student and the teacher, G.. This re-elaboration of the signs produced by the student seems to go far beyond his intended meanings: indeed F. will then spend most of the lesson trying to bridge the gap between the meanings at play, created by G.'s misinterpretation. In F.'s physical classroom, even when the class was co-conducted with G., we have not hardly ever witnessed such a discrepancy between a student's intended meanings and those in the teacher's interpretation of the shared signs. I expect this to be the case because, in the physical classroom, signs are elaborated more gradually by other students as opposed to by the teacher.

Using Speech through the Microphone Uncoordinated with Actions with any Other Tools

A standard practice in F.'s class is to have students "ask each other" when they do not understand something. During the online lesson, F. tries to use the same practice, but mediating students' interactions much more, because of her rules on taking turns speaking by turning on the microphone, only when asked. Towards the end of the discussion, F. uses such a practice to ask Jia. to explain to Gem. something that had been shared by other students: why in the DMSB there is a "39-32" that gives 75 (not 7). This is a kind of ask for a synthesis action. F. rapidly follows up on such an action providing a synthesis, herself, in which she further elaborates on the signs produced by the class.

Jia.: Instead, the 39 is simply without the 5 that then in brought down to the 7.

F.: Jia. is saying that 39 is actually a 395,... the 5 stayed up top.

F.: Then, instead she says that that 32 isn't in fact exactly a 32, but it's 320, there is a zero hiding away a bit there. And that 320 is exactly the sum of the two 160[s] instead in the Canadian [algorithm].

Focusing on the signs in this excerpt, there are at least two sets of signs that are put in relation with one another: one that includes the "bring down" in the DMSB, and one including "hidden" digits (these two sets are highlighted graphically in Figure 5). As shown previously, here, too, different components of the sets of signs are produced and managed across the actors: Jia., like her classmates who spoke before her, explains only orally, without the possibility of marking the board or even pointing to signs on the board; F. repeats and rephrases some of the words uttered by Jia., but because she is not in control of the Jamboard, her signs are also expressed in a single modality, though they refer to numbers written on the board.

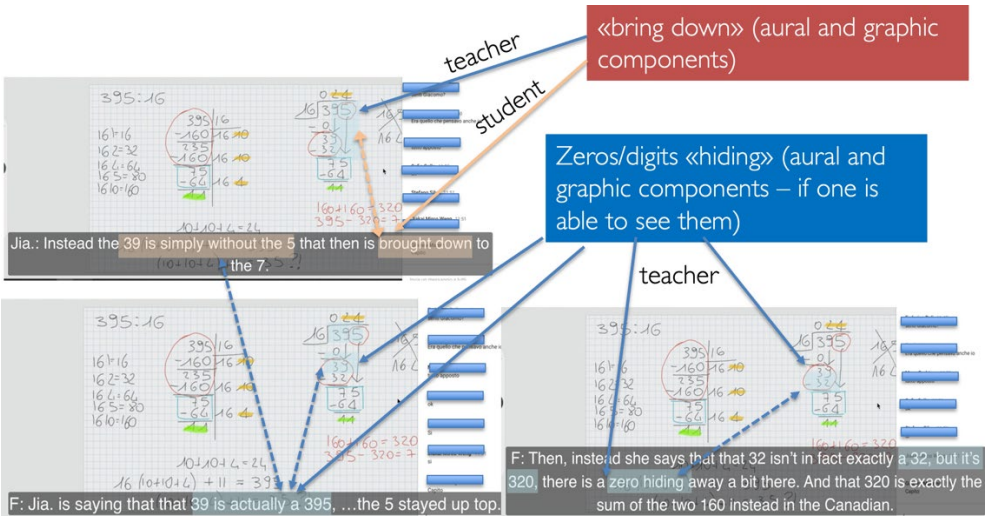


Figure 5. Two sets of signs co-produced by the class and the teachers

Unlike what she did with students, G., is not supporting F.'s synthesis with any online gesturing. Therefore, in F.'s synthesis the connections between visual and verbal components of the sets of signs are left implicit. I tried to highlight some of these connections with the additional markings I made in Figure 5. This surely shows that the ways in which signs are produced and shared on online discussions like this one is quite different than those that take place in physical classrooms. Possibly, a lack of explicit connections, like in the case shown above, can be stimulating for some students, who in this way are not shown everything, but they need to actively follow others' words to visually navigate the page. However, this set up may be problematic for some students, as it surely creates a heavier cognitive burden for them to carry. At the moment, I do not have enough data on the effects of this sort of online teaching practice to discuss it any further.

Using the Microphone for a Vocal Check In

Another interesting technique used frequently by G. and F. is a sort of grunt they emit orally while a student is speaking. The main objective seems to be to let students know they are being listened to and understood. In other words, this "vocal check-in" seems to satisfy a *phatic function*, the means by which two or more speakers reassure themselves that not only are they being listened to, but they are also being understood. The frequent vocal check-ins that occur online seem to substitute for other means through which the phatic function is carried out in physical classrooms.

OVERVIEW OF FINDINGS

The tools used in Meet were the chat, the microphone, the video camera, and, in and Jamboard, the laser highlight, the colored pens and the highlighters. Table 1 presents the main communication techniques, the visible parts of the schemes, the teachers used to communicate with these tools. I will now describe the main potentials and limitations that seemed to be experienced by the teachers and the students, and to mostly influence the emergence and elaboration of signs.

The chat offered the potential to "see behind" students' "raised hands". Indeed, in this online lesson, when students were asked a question, or when they wanted to say something in support or against a classmate's statement, they would signal their desire to intervene by writing in the chat. Scrolling down through the chat, the teachers could quickly get an idea of who was thinking what; perhaps this allowed them to identify relevant signs to pick out and share more rapidly than in the physical

classroom. However, a limitation of having students quickly type their signs into the chat is that the signs emerged in a single modality (written verbal), and usually in a very concise and cryptic form.

The fact that the students could activate the microphone only without the video camera and without having direct access to the Jamboard was another limitation that forced students to share signs in a single modality (aural), when called on. However, this also led some students to attempt to clarify their signs and reasoning through a sort of aural enaction of pointing gestures, giving names to sets of marks on the board.

Writing or highlighting on the Jamboard has the potential of coordinating speech and gestures, like in a physical context, allowing the simultaneous production of sets of signs with graphical and aural components that are immediately accessible to the listeners. However, the fact that, as a class norm, the teacher did not allow students to act on the Jamboard caused the following asymmetry in the management of the written signs: the students could only see them, while the teachers (actually, only G.) could highlight and modify them. This asymmetry forced the students to verbally interpret the written signs, on the one hand, and, on the other hand, it led the teacher to translating them into a mixed verbal and graphical set of signs. The teachers were constantly active (more than in the physical classroom) interpreting the students' signs, guessing at the meanings evoked by the students, adding components and enriching the semiotic bundles, which led to a relatively large discrepancy between Al.'s intended meanings and those in the G.'s interpretation of the shared signs (Figure 4).

technique	*goal*
Open and scroll chat	Obtain an overview of students' participation and search for interesting signs for discussion
Write in chat	Parallel communication between students or between students and the teacher to organize speaking turns, or just to "stay in touch"
Write or highlight in Jamboard	Add and coordinate visual parts of signs in student's speech, to help class follow/focus and show speaker s/he is "understood"
Use mic. for vocal "check-in"	Let students know they are being listened to (phatic function)
Use mic. to call on "invisible students"	Increase participation of students (especially those expected to struggle)
Use mic. to aurally enact a pointing gesture	Get listeners' attention to go somewhere specific on screen

Table 1. Main communication techniques implemented in Jamboard and Google meet to communicate

In general, the analyses also suggested that focalization and provide a synthesis actions prevailed over the others. Frequently right after asking for a synthesis, online the teacher would quickly shift to providing a synthesis herself, while in her physical classroom, F. would carry out this action less frequently. This behavior could be due to a feeling of lack of control over her class that F. experienced online, or simply to the difficulty of bearing silence in the online situation.

DISCUSSION AND CONCLUSION

This study suggests that during the lockdown in Italy, even in a class whose teacher belongs to a research action group and who continued to collaborate with researchers (indeed one was even co-teaching with her!), digital technology is still used rather "naively", as a means to transfer online what the teacher usually implements in her physical classroom, without fully taking advantage of its potential. The analyses suggest that changes in how the discussion was conducted online as opposed to in physical classrooms were due to limitations in how signs were co-produced and shared between students and teachers, but probably also to the norms imposed during the crisis situation. Ruthven's decade-old claim seems to still apply:

> These developments have been readily embraced because they provide relatively simple (if often expensive) enhancements to everyday means of communication and resource use, in and beyond the classroom. These technologies are not strongly framed in didactic terms, and have potential to support activity across the didactic spectrum; nevertheless, in practice they are often appropriated to a reproductive didactic. (Ruthven, 2012, p. 629)

So, it seems natural to question the extent to which the online communication was really responsible for these changes? What could have been done to better exploit the potential of communication supported by digital technology and transform (at least some of) these constraints into opportunities?

A few possibly insightful directions to explore come to mind, concerning the chat and, more in general, shared spaces provided by digital technology. Chats could be used both synchronously and asynchronously with respect to the online discussion to foster students' written argumentations. Indeed, the chat tool offers a means of online communication that privileges written text. This tool could be used according to different schemes, based on forms of communication it is supposed to mediate (e.g., teacher to student, student to teacher; students to student, ...).

Shared "spaces" could be provided, such as text documents or boards shared within small groups of students, in order to promote the production of personal signs that can later be shared with the larger group. For example, breakout rooms as "secure" spaces where students can interact more freely without teacher mediation. In a shared space such as Jamboard, the management could be different, for example, allowing students to access it, write on it and produce online gestures around the signs on it, as was done only by G. in the online lesson. In contexts where the teacher prefers not to give students access to the board, such as F.'s, lessons could involve activities designed to foster the development of verbal language, for example asking for predictions and descriptions of the behavior of interactive digital artifacts (e.g., Baccaglini-Frank et al., 2018). This goes back to the first line of development of digital technologies described by Ruthven, concerning their mediational role with respect to specific mathematical content. While there has been lots of research in this direction over the past decades, why have classroom practices lagged so far behind, making teachers and students fall into such difficult situations (the one described is among the "happiest" I have witnessed during the lockdown) when online education became necessary?

A key might be to look in the direction of professional development. What kind of professional development is needed for teachers to be able to feel more comfortable in online teaching contexts? How should professional development courses be designed? These and similar questions have been raised in different countries and communities, who all feel the need for research on online teaching and learning with and through technology (Bakker et al., 2021). I join this chorus.

NOTES

1. At the Mathematics Department of the University of Pisa the research-action group "Gruppo di Ricerca e Sperimentazione in Didattica della Matematica" (GRSDM) directed by Anna Baccaglini-Frank and Pietro Di Martino has continued to meet online throughout the pandemic.

2. The conceptualization of algorithms as artifacts, and the didactic potential of their synergy are described in Baccaglini-Frank, et al. (2021).

3. This algorithm owes its Italian name to the collaboration between the research group coordinated by Paolo Boero and a Canadian group of teachers.

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DOES THE GENDER MATTER? THE USE OF A DIGITAL TEXTBOOK COMPARED TO PRINTED MATERIALS

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We consider integrated interactive digital tools and interactive tasks as typical potentials of digital mathematics textbooks that distinguish them from printed textbooks. In this research project, the impact on students' achievement compared to analogue printed materials is of particular interest. So far, previous studies have indicated different results between male and female students regarding the learner uptake and benefit supported by the materials. For example, the PISA study showed that boys scored higher in mathematics, while other studies show that girls benefit more from the interactive materials. In the context of the use of a digital textbook, the KomNetMath project is investigating in a pretest-posttest design whether male and female students show a different achievement development with or without a digital textbook. The results show advantages for the female students when using digital materials.

Keywords: Achievement, digital textbook, digital tools, gender, influence of technology.

LEARNING WITH A DIGITAL TEXTBOOK

The latest digital mathematics textbooks integrate features and digital tools that distinguish them from printed textbooks and traditional e-books (Pepin et al., 2015). These variations represent the potential of digital mathematics textbooks. For example, digital mathematics textbooks may include interactive tasks, multimedia elements, opportunities for assessment, personalisation or communication (Choppin et al., 2014; Pepin et al., 2015; Rezat, 2021). A digital textbook should therefore differ from a printed textbook in having dynamic structural elements in addition to static structural elements (Pohl & Schacht, 2019).

These fundamental differences in the structure of digital and printed textbooks are of particular relevance for an investigation into the influence on learning with digital textbooks. This is because the learning opportunities offered by a textbook are shaped by its structure and content (Pepin et al., 2015). The learning opportunities and, thus, the particular textbook have an impact on the organisation, more precisely on the "how" and "what" of the teaching and learning process (Chazan & Yerushalmy, 2014; Pepin et al., 2015; Sievert et al., 2021). This means that the quality of the textbook on a topic can affect the time spent on that topic and thereby the classroom activities. Subsequently, there is research indicating that students' performance correlates with the learning opportunities provided by a textbook (Pepin et al., 2015; Sievert et al., 2021).

The textbook can be classified as an artefact that has a translation function from the intended curriculum to the implemented curriculum, i.e., teachers' actions (Valverde et al., 2002). Although the textbook is the most important resource and the guiding medium for teachers (Fan et al., 2013; Pepin et al., 2015; Valverde et al., 2002), there are very few studies investigating the use of a textbook and its impact on student learning and achievement (Fan et al., 2013). So far, often studies have rather compared different textbooks and neglected the actual usage (Fan et al., 2013). The first studies comparing digital and analogue learning materials regarding the effects on achievement were able to show the first positive effects of the digital materials (Radović et al., 2020; Reinhold et al., 2020). In the study of Reinhold et al. (2020), it was shown that especially low-achieving students could benefit from the digital materials when learning fractions. This was attributed to interactive and adaptive

scaffolds in particular, e.g. individual feedback, which should help to reduce the cognitive load when dealing with the digital materials. Radović et al. (2020) particularly highlight the potential and influence of embedded GeoGebra applets. In addition to a positive influence on students' knowledge, they observed a positive impact on students' perception of learning and their motivation. These findings are in line with the results of a meta-analysis of quantitative studies, which showed that the use of digital tools has positive effects on learning success (Hillmayr et al., 2020).

GENDER DIFFERENCES IN MATHEMATICAL ACHIEVEMENT AND THE USE OF DIGITAL TOOLS AND MATERIALS

Various studies have already identified gender differences between male and female students in terms of their mathematics achievement. For example, in the Programme for International Student Assessment (PISA) study, 15-year-old boys show significantly better results than girls in mathematics competencies in Germany and across all of the Organisation for Economic Co-operation and Development (OECD) countries (OECD, 2019; Reinhold et al., 2019). In Germany, similar results in favour of male students were found in the second national assessment of mathematics and science proficiencies at the end of ninth grade (Schipolowski et al., 2018). The differences between boys and girls were particularly large in the core theme *data and chance* (Schipolowski et al., 2018).

This gender gap is often attributed to psychological factors (Perez Mejias et al., 2021), such as a lower self-image and lower self-efficacy on mathematics and the use of digital media among female students (Fraillon et al., 2014; Gerick et al., 2019; Perez Mejias et al., 2021). Especially when using digital textbooks and digital tools, this can be a decisive factor, as the use demands certain challenges from the users (Rezat, 2021). In particular, when it comes to advanced skills in using digital media and the use of computers, higher self-efficacy of male students has already been identified (Cassidy & Eachus, 2002; Fraillon et al., 2014; Gerick et al., 2019). This does not suggest that boys' computer-related or digital skills are also higher. For example, the ICILS study found that girls performed better in this respect (Fraillon et al., 2014; Gerick et al., 2019). Overall, there often are different and contradictory results on gender disparities, which may also depend on the methodological conduct of the study and on social and cultural reasons reflected, for example, in stereotypes or teacher behaviour (Reinhold et al., 2019).

There also are different results so far as to whether the use of an interactive textbook differs between male and female students, e.g., with regard to the way the integrated elements are used (Hoch, 2020). Regarding the performance in the use of interactive materials, the previously mentioned studies conducted by Reinhold et al. (2020) and Radović et al. (2020) found differences in the performance of male and female students. In the study of Radović et al. (2020), females performed better than boys in the retention test. Reinhold et al. (2020) found that the girls outperformed the boys only among high-achieving students.

RESEARCH QUESTION

The digital textbook provides new opportunities for teaching and learning through its potentials and the integration of digital tools (Chazan et al., 2014; Pepin et al., 2014). The first study results show that students can benefit from digital materials and tools under certain conditions (e.g., Radović et al., 2020; Reinhold et al., 2020). Whether there are differences between male and female students in terms of their use of digital media and tools and the subsequent performance has not yet been definitively clarified, although they often differ in their mathematical competences in various studies (e.g., OECD, 2019; Reinhold et al., 2019; Schipolowski et al., 2018). This research gap is addressed in this study. Thus, the aim of this study is that a differing achievement development can only

attributable to differences between digital and analogue materials in a controlled series of lessons. This is why this study compares the textbook in a digital and an analogue printed version, but with the same content. The topic is of particular relevance because the textbook is considered the most important resource in teaching mathematics (Fan et al., 2013; Pepin et al., 2015; Valverde et al., 2002) and may enable learning not only with but also through technology due to the newly recognised potentials. In summary, the following research question results:

What impact on achievement does the use of the digital textbook have on learner achievement in comparison to the use of analogue printed materials in terms of gender in a teaching series of the theme *data and chance*?

METHOD

Sample and Data Collection

In this study, 93 male students and 86 female students from nine mathematics courses from German secondary schools participated. The students attended grade 10 of the German *Gymnasium* and grade 11 of the German *Gesamtschule* (same level in different school tracks). One person assigned to the gender "diverse". Since this was only one individual, this person is excluded from further statistical analyses. The participating students had a mean age of $M = 15.62$ ($SD = .77$).

The data was collected as part of a series of five lessons on conditional probability in the school year 2020/21. Students completed a test before and after the series of lessons to measure student achievement. The test was developed for this study (see section Design and test instrument). Due to the COVID-19 pandemic, it was not possible to realise a follow-up test under the same conditions in the participating courses.

As part of the *KomNetMath* project, the students and teachers are provided with the digital mathematics textbook *Net-Mathebuch* for a complete school year. Therefore, the students and teachers are already familiar with the digital textbook at the time of data collection and have thus instrumentalised it. In addition, the participating teachers receive regular advanced training on the use of the digital textbook in mathematics lessons.

Material

The digital mathematics textbook called *Net-Mathebuch* (www.m2.net-schulbuch.de) is a freely available digital textbook in Germany. The digital textbook can be used on any device. It covers the contents of the upper secondary school and is not based on a printed original. The digital textbook is structured in chapters and sections, allowing users to choose their own tasks and topics. It is characterised by being a purely digital concept and by providing digital tools throughout the entire textbook, for example, through the integration of *GeoGebra*. The textbook takes up many of the potentials of a digital textbook described by Choppin et al. (2014). For example, there are interactive tasks (e.g., including GeoGebra applets) or feedback options for entering solutions.

For this study, students used the chapter on conditional probabilities with its interactive features in the *Net-Mathebuch* (see Figure 1). The control group used a printed version created for this study. In this version, the chapter with its digital interactive features was adapted. For example, drop-down hints became equivalent printed hint cards and dynamic drag-and-drop tasks became static matching tasks.

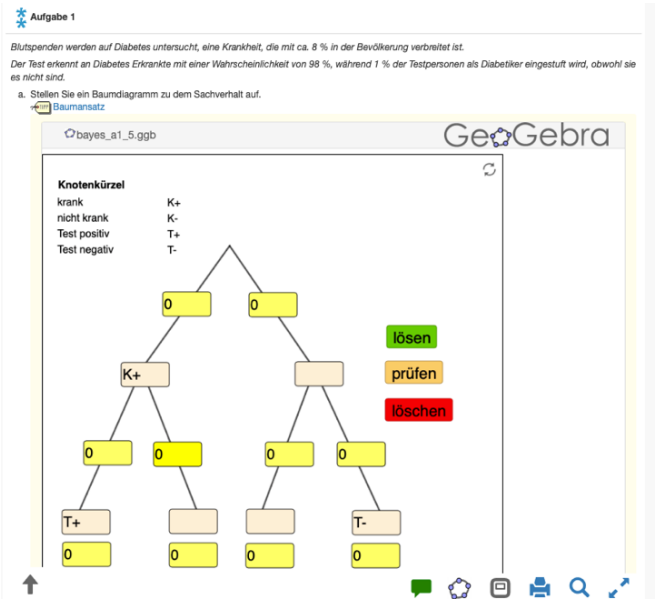

Figure 1. Sample task with feedback options in the chapter on conditional probabilities in the digital textbook *Net-Mathebuch* (www.m2.net-schulbuch.de)

Design and Test Instrument

In this study, a teaching series of five lessons on conditional probabilities was conducted. Each participating course was divided into an experimental and a control group based on the pretest (see Figure 2). The groups have been divided in such a way that two mixed-ability groups were created in which female and male students are equally represented. In the experimental group, the students worked with the digital mathematics textbook Net-Mathebuch. In contrast, the students in the control group used the printed adaptation of the Net-Mathebuch. The teacher of one course taught the same content in both study groups following predefined plans, so that the influence of the teacher on the results of the intervention is minimised. Thus, the students in both groups always work on the same tasks.

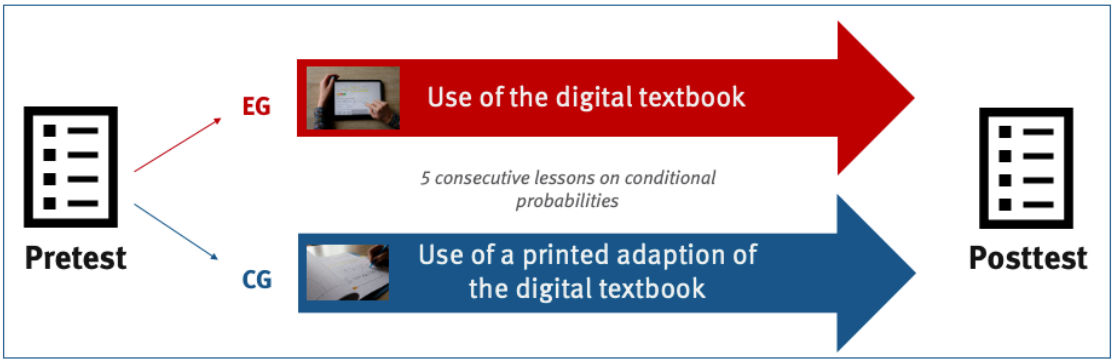

Figure 2. The pretest-posttest design of the study

The pretest and posttest were developed for this study. In total, these two tests contain 21 items. The items have a closed or semi-open response format. The students' answers in the tests were coded dichotomously. Both tests ask about the contents of the lesson series, whereby the pretest, in particular, contains items on prior knowledge in stochastics. Using the one-parameter Rasch model, the data were scaled with the software *Conquest* and item- and person-parameters were estimated

with a one-dimensional model (Rasch, 1960/1980; Wu et al., 2007). According to PISA (OECD, 2012) and Bond and Fox (2007), the weighted mean square fit (WMNSQ) as an index of the item fit for a very good test should be between 0.8 and 1.2. In this test, the values are between 0.93 and 1.10. The EAP/PV-reliability is 0.54 and is sufficient for group comparisons (Lienert & Raatz, 1998). With the person parameters obtained, methods of classical test theory are used in the analysis of the groups.

RESULTS

In the group that used the digital mathematics textbook (see the experimental group in Figure 3), t-tests show different results for male and female students.

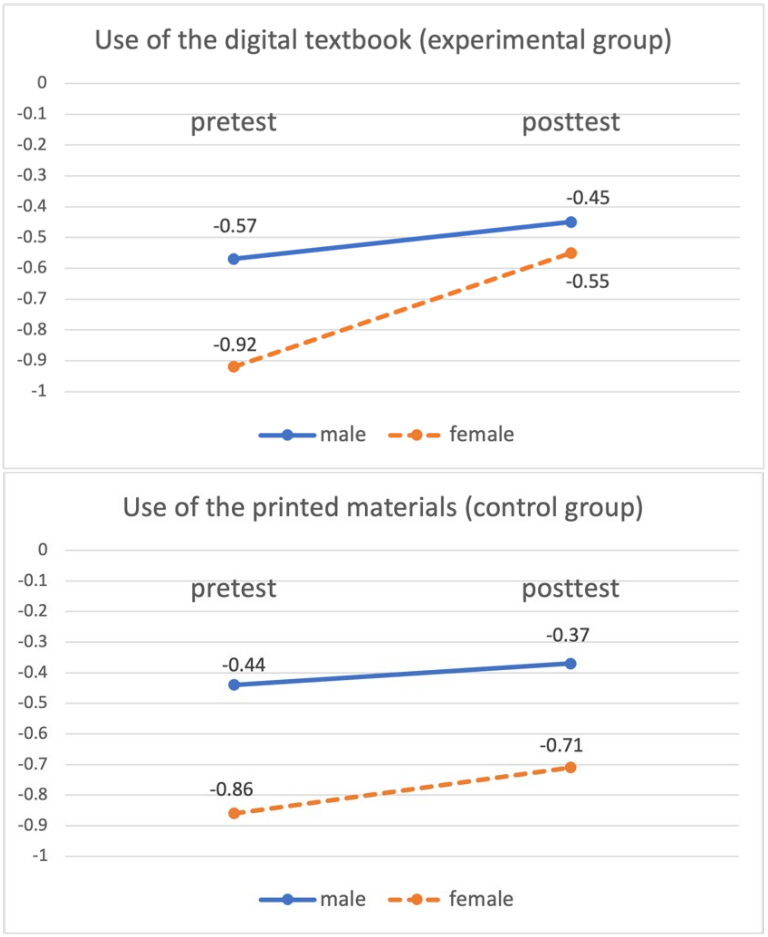

Figure 3. Results of the pretest and posttest in the experimental group and in the control group

The female students (N_{fd} = 40) increased significantly with $t(39) = -2.53$, $p = .016$ and the effect size Cohen's d = .40 represents a small effect. For the male students (N_{md} = 49), no significant effect is depicted with $t(48) = -0.84$, $p = .407$. The results of a two-way repeated measures ANOVA show no significant interaction between the two measurement time points and gender ($F(1, 87) = 0.67$, $p = 0.227$, partial $\eta^2 = .06$). The within-subject factor gender also reveals no significant results ($F(1, 87) = 1.41$, $p = .239$, partial $\eta^2 = .02$). In the control group (see Figure 3), the results for the male students (N_{mp} = 44, $t(43) = -0.43$, $p = .670$) and for the female students (N_{fp} = 46, $t(45) = -1.08$, $p = .285$) do not increase significantly. A two-way repeated measures ANOVA also shows no significant results for the interaction between time of measurement and gender ($F(1, 88) = 0.12$, $p = .732$, partial $\eta^2 = .001$). In contrast, the within-subject factor gender is significant for this control

group ($F(1, 88) = 5.17$, $p = .003$, partial $\eta^2 = .06$). In addition, the achievement of male and female students in both groups already differ significantly at the pretest, $t(177) = -2.54$, $p = .012$).

DISCUSSION

As conducted in the PISA study, male and female students differ from each other in terms of their mathematical performance (OECD, 2019; Reinhold et al., 2019). In this study, the male students already have significantly higher scores in the pretest compared to the female participants. In the control group, male and female students develop comparably, and these differences, in the beginning, persist across both measurement time points when using analogue printed materials. In contrast, in the experimental group, there was no significant difference regarding the genders implied. This finding could be related to the fact that, in this group, the female students improve significantly in contrast to the male students. It is also the only one of the four subgroups to improve significantly. This is consistent with the findings that girls, in particular, can benefit from interactive digital materials (Radović et al., 2020; Reinhold et al., 2020). Hence, this assumption can also be made in this study, as the experimental and control groups only differed in the materials they were given. The same lessons were always held in both groups according to prescribed plans. In addition, the same teacher always taught both comparison groups so that the results are not dependent on the individual teacher and thus on previous experience.

However, since the interaction of gender and measurement time points does not show any significant results, further analyses on gender differences are necessary. For example, it could be examined whether girls were able to answer certain items correctly more often than boys. Further analyses could therefore investigate whether there are differences between items that are interconnected more than other items to the interactive elements used from the book. In this way, it could be analysed whether multidimensional Rasch models are more fitting for the evaluation. It can also be debated whether dichotomous coding fully represents students' solutions. Furthermore, in this study there was no insight into the use of the digital textbook and the printed materials. In Hoch's study (2020), for example, process data showed that girls used the embedded widgets more frequently and then solved the interactive tasks more often in the tests.

Psychological factors have been presented as possible reasons for differences between male and female students. For example, different self-efficacy of boys and girls regarding the use of the digital textbook could be a contributing factor (Fraillon et al., 2014; Gerick et al., 2019; Perez Mejias et al., 2021). Therefore, in this *KomNetMath* project, attitudes and self-efficacy regarding the use of the digital textbook are additionally surveyed (Brnic & Greefrath, 2020). For this purpose, the participating students fill out corresponding questionnaires at the beginning, in the middle and at the end of a school year. The upcoming results of these questionnaires can then be linked to the results of the intervention and analysed. In this way, conclusions can be drawn as to whether there are connections between these psychological factors and the achievement of the genders.

The question arises to what extent the results depend on the chosen topic. This is because it is precisely for the area of *data and chance* that the greatest differences between boys and girls were found in Germany (Schipolowski et al., 2018). On the other hand, this core theme is most suitable for an investigation of whether gender differences in achievement can be minimised with the use of digital materials. Thus, the chosen topic may be one reason why differences between boys and girls can already be seen at the pretest. But since the students had already worked with the digital textbook before the intervention, this seems to be contradictory to the results presented here that girls particularly benefit from the use of the digital textbook. This suggests that other factors also play a role, e.g. the interaction between student and teacher, the actual use of the textbook in class initiated

by the teacher, or the influence of stereotypes that is often cited in the literature (Gerick et al., 2019; Perez Mejias et al., 2021). Through the choice of study design, such factors could be partially minimised so that girls seem to benefit particularly from the digital learning environment. This could be attributed to research indicating that girls have higher computer-related and digital skills than boys, as using a digital textbook has particular challenges (Fraillon et al., 2014; Gerick et al., 2019; Rezat, 2021). In order to confirm the results of the survey, the series of lessons will be conducted with further mathematics courses. This study already gives first indications that differences between the genders can be minimised through the use of a digital textbook with its integrated tools and features.

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MATHEMATICAL THINKING AND SOCIAL CONSTRUCTION OF MEANINGS IN DISTANCE CONTEXTS: THE ROLE OF THE TEACHER

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This paper examines the role of the teacher in fostering students to think mathematically and to socially construct mathematical meanings in distance teaching contexts. Findings from two teaching activities, carried out during the pandemic, are presented and discussed in order to contribute to the discussion on whether and how teachers can foster students' learning when teaching mathematics with technology in distance contexts. The first activity involved twenty-three 9^{th}-grade students in a dice situation aimed at recognising regularities, conjecturing and verifying, communicating online with peers. The second activity was carried out online with a small group of 12^{th}-grade students with the aim to socially construct the meaning of rotation with its main properties through the use of GeoGebra. The analysis of the teacher's intervention during the discussions shows that the teacher's ability in orchestrating the activities allows students to develop their learning.

Keywords: Mathematical thinking, social construction of mathematical meanings, role of the teacher, teacher's intervention, distance teaching contexts.

INTRODUCTION

Many studies in mathematics education show activities in which technologies are used to foster: the observation, study, and experimentation of patterns—systematic attempts in order to determine the nature or principles of regularities (Arzarello, 2016); the production of conjectures and the need to verify them (Swidan & Faggiano, 2021); peer communication and the shared construction of meanings (Bartolini Bussi & Mariotti, 2008; Faggiano et al., 2018). This kind of technology integration can foster the development of mathematical thinking and the social construction of mathematical meanings. Accordingly, as far as the gap of the physical distance is bridged by remote teaching, one of the main challenges in pandemic education is the design and orchestration of activities that cannot fail to effectively integrate technological tools in distance contexts. When teachers are aware of the potential usefulness and effectiveness of technological tools as a pedagogical resource, they can move their attention to re-think teaching practices (Maschietto & Trouche, 2010) with the aim to guide students in thinking mathematically and socially constructing mathematical meanings. Hence, the role of the teacher is fundamental in order to foster students' learning when teaching mathematics with technology (Clark-Wilson et al., 2014), within the mathematical systems or through models of real-world objects.

This paper attempts to contribute to the reflection on remote teaching due to the pandemic, showing that the development of mathematical thinking and the social construction of mathematical meanings can be promoted by the teacher's orchestration of the activities. In particular, the research question we aim to answer is: how can the teacher's orchestration of the activities in technology-rich mathematical contexts foster students' learning? To do that, we focus on the teacher's intervention during the discussions in two case studies developed in distance contexts. Findings show that the students' development of mathematical thinking and the social construction of meanings in remote teaching can be promoted by the teacher's questioning and revoicing, likewise it happens in a traditional educational context.

THEORETICAL FRAMEWORK

A substantial amount of research has proved that the use of technology may allow teachers to create suitable learning environments, with the goal of learning to think mathematically and promoting the construction of meanings for mathematical objects (e.g., Noss & Hoyles, 1996).

According with Schoenfeld (1992), learning to think mathematically means:

> developing mathematical point of view, valuing the processes of mathematisation and abstraction, developing competence with the tools of the trade, using those tools with the goal of understanding structures and mathematical sense-making. (p. 334)

Many studies also deal with the social construction of mathematical meanings (Bishop, 1985) as a process through which the learning of mathematics is developed with understanding –extending and applying previous knowledge to new problems, reflecting about experience and constructing relationship– by means of different kind of activities (e.g., Arzarello, 2016; Swidan & Faggiano, 2021).

Our claim is that effective distance education cannot fail to take this into account: students have to be guided in learning to think mathematically and socially constructing mathematical meanings. The role of the teacher, hence, becomes fundamental in exploiting the affordances of technology in fostering students' learning: teachers can prompt students' mathematical thinking through their orchestration of the activities in technology-rich contexts in the same way as they do it in traditional classroom teaching. For many years, in particular, researchers have highlighted the importance of teachers' questioning (e.g., Wood, 1998; Herbel-Eisenmann & Breyfogle, 2005) and revoicing (Forman et al., 1998; Enyedy et al., 2008), in guiding the students to the desired end, to perform a certain procedure for solving problems, or to facilitate students debate and mathematical argumentations, thus developing a deeper conceptual understanding of mathematics. Carefully listening to the students' responses, the teacher can guide students with questions in the form of "why" and "what do we have?", to focus on their own understanding and to reach the desired solution. With revoicing, which involves the re-uttering of another person's speech through repetition, rephrasing, expansion, and reporting, the teacher tries to share and eventually emphasise one student's idea with the others in the classroom and to add knowledge to what was said. According to this, we believe that through questioning and revoicing, teachers can help students to learn to think mathematically and to socially construct mathematical meanings also in distance teaching contexts. For this reason, in this work we mostly refer to teachers' questioning and revoicing during the collective discussions conducted in the remote teaching platform.

METHODOLOGY

In order to answer our research question, in the next two sections, we present and discuss two teaching activities carried out during the pandemic: the first involved twenty-three 9^{th}-grade students in a dice situation aimed at recognising regularities, conjecturing and verifying, communicating online with peers; the second was carried out online with a small group of 12^{th}-grade students with the aim to socially construct the meaning of rotation in the plane with its main properties through the use of GeoGebra. The two examples are not meant to be compared. Rather, they have been chosen to show that even in the distance contexts caused by the pandemic, the teachers in these two examples succeed in fostering students' learning through the use of the remote teaching platform and, for the second case, also of GeoGebra. We video-recorded the sessions in their entirety including the group work, in which students worked together to solve the tasks, and the general discussion led by the teacher. The videos were transcribed and analysed, together with the students' protocols, focusing on the role

of the teacher in helping students to think mathematically and to socially construct mathematical meanings. In this paper we report on some significant episodes revealing the teachers' orchestration and, in particular, their use of questioning and revoicing.

THINKING MATHEMATICALLY IN A DICE SITUATION

The main general aim of this activity was to help students in learning to think mathematically, and in particular to give meaning to the algebraic language. Through the formulation of appropriate questions, the students behaved like researchers: they investigated the problem under consideration, collected data and formulated hypotheses and conjectures, at the same time feeling the need to argue their own ideas and to discuss them with their peers. For the purpose of this paper, herein we focus on the first part of the activity, which started with group work. Students were asked to think about the arrangement of six white dice and ten grey dice in a specific given 4x4 configuration (Figure 1a) in which the numbers (A and B) on all the dice of each of the two groups were the same, and the difference between them was fixed. Then, the students were asked to observe how the total sum changes if A and B are changed, keeping their difference fixed at two (Figure 1b).

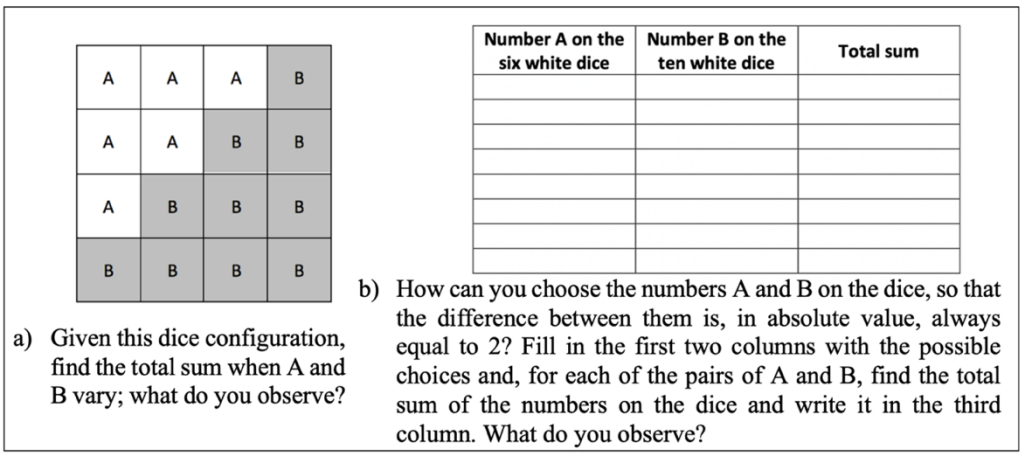


Figure 1. English translation of the tasks given to the students

At the end of the group work, a collective discussion was conducted by the teacher sharing her screen. First of all, she showed the empty table in Figure 1b and asked the students to help her to fill it in (Figure 2a). The order of the pairs was given by the students.

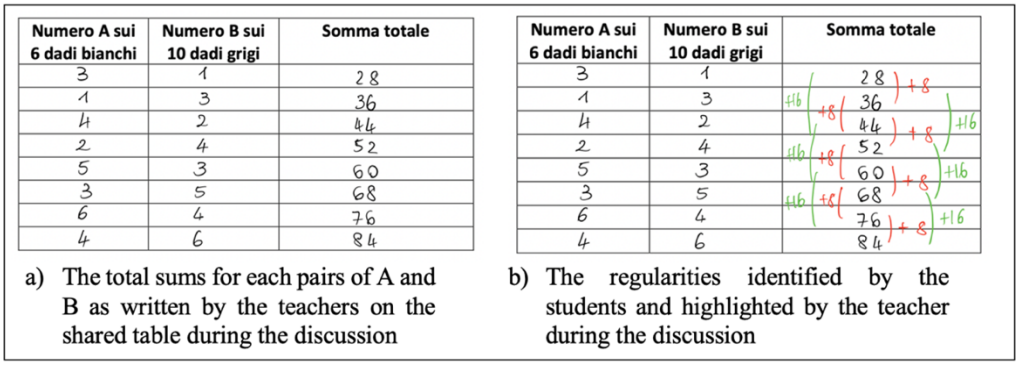


Figure 2. Tables written and shared by the teacher

The following excerpts of the discussion refer to the students' explanations of the observed regularities in the total sums:

1 Teacher: *What do we observe* on the total sums as the numbers A and B vary?

2 Vittorio: We have noticed that the sums of the opposite pairs always have 8 as a difference... That is, for example, (1,3) and (3,1) have sums that have a difference of 8.

3 Caterina: We have noticed that, for example, the pairs (3,1) and (4,2) always have 16 as the difference of the sum ... That is, the difference between the sum of the pair (4,2) and that of the pair (3,1) is equal to 16.

The observations made by the students were reported by the teacher in the shared table (Figure 2b), using different colours to highlight the regularities that they found. Students justified the reason why the difference between the sums of the pairs (3,1) and (4,2) is 16 referring to the fact that, when changing the six white dice from 3 to 4 and the ten grey dice from 1 to 2, the result is that the numbers on all the sixteen dice have been increased by 1. The discussion then moved to give reason to the other observed regularity.

4 Teacher: Ok, then ... Let's try to understand, *why is the difference 8*?

5 Caterina: I believe it is because the difference between 1 and 3 is 2, and since the dice are six and ten, then the difference between 6 and 10 is 4 ... So, swapping the four dice is why a pair has the greater sum of 8, and the other is less than 8.

6 Teacher: *You told me* the difference between 1 and 3 is 2, instead the difference between 6 and 10 is 4… *What should I do*?

7 Caterina: Precisely since it is as if four dice were being exchanged, a pair will have a sum that will be a number greater than 8 ... Because since the difference between 1 and 3 is equal to 2 and the dice that are exchanged are four, it is 8.

8 Teacher: So, *you are telling me that* we are swapping pairs and that when I swap them there are four dice left over, *so what should I do*?

9 Caterina: Precisely, I have to multiply by 2, because it is the difference between the numbers on the dice; that is, it is the difference between 3 and 1.

10 Teacher: *Times 2* ... And then becomes 8.

11 Giorgio: Yes, it seems to me a correct reasoning ...

The teacher then asked if the justification for the difference 8 could be given in another way.

12 Giorgio: We could write A=B+2.

13 Teacher: Ok… But how did you write the sum before?

14 Giorgio: Yes, we can write (B+2)*6+B*10… then B*6+12+B*10… and so becomes 16B+12.

15 Teacher: In which pairs did you find the difference 8?

16 Giorgio: In general, between the opposite ones, that is (3,1) and (1,3).

17 Teacher: But what does opposite mean?

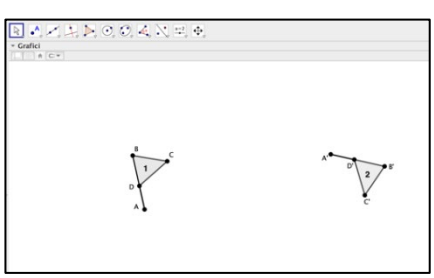
18 Giorgio: Yes, opposite… Those in which first A is greater than 2 with respect to B and then those in which B is greater than 2 with respect to A.

19 Teacher: So, in the sum formula, what should I exchange?

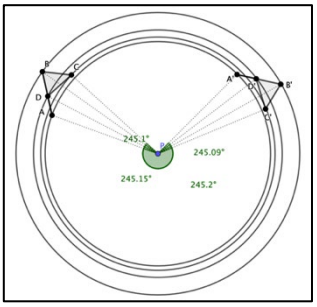
20 Giorgio: Ah, maybe it becomes B*6+A*10… we have 6B+(B+2)*10… So 6B+10B+20… That is 16B+20… We make S2-S1 and then becomes 8… Yes!

SOCIALLY CONSTRUCTING THE MATHEMATICAL MEANING OF ROTATION

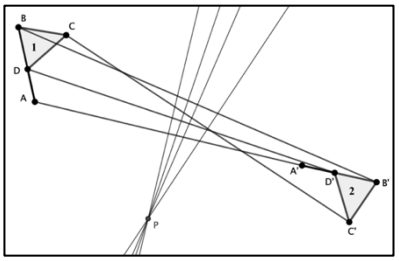
The activity described in this second case aimed at fostering students' social construction of the mathematical meaning of rotation as an isometric transformation, characterised by specific properties. It was composed by a sequence of six tasks involving manipulatives and digital tools (Faggiano & Mennuni, 2020). For the purpose of this paper, herein we focus on the fourth task (Figure 3a). It required the use of GeoGebra and aimed at drawing students' attention to the centre of the rotation as the unique point at the same distance from each pair of corresponding points of the figures. That is, the centre of rotation can be found as the intersection point of the perpendicular bisectors of any two segments joining a point of one figure to the corresponding point of the other figure. Students were given two congruent figures and they were asked to: find the centre of the rotation that transforms one figure into the other; explain how the point has been identified.



a) The task with GeoGebra: find the centre of rotation, given the two rotated figures



b) Valentina's construction as it was shared and explained during the discussion



c) The construction to find the centre as it was built by Valentina during the discussion

Figure 3. English translation of the task given to the students (a) and two Valentina's screenshots, (b) and (c), taken by the video recording of the final discussion led by the teacher

At the beginning of the discussion, none of the students has correctly identified the hidden centre of the rotation, which transformed one figure into the other. One of them (Valentina) explained that she firstly pointed on a point in the GeoGebra plane that seemed to be the centre and then verified if the properties were still satisfied (Figure 3b): she drew the circumferences with centre in the hypothetic centre of rotation, passing through the main points of the first figure, to verify if the corresponding points of the second figure belong to the relative circumferences. It was when she checked if the obtained angles of rotation were all equal that the doubts arose. Then, during the discussion, another student (Pietro) explained that he considered the midpoint between A and A', even if he has then realised that it couldn't be the centre of rotation. The following teacher intervention brought Valentina to think about Pietro's words and so to consider the perpendicular bisectors of the segments joining corresponding pairs of points:

21 Teacher: Try to think for a while... *Valentina was saying* something about distances… Last time you started to see some of the properties of rotation, right?... So, try to focus on the properties for a moment and, if you like, read them backwards, because the properties can help you to find the centre.

22 Valentina: We could draw the segments AA', BB', CC' and DD'… And consider the perpendicular bisectors of each of them… [while speaking, she drew the segments and built their perpendicular bisectors]. Then the centre should be…

this point [she pointed on the common intersection points of the four perpendicular bisectors (Figure 3c is the screenshot of her shared construction at this point)].

23 Teacher: *Why* did you choose these four perpendicular bisectors?

24 Valentina: Since Pietro had considered the midpoint, and since we know that one of the properties of rotation is that of the preservation of the distances of the points of the two figures from the centre, I thought that... The geometric locus of the points equidistant from the extremes of the segment AA', as well as for the other segments, it is the perpendicular bisector. So, if we intersect all the perpendicular bisectors, the point we get could be the centre of this rotation.

RESULTS, ANALYSIS AND DISCUSSION

Our findings present some excerpts, taken by discussions developed in distance contexts, chosen as examples of teachers' orchestrations in which questioning and revoicing are particularly evident and were fundamental to help students in learning to think mathematically and in socially constructing mathematical meaning. In both the presented case studies, the teachers' questions of the form "why" and "what do we have?", guided the students towards the goal of the activity. At the same time, this type of questioning allowed the students to focus on their own understanding of the situation at stake in order to express their thoughts, directing them towards the construction of meaning with the continuous discussion with their classmates.

In the first case, the aim of the teacher was to offer students an activity in which it is necessary to critically observe, make conjectures and seek justifications, thus making students act as researchers. This case is interesting to be analysed because it is an example of an activity in which the students succeeded in thinking mathematically in a dice situation thanks to the ability of the teacher to orchestrate the discussion even using a remote teaching platform. Results, indeed, showed how the proposed activity brought students to make hypotheses, giving space to the arguments. A first example of argumentation to justify their hypothesis, concerning the difference between the sum of pairs such as (3,1) and (4,2), can be seen when students referred to the idea to turn all the sixteen dice to the successive number in order to move from the one pair to the other. Moreover, through the formulation of appropriate questions by the teacher, students thought mathematically, and in particular gave meaning to the algebraic language. The teacher shared and emphasised Caterina's idea with the other students, asking questions such as "what do we observe?" ([1]) or "why the difference is 8?" ([4]) and revoicing Caterina's observations ([6], [8] and [10]). This allowed the students to compare their ideas with one another and facilitate their mathematical argumentation. Indeed, the teacher's revoicing also helped the other students (such as Giorgio ([11]), for instance) to focus on the differences between the sums in the case of swapped numbers in the pairs. The mathematical sense given by Caterina to the swapping situation and the successive teacher's question ("how did you write the sum before?") pushed Giorgio to re-think the situation in terms of the relationship between the formula to calculate the sum and the constraint of the fixed difference between A and B ([14] and [20]). In this way, the algebraic language allowed to re-think the dice situation mathematically.

In the second case, being aware of the potential of the digital tool, both in terms of mathematical meanings and in terms of the students' production of personal meanings, the teacher's aim was to lead, with appropriate questions, the evolution of personal meanings towards the mathematical meanings of rotation. During the discussion, the teacher's use of questioning and revoicing was useful to allow students to focus on a particular aspect of the properties of rotation. This allowed the property

of the preservation of the distances from the centre to be exploited to build the correct procedure for determining the centre of rotation. With the first intervention reported in the excerpt ([21]), indeed, the teacher tried to bring to the fore the students' personal meanings in order to lead their evolution towards the mathematical meanings. In fact, she pushed the students to connect Pietro's idea, considering the midpoint of the segments obtained by joining pairs of corresponding points, with the property of the rotation explained by Valentina on the preservation of the distances from the centre. The teacher's intervention of revoicing added some knowledge to what was said: she suggested focusing on the properties they have already gained and to read them backwards, adding that "the properties can help you to find the centre". Thanks to her added observation, the students focused their attention on the previous shared meanings and succeeded in finding the centre of rotation ([22]). Furthermore, the teacher's request to let Valentina share her reasoning was also fundamental for all the other students. It allowed them to generalize the property of equidistance, characterizing the rotation, starting from the idea of having to use the midpoint and arriving at the use of perpendicular bisectors. This prompted Valentina to determine the centre of rotation as the intersection point of any two of them ([24]). Therefore, the teacher's revoicing and questioning intervention resulted crucial in order to guide the students to accomplish the task and brought Valentina, and then also the other students, to realize that they must use the perpendicular bisectors. Finally, the teacher's "why" question ([23]) concerning the role of the perpendicular bisectors helped the students to gain a full comprehension of the properties of the rotation and in particular of the centre.

CONCLUSION

In this paper, we have shown how the teacher's questioning and revoicing can help students in learning to think mathematically and in socially constructing mathematical meanings in technology-rich distance contexts, likewise happening in traditional teaching. To do this, according to the research results concerning teachers' intervention presented in the theoretical framework section, we identified some episodes with "why" and "what do we have" questions and revoicing. These episodes come from two different teaching activities carried out during the pandemic with different teachers and are exemplary to answer our research question. The first teaching activity, which involved twenty-three 9^{th}-grade students, aimed at offering students a situation to be investigated, formulating hypothesis and conjecturing. Thanks to the teacher's questioning and revoicing, the students succeeded in giving meaning to the algebraic language as a mathematical tool to explore the given dice situation, even if the entire activity was developed in a distance context. The aim of the second activity, carried out with 12^{th}-grade students, was to socially construct the meaning of rotation with its main properties. It was based on the use of GeoGebra and developed online. The episodes presented in this case exemplify how the teacher can foster the students' social construction of meanings through an appropriate orchestration and, in particular, again, through the use of "why" and "what do we have" questions and revoicing, even in distance context. Further investigations are needed in order to explore how the affordances of technology-rich environments can be exploited.

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CHILDREN IN MOVEMENT TOWARDS STEAM: CODING AND SHAPES AT KINDERGARTEN

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In this paper, we present a project that involved two classes of kindergarten children and their teachers in a teaching experiment focused on the development of spatial and computational thinking using a simple robot (Blue Bot). We focus on the design of activities, which also make use of playful movement experiences, with the aim of creating an inclusive environment in which children aged three to six can explore mathematically relevant ideas such as length, direction, and shape. We will discuss the initial insights coming from the teaching experiment on how children approached the length of straight paths.

Keywords: Bodily movement, coding, computational thinking, STEAM, spatial thinking.

FRAMING THE STUDY

There is a growing interest in mathematics learning in early childhood (e.g., Clements & Sarama, 2007), but the number of research studies focused on the use of digital tools in mathematics sessions for young children is still limited (Carlsen et al., 2016). Nevertheless, young generations are born in a digital world, and it is likely that pre-schoolers become familiar with digital devices before they are exposed to books (Hopkins et al., 2013). Balanskat and Engelhardt (2015) further highlight that many of today's students will be involved in future developments of technology, which is important for society. Programming skills, therefore, became more and more relevant for the 21st-century skills required for future citizens and came to be integrated into the curricula of many countries as they are related to skills like problem-solving, creativity, and logical thinking, with which learners need to be equipped in the digital world nowadays.

Schools have then increasingly integrated programming into other subjects, like mathematics, a concept that is not completely new as Papert (1980) already proposed to use programming in mathematics education, intending to provide different environments for the learning of mathematics and motivate students to engage with mathematics. Papert, for example, developed a Logo environment that required learners to program a computer to move a little turtle on the screen. Later, Benitti (2012) wrote a literature review in which he analysed the potential of robotics in schools and found that, in the examined studies, robots were useful to understand STEM concepts. Benitti and Spolaôr (2017) further underline that the potential use of robots within the mathematics classroom can be seen as a support tool. A recent review specifically investigates the use of programming in mathematics education for students aged 6 to 16 (Forsström & Kaufmann, 2018).

Concerned with mathematical cognition with very young children, several studies have been investigating early learning of number and number sense using multitouch applications (e.g., Sinclair et al., 2016; Ferrara & Savioli, 2018). In such a digital environment, for example, specific gestures are used to create and manipulate numbers, but children's perceptual and bodily engagement is enriched by auditory and visual responses to touch. The usage of simple robots, like Bee-Bots, was discussed in the field as a manner to foster the development of children's geometrical understanding and to create occasions for early steps into computational thinking through coding activities. Studies show the strength of this technology to work on spatial abilities at the end of kindergarten, with 5-

year-old children (Sabena, 2015), and to work on definitions of simple geometric figures, for example, the rectangle and the square, in primary schools (Bartolini Bussi & Baccaglini-Frank, 2015). What we find intriguing about using this artefact within the mathematics classroom is the way that starting from playing it, children can move their very first steps into 3D explorations in space with their bodies, imitating the robot's movement or comparing their own movement with the robot's. This opens room for processes of understanding and communicating about movement, direction, and path of the robot, and their relationship with spatial thinking and shape in mathematics.

In this paper, we want to contribute to this line of research about robotics and mathematics with very young learners by presenting insights from a teaching experiment designed and carried out as part of the project "Children in Movement towards STEAM", whose target are children aged 3 to 6 and their families. The intervention aimed primarily at introducing kindergarten children to coding and mathematical thinking, as a tool to start *making sense* of the complexity and variety of experiences they live and as a first approach to computational thinking at school. We will focus on the structure and objectives of the activities and offer, through a brief classroom excerpt, some initial discussion on how preliminary activities involving children's bodily movements created the ground for further mathematical investigations into coding activities.

CHILDREN IN MOVEMENT TOWARDS STEAM

"Children in Movement towards STEAM" aims at engaging kindergarten children in laboratory activities about mathematics and robotics as an approach to the development of mathematical and computational thinking. The specific reference to STEAM (Science, Technology, Engineering, Art and Mathematics) helps to frame the project from a wider perspective, which takes into account the interdisciplinary nature of mathematics as a discipline that allows developing critical thinking and problem-solving, and argumentation skills in a variety of contexts. The project wants to nourish a vision of the cultural value of mathematical-scientific knowledge for learners to become aware citizens. Also, it makes room for creativity in the teaching and learning of mathematics, a dimension that is often neglected but is significant to engage in learners from an early age and to work on the relationships between mathematics and other sciences as well as art.

While mathematical digital competency (Geraniou & Jankvist, 2019)—the ability and awareness of using instruments in various contexts and engaging in mathematical discourses and solving mathematical problems with digital devices—is generally discussed for older students, it is also apparent that young children develop with extraordinary ease fascination for and mastery of digital devices. It is a matter of concern for educators that this pre-disposition is somehow directed, during the school years, towards a critical use of digital devices. This can be achieved through the design of mathematical activities that account for the playful engagement with the instrument while valuing the relationships with it as one that includes questioning and discovery in manners that are typical of the scientific process. Moreover, taking a multimodal approach to cognition (e.g., Ferrara, 2014), all the modalities along which a mathematical activity develops come to constitute the learning process. In line with this idea, a design principle to consider is to incorporate and value bodily, imaginative, and semiotic aspects into any activity.

Concerning tool use, we take as a reference the idea of mathematical instrument "as a material and semiotic device together with a set of embodied practices that enable the user to produce, transform, or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics" (Nemirovsky et al., 2013, p. 376). This definition wants to embrace the complexity of learning to use a new tool and encompasses a non-dualistic approach to tool use, which values the minute interactions that come to constitute the experience of

playing an instrument. The expression "playing an instrument" is purposely used by the authors to evoke musical instruments, for which the ability of playing is indiscernible from the fluency of using the instrument and some knowledge of music.

Nemirovsky and colleagues (2013) study how subjects interact with mathematical instruments in informal learning settings (museums) inside a semi-structured environment that is quite different from a kindergarten classroom. However, we see their perspective as appropriate to investigate tool use in our context. First, these researchers conceptualize tool use to the extent to which tools get incorporated into one's lived experience. This also means that one's investment in tool use emerges out of many aspects and that the tool (thoughts regarding it, sensations felt when using it, etc.) might permeate moments that are temporally far from the actual use. Secondly, this perspective allows us to move away from an instructional perspective on tool use, towards a vision of tools as occasions for meaningful encounters with mathematical concepts. In this direction, a second design principle that was crucial in the context of kindergarten activities is to use narrative as part of the teaching story, to engage children in discovery and reasoning and to raise their motivation.

After all, language and mathematics are the basis of computational thinking, which has a specific role in the National guidelines for the curriculum of the primary cycle (kindergarten to junior high school in Italy; Ministero dell'Istruzione, dell'Università e della Ricerca, 2017) in line with the curricula in other countries. Coding and computational thinking are associated with logical, analytic, and creative thinking because they allow for problem-solving by constructing procedures, establishing connections and planning strategies, and intervening every day in facing and solving problems. Regarding the mathematical content, the guidelines specifically highlight that since early childhood, children develop spatial reasoning, learn to describe the distance and location of objects in space in their own words, and discover geometrical concepts like those of direction and angle.

The project activities involved the kindergarten children using little robots (Blue Bots) to walk along open and closed paths (segment lines, L- or U-like lines, squares). The robots need to be programmed accordingly. While, in this manner, children learn coding and explore relations between the shape of the path and the corresponding code (as an approach to computational thinking), the aim is to stimulate an initial understanding of squared paths as shapes that satisfy certain mathematical properties. We can, for example, articulate on a square-like path saying that its sides must be formed by the same number of steps, and its turns (90° rotations) must occur in the same direction. Or describe it as the repetition of the same sequence (e.g., forward-forward-right turn) four times. These ways of seeing (and speaking of) the square allow for thinking of the shape in terms of spatial properties (e.g., side and angle equivalence). In addition, the activities implicate aspects of direction and orientation as well as of movement in space, introducing children to the capacity of reasoning on spatial relationships and the development of spatial thinking.

PARTICIPANTS AND METHOD

Three teachers from two different schools based in the surroundings of Torino (Italy) have been involved in the design of the project activities during the initial phases of work and have conducted a teaching experiment with their respective classes. The teaching experiment consists of 4 activities, each carried out in two 45-minute-sessions with a group of 12 children (half of the class), for a total of 9 sessions per group. The children in each group are 3 to 6 years old. A kit of six Blue Bots was available for each group in every school. Resources and materials have been designed to guide teachers' work in the classroom, but they do not have a fixed structure; rather, they serve choices by the teachers along the way. The activities have been refined and redesigned during the whole duration of the experiment, by considering the children's responses to specific tasks and associated difficulties

in managing the group of students. In-between the sessions, the teachers were free to work on collateral activities that do not directly face the mathematical content but that complement it in some fashion. For example, in one of those activities, children were asked to produce drawings associated with bodily experiences focused on how we step during walking (something that anticipates later work on the way that the robots move when they must cover a certain distance).

The children's parents took part in a presentation of the project and its purposes, to raise awareness and interest towards the relevance of the teaching experiment and, at the same time, of mathematical and scientific knowledge as a thinking and problem-solving means that can help children to face the complexity of their experiences in the world. They also had a role later when asked to partake in a final, collective digital creation to be shared with the researchers and all the other families. Parents' involvement in the project is seen as a crucial point to sensitize families not only towards the value of the children's mathematical experiences but also towards the value of the relationship between parents and children concerning the learning dimension.

The authors of this paper participated in the conceptualization and design of the project (the third author is a primary school teacher with huge experience in curricula and design-based research activities). Two researchers (the first and second authors) were present during the intervention as active observers, one for each class. They could therefore interact with the teacher and the children, giving support to the teachers during the activities. The sessions were video recorded with a mobile camera, and all the written productions (mainly drawings) created by learners were collected. Weekly meetings between the researchers allowed to introduce variations in the implementation and refinements of the initial draft of the activities. After any activity, an individual interview with each teacher was also carried out to have information about the state of the art of the intervention and indications about its development. Some of the productions by the children, together with photos and short video pieces, were used to create an online Padlet board, weekly updated by the teachers and made accessible to all the families to share the flavour and some content of the activities with parents. In the following, we focus on the main ideas that characterized the creation of the activities. We will then discuss some insights coming from the teaching experiments.

THE TECHNOLOGY: BLUE BOT

Blue Bot is a simple bee-shaped robot (Figure 1). It can be moved by pressing a sequence of movement commands through orange buttons with arrows on the bee's back: forward, backward (about 15 centimetres), left, and right (90-degree turns). Pressing the green button "GO" will make the robot move accordingly to the sequence that has been programmed. A one-second pause button can be used. A delete button (showing an "X") allows learners to clear out their commands and start a new sequence from scratch.

Figure 1. Blue Bot (front and above view) and the buttons with the commands on its back

As Bartolini Bussi and Baccaglini-Frank (2015) point out, many significant processes that are typically mathematical or related to computer science emerge out of playing with this device:

counting (the number of commands), measuring (the length of a step or the path, the total distance travelled by the robot), exploring space, constructing frames of reference, coordinating spatial perspectives, programming, planning, and debugging. The trajectories that can be walked by the robot are broken lines and possibly polygons with 90-degree-angles. For example, a square-trajectory with each side 2 steps long is walked when a sequence of forward-forward-right, repeated four times, is programmed. The device resembles the real little animal but has hybrid characteristics of a robotic creature: it makes sounds as it moves and stops. This helps the children develop an affective relationship with the tool and care about it and its peculiar behaviour.

OVERVIEW OF THE ACTIVITIES

The activities are learner-centred and engage the children through making, bodily actions, senses, movement, and diagramming to foster their participation and motivation in playing with mathematics. They often challenge children to work in groups and collectively to promote a vision of mathematics as an activity with socio-cultural value, in and outside of the classroom context. The children are also part of heterogeneous groups to foster collaboration between different age levels. During the teaching experiment, different modalities and contexts were alternated. We used a set of introductory activities focused on movement in between points of reference within a space of the school, flowers previously coloured by the children and placed on the floor. In such activities, the children initially explored free movements and then were asked to control or limit their manners of stepping from one flower to another, with tasks proposed by the teacher in a playful environment.

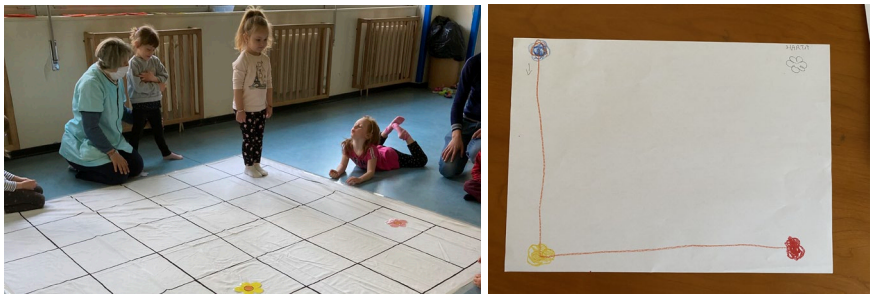

Figure 2. Activities involving body movements and drawings throughout the experiment

After this phase, the teacher removed all the flowers but two and led a collective discussion that focused on how one can move from one flower to the other. Great attention was given to understanding one own's way of moving and eventually the number of steps, and the children were encouraged to share their thoughts verbally or express them through a drawing (Figure 2). The device was introduced through a video prompt and a treasure hunt on site. The video, created for the children in Powtoon, presented a character, a little girl named Alice, who asked the children to help her to look for her special bee-friends because they were lost and hid in the school (Figure 3, left). This was done to create a narrative storyline that connected different moments of the experiment and of providing the children with an objective that guided their interactions with the device and introduced problems that the children had to solve. In the storyline, indeed, Alice was worried about the bees to be able to return home and engaged the children to understand and teach them how to move from one flower to another.

The Blue Bots were then investigated by the children regarding how they look and how they move. After this examination phase, the children shifted to observing the kind of movement of the robots and exploring the presence of buttons and their role in that. At this point, the children programmed the robots for the first time, to have them moving along straight segment lines of different lengths and started guessing about the number of necessary steps or commands.

Figure 3. A frame of the video introducing the treasure hunt; using arrows to describe movements

A general idea of the design was that of alternating moments in which the tool was present and in use and others in which it was only recalled, and body explorations in space were prevalent in the activity. Following the existing literature, a considerable amount of time was devoted to comparing the children's movement and that of the robot. This was done to 1) promote a multimodal approach to the activities and 2) to provide the children with initial insights into both ways of moving and a shared vocabulary in the classroom on how to describe directions, paths and orientation.

In the last part of the teaching experiment, arrows were used to codify the movements of both the bees and the children (Figure 3, right) through an ordered sequence of arrows. In the progression of tasks, a square path was gradually built by adding complexity: from the request to interpret a given code (forward-forward-turning) to then move the bee from one flower to another (passing through a third flower in a way that the bee can travel an "L" shaped path, then a "U" shaped path). In so doing, the teacher focused on a back-and-forth movement from imagining the code (or robot motion) to impersonating that code/movement or coding the robot, which allowed for reasoning on the tool in terms of bodily interactions (Ferrara, 2014; Nemirovsky et al., 2013).

PRELIMINARY INSIGHTS FROM THE ACTIVITIES

We present an episode to discuss initial insights coming from the teaching experiment, which informed the design of the activity. During the first activity, after the children have compared the different lengths of the paths between the flowers using their footsteps, the teacher gives the children the cut-out figures of a foot and flowers to glue on a sheet of paper to describe the previous bodily experience. Pairs of children of different ages are now working on the task. The researcher (R) approaches a pair of girls (Camilla, C, 3, and Mia, M, 5) who have already positioned two sets of flowers on a sheet (see Figure 4) and asks them:

R: What did you do?

M: We made longer paths.

R: Which is the longer path?

M: This (*points, Figure 4, left*)

R: Because I see two paths…

C: And this is the shorter one! (*points to the shorter path)*

R: Why that one (*points to the sheet*) is the longer one?

C: This one!

M, R: Because it has more feet

C: Yes, this is the longer one, and this is a little shorter (*points to the two paths*).

R: And how many steps are there?

M: One, two, three, four. One, two (*counts the feet on the two paths, Figure 4, middle*)

R: So, how can I do to see if one path is longer or shorter?

M: It shows, and if you turn it, you see it better (*shows the sheet to the camera, Figure 4, right*)

Figure 4. Mia and Camilla comparing lengths of the two paths on the paper sheet

The two children work together on a task that is preparatory to coding activities that require using a sequence of "steps" to describe and create a movement in space. The "foot" prepares the ground for the one-to-one relationship with the Blue Bot movement embedded by the code. We discuss this episode in terms of the aspects that we see as enriching the experiences of the children: the analysis is meant to highlight how the girls *make* and *make sense* of their drawing. In the brief interaction with the researcher, the two girls use the created representation to compare the lengths of the paths. They produce a first argumentation about the contextual experience of the number of feet; that is the paths are of different lengths. At the end of the brief excerpt, Mia changes the drawing position to allow the researcher "to see it better". This funny moment speaks directly to the way in which the girl comes to inhabit the representation: for her, the privileged position to look at the drawing is the one in which they explored the movement from one flower to another. After, Mia exchanges the position of a blue and red flower to match the colours. Then, the girls add a much shorter path to their drawing (one foot only). They notice details that count in the description of a movement, even if only to adhere to the previous bodily activity, distinguishing between the starting point and the ending point through colour; they also explore new variations, using the foot as a measuring unit. The two children collaborate, despite the difference in age, and converge on a common narrative that establishes a comparison between the two paths, using numbers and a first measuring unit (the foot). Later in the teaching experiment, when the children programmed the Blue Bot to move it from flower to flower (a straight path across the classroom floor), we observed peculiar behaviour. Some children scanned the space in between and pressed buttons on the robot as they moved their eyes to the farthest flower as if they were imagining a movement happening in front of them. Further discussion is needed on how such activities can foster the use of the tool and promote spatial thinking and will constitute future research.

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SPATIAL AND COMPUTATIONAL THINKING AT KINDERGARTEN THROUGH THE AID OF AN EDUCATIONAL ROBOT

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Keywords: Bodily movement, coding, computational thinking, shape, spatial thinking.

FRAMING THE STUDY

This workshop discussed insights from a research project focused on the development of spatial and computational thinking with very young children, with the aim to introduce them to STEM (e.g., Benitti, 2012). In the project, two teaching experiments were carried out in two kindergarten classes (children aged 3 to 6), and a Blue Bot was used to create opportunities for mathematical explorations for the children in a playful environment. The Blue Bot is a little bee-shaped robot, which can be programmed to move by pressing sequences of commands (Figure 1, left and middle). Following Bartolini-Bussi and Baccaglini Frank (2015), significant processes that are typically mathematical or computer science-related emerge from play with this device, like counting, measuring, programming.

ACTIVITY DESIGN

In the initial part of the workshop, the principles that guided the activity design were presented to the participants. These principles are inspired by embodied cognition theories, which value the body in the teaching and learning of mathematics, and concern: the relationships between the children bodily movement and the robot movement; the interplay of imagining and observing, and of doing and creating; the passage from movement to trajectory to code, and vice versa; the multimodality of mathematical cognition (Ferrara, 2014). As an example, one of the first activities preceding the use of the tool involved bodily movement in space. Printed paper flowers of different colours were placed on the floor. The children first moved freely from one flower to the other, then the teacher asked them to perform variations of movement (faster, slower, with big steps, walking from one specific colour to another). This was done to raise awareness on bodily motion, explore constraints of movement, and share a common vocabulary to talk about movement. Next, the teacher turned the children attention to linear paths connecting two flowers, working on the comparison of their lengths by means of step-counting. The activity primed new activities that involved the robot and focused on its peculiar way of moving.

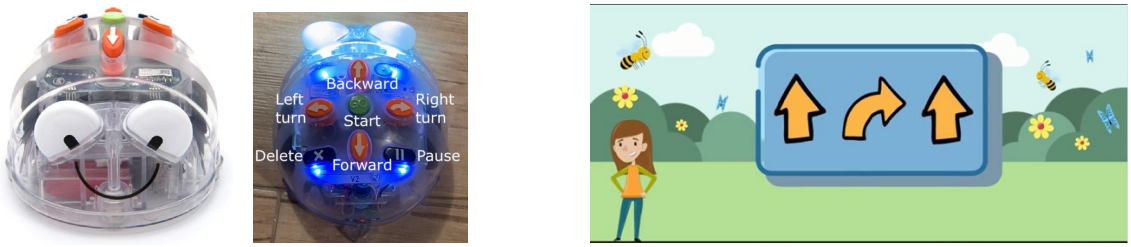

Figure 1. Blue Bot (front and back); the code discussed in the episode from the classroom

FROM THE CLASSROOM

In the second part of the workshop, a video of a brief classroom situation was watched, in which the children interact with the teacher to solve a task. In the video, the children have been exposed to a

three command-code (Figure 1, right) and are asked to think of the robot movement. The teacher asks each child: “For you, which path will the bee follow?”. One of the children, Samuele, is at the centre of the classroom in front of his classmates, with the code captured by three plastic arrows positioned on a paper sheet on the floor. The bee bot is at his disposal on the floor. Samuele gestures on the floor, creating a shape like the one sketched in Figure 2, and answers: “Straight, then crooked, then straight again.” He repeats the same path three times. But Giovanni disagrees, so the teacher involves him in the discussion. Giovanni stands up and explains why: “Yes, because straight, then it turns, then it comes straight again”. As possibilities of the robot movement are discussed, the teacher asks other children to participate with their thoughts, until she involves a third child, Lorenzo, who before was gesturing the movement trajectory in the air. Lorenzo is asked to move as if he was the Blue Bot, and a conflict emerges. Lorenzo walks along a shape like the one traced by Samuele, contrasting Giovanni’s idea again. The Blue Bot is programmed by Samuele under the request of the teacher. But, as soon as the robot stops moving, Samuele exclaims: “No!”, and annoyed lifts it up, convinced that it has not moved as he has programmed it.

Figure 2. Samuele gesturing the path he imagined

The dialogue continues, and Samuele and Giovanni discuss the code to reason about the movement of the bee bot. Giovanni tries to convince Samuele: “But Alice’s code, Alice’s code is like this: straight, turn, straight (*pointing to the three arrows, looking at Samuele*), not like this”. This puzzles Samuele, who struggles with the gap between what he is imagining and what he sees.

Focus on this episode engaged the participants in a rich discussion concerned with the classroom dynamics that were nurtured by the activity design. We point out two main aspects raised in the discussion: 1) the turning arrow requires a change in perspective implying a rotation instead of a step in the movement; 2) the technology is somehow troubled on the way it works, appearing to do what it prefers instead of what the children want. These aspects appeared problematic with respect to the children’s understanding of temporality and spatial displacement, which are embedded in the code, and are worth of further research. On a theoretical level, another key point was the difference between the linearity and discreteness of the code, versus the freedom and continuity of the bodily movements. The participants also questioned the role of the teacher in exploiting the different registers which are used to imagine, speak of, and enact the robot movements, another promising line of investigation.

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A REVIEW ON *ALLGEMEINBILDUNG* AND MATHEMATICAL LITERACY IN RELATION TO DIGITAL TECHNOLOGIES IN MATHEMATICS EDUCATION

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In Denmark, the concept of Allgemeinbildung is embedded in the curriculum, but it is difficult for mathematics teachers to translate this aim into actual teaching practices. The inclusion of digital technologies in mathematics teaching might comprise new potentials for Allgemeinbildung. Through a hermeneutic literature review, this paper investigates how the inclusion of digital technologies in mathematics education may contribute to Allgemeinbildung/mathematical literacy. 113 search results were transferred to abstract screening, which resulted in all 20 texts to review. These were categorised into four groups: Mathematics and technologies for work-life, for everyday life, critical aspects, and digital literacy concerning mathematical literacy. It is concluded that the inclusion of digital technologies in mathematics education comprises potentials and challenges for mathematical literacy and Allgemeinbildung. Directions for further research are suggested.

Keywords: Allgemeinbildung, digital technologies, digital tools, mathematical literacy, numeracy.

INTRODUCTION

In Denmark, it is a task for elementary education to contribute to students' citizenship (Blomhøj, 2001; Niss, 2000). The concept *Allgemeinbildung* is embedded in the national curriculum, and it has been for decades (Hansen, 2009). It is, however, difficult to translate into actual teaching practices. Denmark, at the same time, is far ahead with the inclusion of digital technologies, both in teaching in general (Bundsgaard et al., 2019), and in mathematics teaching particularly (e.g. Jankvist et al., 2019). The question is, however, if this comprises new potentials for students' *Allgemeinbildung*.

The notion of *Allgemeinbildung* is primarily employed in the Scandinavian countries and Germany, and it is difficult to translate into English (Biehler, 2019). In Germany, there is a rich and old tradition to discuss the notion concerning mathematics education (e.g. Vohns, 2018; Biehler, 2019; Jahnke, 2019). Even though the notion cannot be translated properly, there are significant similarities with that of Mathematical Literacy (e.g. Niss in Biehler, 2019). E.g. in "…fostering societal participation and active citizenship (…) a goal that is prominently addressed in both conceptions..." (Vohns, 2017, p. 1). This paper shows the results of a literature review of both *Allgemeinbildung* and Mathematical literacy concerning the inclusion of digital technologies in mathematics education. This paper aims to answer the research question: According to the research literature, how may the inclusion of digital technologies in mathematics (primary and lower secondary) education contribute to *Allgemeinbildung*/mathematical literacy?

Allgemeinbildung and Mathematical Literacy

As stated initially, *Allgemeinbildung* is an aim for mathematics education in the Danish school system. The purpose of *Allgemeinbildung* is that a democratic society needs Allgemeinbildete citizens (Niss, 2000). *Allgemeinbildung* connotates "holistic self-enculturation" and is contrasted to vocational *Bildung* (vocational education), which prepares for specific vocations. It concerns the formation of human beings in a society (e.g. Vohns, 2018; Biehler, 2019; Neubrand, 2015) and

comprises that subjects see a meaning in the way this formation takes place as a self-reflecting process (Vohns, 2021). Heinrich Winter is a prominent voice in the German literature on mathematics and *Allgemeinbildung* (Biehler, 2019; Vohns, 2018). He explicates *Allgemeinbildung* as "...competencies and knowledge that are essential to every human being as an individual and as a member of society independent of his/her gender, religion, (future) profession, etc." (Winter, 1995 translated in Biehler, 2019, p. 153). The substance of *Allgemeinbildung* focuses on the fundamentals of understanding nature, culture and society (Niss, 2019, as cited in Biehler, 2019).

The concept of mathematical literacy stems from English speaking mathematics education community. It first occeurs in USA in mid-fourties (Niss & Jablonka, 2020). Mathematical literacy is defined in the latest PISA framework as:

> an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st-century citizens. (OECD, 2018, p. 7)

Jablonka (2003) states that promoting mathematical literacy, at the same time, is promoting certain social practices: It is always mathematical literacy for something. She elucidates the different social practices: Mathematical literacy for, respectively, developing human capital, cultural identity, social change, environmental awareness and evaluating mathematics.

There is a long list of related literacies: numeracy, quantitative literacy, financial literacy, statistical literacy, mathemacy, matheracy and critical mathematical literacy. Some are related to specific content. Others are sometimes used as synonyms to mathematical literacy, sometimes definitions are distinguished (Niss & Jablonka, 2020).

Even though mathematical literacy and *Allgemeinbildung* differs in meaning and origin, the two concepts address the same issues. According to Biehler (2019), mathematical literacy would be a subset of *Allgemeinbildung*. Diverging from mathematical literacy, *Allgemeinbildung* addresses the development of individuals' personalities. Though, mathematical literacy and *Allgemeinbildung* intersect on the theme of active citizenship and on preparing students for the future as members of a society (Vohns, 2017).

METHOD

To answer the research question, I conducted a hermeneutic literature review (Boell & Cecez-Kecmanovic, 2014). I selected four databases relevant to mathematics education research: Web of Science, Eric Proquest, Scopus and SpringerLink. The ladder was also pertinent to secure getting German-language literature. It is relevant because of the tradition for *Allgemeinbildung* in German mathematics education research. For the same reason I reviewed all issues of *Mittelungen der Gesellschaft für Didaktik der Mathematik.* The phrases *Allgemeinbildung,* mathematics, mathematical literacy and digital technologies were combined with Boolean operators and truncations. 113 results were transferred to abstract screening. In this process, I have focused on primary and secondary education. It excludes education for certain vocations such as engineering and financial education. Both digital technologies and different notions of mathematical literacy or *Allgemeinbildung* should be part of the abstract. Because quantitative literacy and numeracy sometimes are used as synonyms for mathematical literacy, I was open to this use in the screening. Mathematical literacy is sometimes used only as a synonym for mathematical knowledge (Jablonka,

2003). In these cases, the literature was excluded. References were used for snowballing. It resulted in 18 texts for review. In the review process, *Allgemeinbildung* and mathematical literacy was handled as synonyms and did not give rise to a subdivision of the results. Though, the balance between the amount of literature addressing, respectively, *Allgemeinbildung* and mathematical literacy, is addressed in the discussion of this paper.

CATEGORISATION OF THE IDENTIFIED LITERATURE

Three categories related to different spheres of life, respectively work-life, everyday life and critical aspect related to citizenship could be identified. One category addressed a discussion about digital literacy.

The group of literature about stundents' future work-life identifies how digital technologies changed mathematical practices in workplaces, and it discusses curricular consequences. The group of literature about preparation for everyday life comprises considerations about how the inclusion of digital technologies in mathematics teaching can enhance numeracy practices and how digitalisation affects what future needs mathematics teaching should respond to. The literature about critical aspects of citizenship problematises the digitalisation and the demathematisation of society. The last group discusses the notion of digital literacy or *Digitale Bildung* concerning mathematics teaching. Two results treated mathematics and technologies as aspects of both work-life and everyday life, Gravemeijer et al. (2017) and Geiger et al. (2015a).

Mathematics and technology: Preparation for work-life	**Mathematics and technology: Preparation for everyday life**	**Mathematics and technology: Critical aspects**	**Digital literacy and mathematical literacy - potentials and constraints**
Geiger et al. (2015a) Gravemeijer et al. (2017) Hoyles et al. (2010) Zevenberger (2004)	Geiger et al. (2015a) Geiger et al. (2015b) Geiger et al. (2020) Gravemeijer et al. (2017) Novita and Herman (2021) Steen (1999) Steen (2001)	Forgasz et al. (2010) Gellert & Jablonka (2007) Gellert & Jablonka (2009) Keitel et al. (1993) Straehler-Pohl (2017)	Dofkavá (2016) Hirscher (2018) Nocar et al. (2019) Vohns (2021)

Table 1. An overview of the literature ordered in four categories

Mathematics and Technologies – Preparation for Work-Life

Today's society is deeply influenced by digital technologies and digitisation, and there is a paradox in the fact that mathematics at the same time is invisible and pervasive. The role of mathematics in society is growing, and mathematics is increasingly carried out by machines. Even though mathematics has this central role in society, only a few master it (Gravemeijer et al., 2017). Technologies have transformed the mathematics used in workplaces (Geiger et al., 2015a; Gravemeijer et al., 2017). According to Zevenberger (2004), the technologising of workplaces and society influences how young people work mathematically and must influence mathematics teaching: the focus of accuracy and precision in calculations is redundant, and there must be a retheorising of mathematics education to promote more global aspects of problem-solving. Gravemeijer et al. (2017) point out that, on the one hand, calculations in real life are carried out by computers. On the other hand, calculations are the main focus of school mathematics. Instead, the mathematics taught should focus on three overall mathematical competencies: 1) Applying and modelling, such as recognising

where mathematics is applicable and translating practical problems into mathematics. 2) Understanding. A conceptual mathematical understanding is needed to understand the underlying and hidden procedures of digital technologies. 3) Checking. It is necessary to evaluate the computer-based results. Not by repeating the calculations by hand but by checking if the results seem plausible. Working with computers requires that phenomena from the real world translate into numerical quantities. There is a need for deep understanding of how these processes are carried out and an awareness of which information gets lost in such translations (Gravemeijer et al., 2017). Gravemeijer et al. (2017) stress the need for not just handling the artefact but for an explicit focus on the instrumentation.

Hoyles et al. (2010) derive the term Techno-mathematical Literacy from mathematical literacy to cope with the needs for modern work life and to address the specific needs for mathematics in this context where mathematics is expressed through an artefact.

Mathematics and Technologies – Preparation for Everyday Life

Steen (1999, 2001) points out that society is "data-drenched". Therefore, it stresses the need for citizens who can cope with numerical information. Numeracy practices can be enhanced by the efficient integration of digital tools in mathematics teaching practice. The "collection, recording, and analysis of real-world data are real; comparing tools in mathematics teaching can enhance numeracy practice the features of relevant data sets; critiquing a situation or making judgements." (Geiger et al., 2015b, p. 538). Digital technologies influence how mathematics is relevant in society. Since big data is growing, the need for statistical literacy is increasingly important (Gravemeijer et al., 2017; Geiger et al., 2020). Also, the need for space geometry becomes relevant to handle 3D technologies (Gravemeijer et al., 2017).

Geiger et al. (2015a) investigate empirically how the use of digital tools can support numeracy teaching and learning. They focus on students' development of "technology-integrated mathematical capacities" and how these can prepare them for future work and citizenship. For this purpose, they present a model that integrates the context with mathematical knowledge, dispositions and tools. These relations are embedded in a critical orientation "as the fundamental purpose of numeracy in practice is that it empowers individuals with the capacities to evaluate and to make judgements and decisions about their options and opportunities in the lived-in world." (Geiger et al., 2015a, p. 1123). In a Design-Based Research-study, students investigate an overall question of what level of physical activity is required to maintain good health. The students used different digital tools to track, gather and represent data relevant to their question. In addition, the digital tools gave rise to reflections on what measures were necessary and critical examination of their situation.

Novita and Herman (2021) find, in a literature review, that consistent use of technology might play an essential role in developing mathematical literacy. They point out that integration of digital technologies in mathematics teaching should be accompanied by pedagogical considerations about three different uses of technology in mathematics teaching: instructive, manipulable and constructive digital technologies.

Mathematics and Technology – Critical Aspects

In the following perspectives, the ways technology and mathematics influence society are addressed as a democratic problem. Mathematics and technologies act as black boxes in ways that make them invisible and impenetrable for citizens.

Because of technological development, mathematics affects all parts of society. Mathematics is mostly invisible or just recognised on the surface, and the technology functions as black boxes. A

demathematisation of society is taking place as a consequence of technological development. The mathematical skills needed before are now taken over by technology. This concerns with social availability of mathematical knowledge. The power relationship between constructors, operators and consumers of technologies is problematic for democracy (Gellert & Jablonka, 2007; 2009). Keitel et al. (1993) call this black-boxing of mathematics and technology for implicit mathematics. Thus, specific needs for mathematics education emerges and influences the way mathematics and technology should be taught. To develop democratic competence, it is not mathematical nor technological knowledge that is needed. "An extension of mathematical or technological knowledge does not automatically lead to a reflection about the use or function of technologies nor about the underlying mathematical models" (p. 270). Instead, reflective knowledge is needed.

Straehler-Pohl (2017) stresses the need for recovery of critical distance toward the demathematisation on the school agenda. For this purpose, he suggests three agendas for school mathematics: 1) reflective knowledge (as stated by Keitel et al., 1993) on the uses of technology and mathematics, 2) let students experience the opportunity to reject the use of mathematics for solving problems, and 3) "the entanglement of de|mathematisation with capitalism should be explicitly brought on the agenda of mathematics classrooms" (p. 17).

Based on the perspective that all citizens should have access to powerful mathematical ideas, Forgasz et al. (2010) examine issues of equity concerning mathematical learning with digital technologies. They focus on access and agency and conclude that the availability of resources for mathematical learning with digital technologies varies according to economic status and cultural and educational values and beliefs.

Digital Literacy and Mathematical Literacy – Potentials and Constraints

In the following perspectives, digital literacy (or digital *Bildung*) as an objective in mathematics education is discussed. The discussion relates to curricular reforms and public debate about implementing digital literacy. Dofkavá (2016) and Nocar et al. (2019) recognises great potentials for mathematical and digital literacy to complement each other. Their finding is made in the context of curricular reforms in the Czech Republic. In the German debate, Vohns (2021) and Hirscher (2018) criticises the terminology of *Digitale Bildung* (equivalent to the notion of digital literacy). Their criticism concerns the meaning of *Bildung.*

Bildung is tied to the subject and human consciousness (Hirscher, 2018). *Bildung* is more than learning and requires reflective processes and awareness about self-development (Vohns, 2021). Hirsher (2018) promotes a model that combines methodology, proficiency and reflection related to the implementation of media in mathematics education. Vohns (2021) points to the constraint about digital *Bildung* through 7 theses. Some of these relate to diverging understandings in policy, mathematics education and mathematics communities. Social implications and the development of "critical digital literacy" are, according to Vohns (2021), underexposed.

DISCUSSION

The initial aim of this review was to answer the research question: According to the research literature, how may the inclusion of digital technologies in mathematics (primary and lower secondary) education contribute to *Allgemeinbildung*/mathematical literacy? The research literature could be classified into four groups. The first three relate to different spheres of life: work life, everyday life and citizenship. The last group relates to a present discussion about including digital literacy in mathematics education. Digitalisation changes society and the way mathematics is practised. This is a common point for all four headings.

The way mathematics is practised in workplaces changes the needs in mathematics education. Mathematics teaching should focus on global aspects such as problem-solving and modelling and making sense of results given by the digital tools. To enhance numeracy practices of everyday life, digital technologies comprise potentials. Digital technologies may contribute to students exploring questions about a good and healthy life. The technologies support the students in flexible reasoning and investigations of different measures. The digitalisation of society also creates new educational needs. E.g. statistical literacy becomes necessary to cope with a data drenched society. As for critical aspects, digitalisation challenges democracy. Due to digitalisation, society is "dematematised". Mathematics and technology function as black boxes and becomes invisible powers. Thus, mathematics education must enhance student's reflective knowing in terms of mathematics and technology in society.

The notion of digital literacy is trending. This raises questions about the relationship to mathematics teaching. The literature points out joint potentials for mathematical and digital literacy. Yet, the literature is not very specific about actual teaching practice. The German literature accentuates that *Digitale Bildung* does not embrace the richness of the meanings of *Bildung* (or *Allgemeinbildung*).

The inclusion of digital technologies in mathematics teaching does comprise potentials for mathematical literacy (or numeracy). But digitalisation of society does also comprise some educational challenges related to democracy.

This review comprises a much greater amount of English-language literature. This means an emphasis on mathematical literacy (or synonyms to mathematical literacy). *Allgemeinbildung* comprises some other facets, e.g. self-enculturation, development of student's personalities and self-reflection. If these facets of *Allgemeinbildung* shall be addressed in connection with the digitalisation of mathematics teaching, then there is a need for further research.

Only one article comprised empirical research connected to actual mathematics teaching. As stated initially, *Allgemeinbildung* is difficult and demanding to turn into actual teaching practice. Therefore, further empirical research on the connection between *Allgemeinbildung* and digital technologies in mathematics education is needed.

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TEACHER DEVELOPMENT IN COMPUTATIONAL THINKING AND STUDENT PERFORMANCE IN MATHEMATICS: A PROXY-BASED TIMSS STUDY

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This paper draws on data from TIMSS 2019 to investigate an approximate relation between Danish, Swedish, and English teachers' in-service training in computational thinking (CT) and their students' TIMSS scores as compared to students whose teachers had only received training in digital technology (DT) or problem solving (PS). To that end, we developed a proxy for CT combining PS and DT, based on recent definitions of CT in mathematics education. The results indicate that, in England, in-service CT training is associated with systematically better student TIMSS scores as compared to students whose teachers received only DT or PS training, but the best scores were achieved by teachers who did not participate in any of the three forms of training. In Denmark, CT training was found to have a negative effect on TIMMS scores while in Sweden, the effect of CT training was difficult to assess.

Keywords: Computational thinking, mathematics teachers, professional development, TIMSS.

INTRODUCTION

Following Papert's (1980) work during the 1970s, ideas linking computational thinking (CT) and mathematics education have recently attracted renewed interest. The CT research literature has grown steadily since Wing (2006) reintroduced the concept, and several countries, including Sweden and France, have already revised mathematics curricula to incorporate programming/CT. Then as now, the integration of CT in mathematics is based on the assumption that this enables students to more fully engage with and express mathematical ideas (Papert, 1980; Benton et al., 2018), ultimately improving their mathematical proficiency. However, implementing CT in mathematics education entails a number of significant challenges, including how best to prepare teachers to teach CT, what specific links should be established between CT and mathematics, and what pedagogical approaches should be applied. Among these, a central challenge is the need for pre- and in-service teacher training to teach the subject. Although it is often argued that there are significant overlaps between CT and mathematics—for example, in relation to logical structures and modelling (Ejsing-Duun & Misfeldt, 2015; Pérez, 2018)—existing research suggests that mathematics teachers cannot be expected to teach CT or programming without additional inputs beyond their pre-service training (Misfeldt et al., 2019). Although this need is widely acknowledged (Bocconi et al., 2016), most research to date has focused on issues related to teachers' beliefs about CT in mathematics, their readiness to teach CT, and effective formats for training multiple teachers quickly. However, less attention seems to have been paid to the question of whether CT training for mathematics teachers actually has any positive effect on students' mathematical achievement. As the integration of CT in mathematics curricula is relatively recent, data supporting direct measurement of any such effect remain scarce. However, researchers continue to make progress toward defining the components of CT in mathematics education contexts (Pérez, 2018; Weintrop et al., 2016; Kallia et al., 2021). Building on this work,

the present paper explores 1) whether CT can be related to aspects of the available TIMSS data and 2) the nature of the relationship between mathematics teachers' CT-related professional development and students' Trends in International Mathematics and Science Study (TIMSS) scores in Denmark, Sweden, and England. To that end, we constructed a CT proxy, building on definitions of CT in mathematics education by Kallia et al. (2021) and others. Using this proxy, we identified teachers in Denmark, England, and Sweden who have received in-service training in PS and DT and compared their students' TIMSS scores to those of students whose teachers have received in-service training in digital tools (DT), problem solving (PS), or none of these. The study addresses the following research questions.

RQ1: How can we relate CT to PS and DT elements of the TIMSS teacher survey in order to construct a proxy for in-service CT training?

RQ2: Using this proxy, what can we learn about the relationship between mathematics teachers' CT-related professional development and their students' performance in Denmark, Sweden, and England?

To begin, we summarize existing research on CT training for teachers. We then introduce our definition of CT and how this informs our choice of data, hypotheses, and analytic approach. Finally, we present our results regarding the relationship between students' TIMSS scores in mathematics for teachers trained in CT as defined and the TIMSS scores of students whose teachers have only received training in PS or DT, or in none of these. In conclusion, we reflect on the strengths and limitations of our approach and what further data are needed to develop a more precise understanding of these effects and the underlying mechanisms.

EXISTING RESEARCH ON TEACHERS' CT TRAINING

As interest in CT as a K-12 subject has increased in recent years, a growing body of literature has focused on CT as an element of teachers' professional development. Given the scale of teachers' training needs, a number of these studies have explored the potential of online formats (Toikkanen & Leinonen, 2017; Sentence & Humphrey, 2015). In one Finnish study, Toikkanen and Leinonen (2017) found that teachers regarded MOOCS as an appropriate format for the development and sharing of CT teaching materials, while an English study (Sentance & Humphrey, 2015) showed that teachers there favored blended formats combining online and face-to-face elements. The Finnish study reported low completion rates of less than 20% for purely online formats (Toikkanen & Leinonen, 2017), indicating that such formats may be problematic.

Another strand of this research addresses beliefs about CT and perceived readiness to teach it among pre-service teachers with no formal training in the area. There is evidence that pre-service teachers differ widely in their understanding of CT, and that CT is often conflated with the more general use of technology or programming (Yadav et al., 2014; Bower & Falkner, 2015; Carbrera, 2019) while failing to consider other aspects such as problem solving. Yadav et al. (2014) found that teachers who had received CT training were more confident about teaching CT to their students and had developed a better understanding of CT as a teaching topic. In their overview study, Bocconi et al. (2016) found significant differences across European countries in terms of responsibility for teacher CT training, how it is organized, and its format and content. However, little is known about the effects of such initiatives on teaching or on student learning. This is a significant issue for mathematics education, as arguments for incorporating CT relate to benefits for learning (Papert, 1980; Benton et al., 2018). As an approximate measure based on TIMSS data, we constructed a proxy of CT encompassing problem solving and use of digital technology in mathematics, as described below.

THEORETICAL BACKGROUND: CT AND ITS RELATION TO MATHEMATICS

Since Wing (2006) reintroduced and redefined CT, several definitions of CT have emerged. While Wing's definition treated CT largely as a topic in its own right, other definitions relate specifically to CT in mathematics education contexts (e.g., Barr & Stephenson, 2011; Weintrop et al., 2016; Pérez, 2018; Kallia et al., 2021). Wing's seminal paper stimulated renewed interest in CT, but many definitions within mathematics education and beyond incorporate problem-solving and the use of digital technologies. Wing (2006) argued that computer science provides a theoretical basis for exploring how computers and other digital technologies can support problem-solving by decomposing and representing the problem and evaluating preliminary solutions. Barr and Stephenson (2011) argued that while most K-12 education curricula include problem solving, CT can help teachers and students to understand and solve problems using computation and digital tools. Weintrop et al.'s (2016) taxonomy of CT in mathematics and science education identifies computational problem solving as a key practice and argues that the use of digital technologies is a key element of CT. In their taxonomy, computational problem solving includes preparing problems for solution by computational means, assessing different computational solutions, and debugging flawed computational solutions to mathematical problems (Weintrop et al., 2016). In all of these examples, PS and DT are viewed as crucial components of CT. In a recent Delphi study, Kallia et al. (2021) asked 25 mathematics and computer science experts about the nature of CT in mathematics education. Their answers referred most frequently to problem solving and the use of technologies to solve problems. While it can be argued that CT includes elements other than PS and DT, it seems that these are considered central. Based on the definitions of CT outlined above, we therefore viewed the combination of PS and DT as a reasonable (though imperfect) proxy for CT. In the next section, we describe the TIMSS data used to address our research questions. We specify how this CT proxy relates to the TIMSS survey data, and we describe how we examined the relation between teachers' in-service CT training and their students' performance.

DATA AND METHOD

As stated in the Introduction, the present analysis is based on TIMSS data from Denmark, Sweden, and England. We were motivated in part by our involvement in an ongoing research project comparing the implementation of programming and CT in those three countries (Misfeldt et al., 2019). The comparison is interesting because the three differ significantly in terms of how long CT has formed part of the curriculum and how it is integrated in the school system. England was the first in Europe to do so, having implemented computing as a new standalone subject in 2014. Sweden incorporated programming as part of the curriculum's core algebra and problem-solving content in 2018 (Misfeldt et al., 2019). Denmark has not yet implemented CT in compulsory schools, but a pilot project currently running in 46 schools will inform the final decision regarding implementation format. These distinct settings provide an interesting context in which to explore differences and similarities using the CT proxy. Investigating the relation between mathematics teachers' in-service training in CT and students' TIMSS scores clearly requires both student and teacher data. Below, we describe the clustered nature of these data sources, how they are collected in TIMSS, and how we used the CT proxy to address our second research question.

Measurement of Student Achievement

The present study draws on TIMSS data from 2019 (Mullis et al., 2020). TIMSS' stated mission is to assess 4th and 8th grade students' mathematics and science achievements internationally, based on an elaborate design involving approximately 175 distinct mathematical items (Martin et al., 2017). In a

matrix sampling design, all students are randomly attributed a subset of the items. Using multiple imputation methods, students' mathematics achievements are then estimated, and these values serve as the outcome measure in the analysis. Achievement is assessed on a scale constructed by the IEA to facilitate comparison across countries and over time; the center point (500) corresponds to the TIMSS mean in 1995, when the scale was developed (Mullis et al., 2020).

Measurement of Teacher Training

The inclusion of a teacher survey enables TIMSS to measure whether teachers have received training within the past two years, and in what sub-branches of their subject. In the case of mathematics, sub-branches include mathematical content, pedagogy/instruction, curriculum, integrating technology into mathematics instruction, improving students' critical thinking and problem-solving skills, assessment, and addressing the needs of individual students (TIMSS teacher questionnaire, 2019, p. 11). TIMSS data on student achievement can be paired with data from the teacher survey, including mathematics teachers' training. In the next section, we examine whether these sub-branch data in combination with data on student CT achievement provide an adequate proxy for exploring the relation between mathematics teachers' CT training and students' mathematical achievement.

Toward a TIMSS-Based Proxy for Teacher Training in CT

As indicated above, TIMSS does not explicitly measure mathematics teachers' training in CT. However, it does measure the seven sub-branches of mathematics education listed above, including "improving students' critical thinking or problem-solving skills" (PS) and "integrating technology into mathematics instruction" (DT). As previously argued, PS and DT can be considered central elements of CT in mathematics education; for that reason, we sought to identify teachers from Denmark, Sweden, and England who had received training in the two central elements of CT in the two years prior to TIMSS. The TIMSS data set also facilitated investigation of the relation between teachers' training in various configurations of PS and DT (including both and none) and their students' achievements in mathematics. Figure 1 below visualizes this approach.

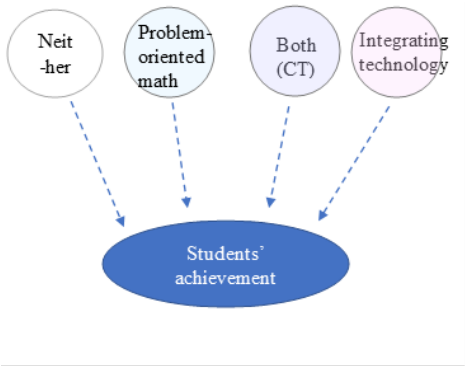

Figure 1. Overview of research design.

Table 1 below shows the distribution of teachers across the four categories in the three countries.

	DT	PS	CT	None
DK	10	8	12	96
SE	25	8	13	91
ENG	2	28	10	19

Table 1. Number of teachers who participated in professional development in DK, SE, and ENG

In Denmark and Sweden particularly, most teachers had not participated in any of the three forms of professional development. In England, most teachers had participated in problem-oriented math, but about a third had not participated in any such training. Using a data set originally developed for a completely different purpose naturally has several limitations, and the advantages and disadvantages of this approach will be discussed toward the end of this paper.

Variables

We were mainly interested in the effect on student achievement of teachers' participation in both types of professional development. The variables are listed in this table.

Analysis: Multilevel Model

The analysis employed three linear multilevel models of data from England, Sweden, and Denmark. This multilevel modelling enabled us to identify data clustering at two levels: classes and students (Snijder & Bosker, 2012). This mirrored the clustered design of the TIMSS survey, enabling us to test our cross-level hypothesis regarding the relation between teacher professional development (class-level) and student achievement (individual-level). In the model, all variables have been grand-mean centralized (unless already dummies) to make the results easier to interpret, as the estimates then show the effect for the grand-average (Gelman & Hill, 2007). Because TIMSS uses a multiple imputation method to assess students' mathematics achievements, the model was re-estimated using each of the five plausible values from the publicly available data (IEA) (Laukaityte & Wiberg, 2017). To average the effect and calculate standard errors according to the imputation structure, we estimated the model using the EdSurvey R-package. Incomplete observations have been removed listwise. The final Danish dataset comprised 1,976 students, clustered in 126 classes; the Swedish dataset comprised 2,422 students, clustered in 137 classes; and the English dataset comprised 1,171 students in 59 classes. The mean number of observations per teacher's class was 18.

RESULTS

We compared the estimated effects of professional development in the three countries. As these estimates are based on the reference model, they deviate slightly from the TIMSS mean in the IEA report (Mullis et al., 2020). Our main focus was *the change* associated with teacher participation in the different forms of professional development for the average student in each country (rather than the estimate itself). Table 2 summarizes the results of the analysis, which are elaborated below.

Professional development	**DK**	**SE**	**ENG**
Average for reference group (intercept):	530.52*	517.82*	570.16*
Standard error	*4.68*	*517.82*	*9.10*
Problem solving (effect):	2.78	8.95	-13.90
Standard error	*10.11*	*7.73*	*8.92*
Use of digital tools (effect):	-4.56	15.64*	-35.53*
Standard error	*10.05*	*7.79*	*9.65*
Computational thinking (interaction effect):	-19.15	-0.95	44.50*
Standard error	*18.32*	*12.86*	*17.53*
**Indicates significance at 5% level*			

Table 2. Effects of in-service training across the three countries

In Denmark, students' estimated TIMSS scores were highest in classes where teachers had participated in PS-related professional development (533.29). These were also the only students to score higher than those whose teachers received none of the three forms of training. Participation in CT-related professional development in Denmark was associated with the lowest estimated TIMSS score (509.58). None of the estimated effects of professional development in Denmark was significant at the 5% level and must therefore be considered unsystematic.

In Sweden, estimated TIMSS scores were highest in classes where teachers had participated in CT-related professional development (541.46). Sweden was the only country where this form of professional development produced higher estimates than non-participation in any of the three forms. However, as the effect was not significant at the 5% level, and the standard error was 12 times the effect size, it is highly unsystematic. The estimated effect of participation in DT-related professional development was significant at the 5%-level (with a standard error of 7.79), implying a systematically positive effect on students' TIMSS scores.

In England, estimated TIMSS scores were highest where teachers did not participate in any of the three forms of professional development (570.16), closely followed by teachers who participated in CT-related professional development (565.24). Teacher participation exclusively in DT-related professional development had the lowest estimated effect on student achievement (534.63). Both DT and CT were significant at the 5% level, indicating that the effect is systematic.

As CT is included as an interaction, the effect reported in Table 2 refers to the change in students' scores when their teachers participated in both PS and DT rather than in only one of these. The significant effect of teachers' CT training in England implies that it differs from the sum of PS and DT training. As the table shows, these results are quite different across countries.

DISCUSSION AND CONCLUSION

The present findings suggest that in-service CT training systematically achieves better TIMSS scores in England as compared to DT or PS training alone but is still inferior to non-participation in any of the three forms of training. The effect of CT training is negative in Denmark and is difficult to assess in Sweden. Although this CT proxy based on the available TIMSS data does not reflect all elements of CT, there are notable differences across these three countries, which have adopted very different approaches to implementing CT in their respective curricula. While this suggests that the differences across countries relate to CT implementation stage and approach, several issues arise regarding the proxy-based approach adopted here. While previous research on CT-related teacher training focused primarily on training formats and teachers' preconceptions, the present study instead addressed the issue of *content.* Nevertheless, the issues of format, teachers' beliefs, preconceptions, and related factors seem likely to affect student performance, but these data were not available from the TIMSS survey. Additionally, the mere co-occurrence of PS and DT cannot be regarded as equivalent to CT, which commonly entails a more symbiotic integration of PS and DT. Ideally, the survey items should be chosen and developed to correspond more closely to the relevant measures (in this case, CT) rather than having to subsequently compile selected parts of the data to align with a concept unrelated to the original survey design.

What, then, is the point of such an analysis? First, in the absence of ideal data sources, this experimental research design pragmatically addresses a key question: the extent to which in-service CT training for mathematics teachers can positively affect students' mathematical proficiency. We believe that ongoing theoretical advances in our understanding of CT in mathematics education can help to clarify whether and how existing data sources can provide a meaningful basis for addressing this question. While these advances can and should inform future large-scale surveys to determine

how in-service CT training of mathematics teachers affects students' mathematical achievement, such studies demand substantial researcher and practitioner resources. Moreover, while such studies may be able to measure the effects of future in-service CT programs, they cannot investigate the effects of past initiatives. However, by building on theory-informed proxy constructs of the kind described here, we can hope to approximate and indicatively measure the effects of such initiatives. By prompting reflection on how data fall short of the ideal, such experiments can, in turn, inform future decisions about the data needed to answer these and other important questions.

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THE USE OF DIGITAL TECHNOLOGIES FOR MATHEMATICAL THINKING COMPETENCY

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Mathematical thinking is about the processes carried out when doing mathematical activities, and digital technologies offer many opportunities for supporting such activities in the teaching and learning of mathematics. This paper presents the results of an initial literature review, at the moment including nine studies, on the use of dynamic geometry environments (DGE) and Computer Algebra Systems (CAS) in relation to learners' mathematical thinking and mathematical thinking competency. The studies are analysed from the perspectives of Thinking Mathematically and the mathematical thinking competency of the Danish competency framework. The results indicate that when using CAS, mathematical thinking and mathematical thinking competency is activated prior to and after the CAS use. Differently, when using features of DGE such as dragging and measuring, mathematical thinking and mathematical thinking competency can be activated during the enquiries in DGE.

Keywords: Computer algebra systems, dynamic geometry environments, mathematical competency, mathematical thinking.

INTRODUCTION

Since the 1990s, the interest in using digital technologies for mathematics education has increased, both in the classrooms and in the field of mathematics education research (e.g., Trouche et al., 2013). Simultaneously, a shift from focussing on mathematical skills and knowledge to mathematical competencies has been taking place (Niss et al., 2016). To investigate the use of digital technologies in relation to mathematical competencies, specific competencies of the Danish competency framework (KOM) (Niss & Højgaard, 2019) have been focal points to specify the study, for instance, the reasoning competency (Højsted, 2020) and the representation competency (Pedersen et al., 2021).

The activities of mathematical thinking cover a wide spectrum of mathematical processes (e.g., Drijvers et al., 2019; Mason et al., 2010; Tall, 1991). The mathematical thinking competency of the KOM framework focus on the activities involved in mathematical inquiries (Niss & Højgaard, 2019), which can be interpreted somewhat narrower than the general term of mathematical thinking. With the heavy introduction and emphasis on digital technologies in mathematics education, this paper reports on a pilot literature study with the purpose of investigating the use of Dynamic Geometry Environments (DGE) and Computer Algebra Systems (CAS) in relation to the mathematical thinking competency. The two phases, *entry* and *review* from the Thinking Mathematically framework (Mason et al., 2010) have the potential for investigating the processes related to the mathematical thinking competency. Therefore, this paper addresses the question: *Which features of DGE and CAS can be identified, in mathematics education literature, as useful tools for mathematical thinking competency through the lens of 'entry' and 'review' from the Thinking Mathematically framework?*

First, I account for the mathematical thinking competency and the theoretical background of the study. Then I explain the review process for the initial literature search, on which this paper is based. Finally, I analyse the literature results with the notion of entry and review of Thinking mathematically (Mason et al., 2010) in relation to the mathematical thinking competency (Niss & Højgaard, 2019).

THE MATHEMATICAL THINKING COMPETENCY OF THE KOM FRAMEWORK

The KOM framework is a characterization of mastering mathematics across educational levels and subject matter (Niss & Højgaard, 2019). The framework consists of eight distinct but non-disjoint competencies, where a mathematical competency is defined as "someone's insightful readiness to act appropriately in response to a *specific sort* of mathematical *challenge* in given situations" (Niss & Højgaard, 2019, p. 5). Other than the mathematical thinking competency, the remaining seven competencies are the reasoning, problem handling, modelling, representation, symbol and formalism, communication, and aids and tool competencies.

The mathematical thinking competency concerns the activities involved when engaging in mathematical inquiry. It comprises the ability to relate and pose generic questions characteristic of mathematics and to relate to the nature of the answers to such questions. Furthermore, the competency comprises the ability to distinguish between different types and roles of mathematical statements, such as definitions and if-then, for all and existence claims, as well as navigate with regard to the logical connectives and quantifiers (Niss & Højgaard, 2019). If you cannot distinguish between the mathematical claims, it can be difficult to understand what an answer to a given question would be. Moreover, the competency involves relating to the varying scope of a mathematical concept in different contexts. For instance, the scope of the number concept is expanded from the domain of natural numbers to rational, real and complex numbers. Finally, the competency includes the ability to propose generalizations of claims and abstractions of concepts and theories, as well as to relate to generalization and abstraction as processes of mathematical activity (Niss & Højgaard, 2019).

Due to the distinction between the eight mathematical competencies in the KOM framework, reasoning, problem handling, modelling, dealing with representations etc., are not part of the mathematical thinking competency, but interfere with it when doing and dealing with mathematics. For instance, the mathematical thinking competency is strongly interwoven with the problem handling and the reasoning competencies (Niss & Højgaard, 2019). Thus, at times, it can be difficult to distinguish these three competencies during mathematical activities. However, this distinction is an important point of the KOM framework.

Considering the role of digital technologies, Niss (2016) argues that digital technologies might enhance or replace mathematical competencies. For instance, digital technologies can help students experience mathematical processes and phenomena, and be part of explorative platforms for investigating mathematical concepts. With multiple representation views and features such as dragging and measuring, DGE and CAS have great potential for studying mathematical concepts and their relations (Pedersen et al., 2021). Digital technologies can produce static and dynamic representations of mathematical concepts and processes, create connections between different representations of the same concept and help perform symbolic manipulations (Niss, 2016). However, because digital technologies are able to perform mathematical activities for students, the tools can easily replace parts of the mathematical competencies rather than enhance them, if not used carefully (Pedersen et al., 2021).

MATHEMATICAL THINKING

Mathematical thinking is a broad term and is often related to mathematical learning and reasoning. One of the main contributions to the subject of mathematical thinking is 'Thinking Mathematically' by John Mason and colleagues (2010). This book is practice-oriented, focusing on the reader to use the given tools and take the time to work on the problems given throughout the book. In the preface of the first edition, the authors explain thinking mathematically as mathematical processes to carry out when doing mathematics (Mason et al., 2010). In their second edition, they change this view on

processes to be about natural powers, which, in the authors' view, we as human beings all possess. However, the powers need to be provoked for us to use and develop in contexts of mathematical thinking (Mason et al., 2010). In total, they present 10 of these powers, here specializing and generalizing are two out of four main powers. Specializing is to try with examples and to consider special or simpler cases of the problem. Generalizing is the move from the few instances, the special cases, to guessing about the relationship for a class of cases. These two processes are intertwined, which the authors illustrate with the slogan: "seeing the particular in the general; seeing the general through the particular" (Mason et al., 2010, p. 232). I shall return to the other two main powers shortly.

The ten natural powers are part of three phases of work when faced with a question. Each of these three phases is complex and comprises sub-activities. Roughly speaking, the first phase, entry, is to really read and understand the problem. It is often in this phase the first cases of specializing are used, as specializing can help to understand the core of the question and set the foundation for the second phase, attack. The attack phase starts when you really understand the question and it has become your own and ends when you have resolved or abandoned it. In this phase, the two other main powers, conjecturing and convincing, come into play. Based on specializing, you can now start conjecturing on and convincing yourself and others of the solution to the given question. The third phase is the review phase, which is to look back and reflect upon the resolution and strategies, as well as to extend the question for new contexts. It is often in this phase when generalizing takes place. The phases and natural powers do not follow strict linear working progress, as illustrated in Figure 1.

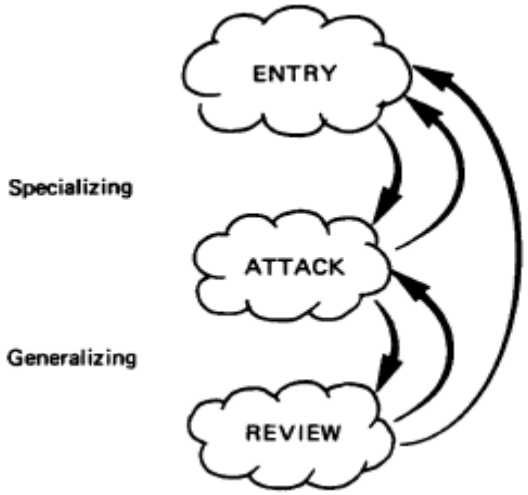

Figure 3: Three phases of work when tackling a question (Mason et al., 2010, p. 26)

Tall (1991) defines a similar cycle of activities in advanced mathematical thinking, going from considering a problem to formulating conjectures and finally to refinement and proving. He states that these activities also are part of elementary mathematical thinking, referring to Mason and colleagues' work, but distinguishes the two forms of mathematical thinking by viewing it as the transition "from *describing* to *defining* and from *convincing* to *proving* in a logical manner based on those definitions" (Tall, 1991, p. 20). Dreyfus (1991) consider advanced mathematical thinking and how it differs from elementary mathematical thinking in the same way. In addition, Dreyfus consider mathematical thinking as a set of different learning processes, with representing, abstracting, generalizing and synthesising as the main ones. Yet, another model of mathematical thinking is the triad consisting of problem solving, modelling and abstraction, presented by Drijvers and colleagues (2019). Thus, considering mathematical thinking as presented in mathematics education research, it includes many different mathematical activities. In relation to the KOM framework and the mathematical thinking competency, mathematical thinking, as a general term, covers much more than

just the mathematical thinking competency. The different views on mathematical thinking also include aspects of the other competencies of the KOM framework. Thus, to specify the broad perspective of mathematical thinking in relation to the mathematical thinking competency, I find that the entry phase including the natural power of specialising and the review phase with generalizing (Mason et al., 2010) are more in line with the aspects of the mathematical thinking competency focusing on the nature of mathematical questions and answers, on the conditions and on generalisation and abstraction. Whereas the attack phase, including the natural powers of conjecturing and convincing, is more related to the problem handling and the reasoning competency.

REVIEW METHOD

I conducted the first initial literature search, on which this paper is based, in WebofScience in spring 2021. With the focus on educational uses of CAS and DGE in relation to mathematical thinking, I searched for either of the following words in title or abstract: "Dynamic software", "Dynamic geometry", DGS, DGE, GeoGebra, CAS, "Computer algebra", "Symbolic calculator", "Graphical calculator", Nspire, Maple, and for "mathematical thinking" also in title or abstract. To make sure to get results of mathematics education, I combined the above search words with math* AND educ* as Topic. The search was restricted to English, but otherwise, there were no restrictions regarding type or year of publication. This search gave 16 results, of which nine were included for the review. Screenings of title and abstract (2 excluded) and of full text (5 excluded) followed the two inclusion criteria:

1. The study uses some kind of definition or explanation of 'mathematical thinking' (implicit or explicit). The study should focus on mathematical thinking as a means or a goal and not only use the term in an unspecified manner.
2. The study uses DGE or CAS. DGE and CAS should be part of the focus for the study, and not just be part of the empirical setting with the study focusing on other aspects.

The included studies were classified in relation to whether the study is theoretical or empirical; the purpose; research method and objects; mathematical content; name and type of tool; which theoretical construct/approaches are used, if any; and implicit or explicit definition of the term "mathematical thinking". Subsequently, the studies were coded from the perspective of Mason and colleagues' (2010) terminology of phases of work. However, as I do not find the attack phase the most relevant for the mathematical thinking competency, the analysis focuses on how DGE and CAS are used for the entry and the review phase of thinking mathematically. This was done by analysing where the tool was used in the problem-solving or investigating process. If the tool was used as part of investigating a given problem or a certain mathematical concept, it was coded as *the entry phase*. If the tool was used for finishing or studying a resolution to a problem, or for expanding or generalising, it was coded as *the review phase*. In this way, I attempt to keep the focus on the mathematical thinking competency and avoid analysing tool use for competencies such as reasoning and problem handling.

ANALYSIS AND RESULTS

All nine included studies are empirical with participants from lower secondary school, pre and post teacher programmes and first year of engineering. All nine papers are peer-reviewed, eight are articles in journals of mathematics education, and the last is a conference paper. Three of the studies make use of CAS, and the remaining six use DGE. In the following, I analyse which papers illustrate tool use as part of the entry phase (five studies), and as part of the review phase (all nine studies).

The entry phase and the use of DGE and CAS for mathematical thinking

Two of the studies (Ismail et al., 2014; Zeynivandnezhad & Bates, 2018), which use Maxima, a CAS program, apply the mathematical thinking powers (Mason et al., 2010) as the framework for their analyses on students' mathematical thinking when using CAS for solving problems with differential equations. In both studies (Ismail et al., 2014; Zeynivandnezhad & Bates, 2018), the authors find that the students use specializing powers to identify which procedures and commands to use for solving the differential equation. Hence, the use of CAS becomes a part of the attack phase to solve the task and justify the solution rather than part of the entry phase itself. In these cases, Zeynivandnezhad and Bates (2018) find that the students, transfer their procedures of pen-and-pencil to the CAS environment. Hence, the students' mathematical thinking of the entry phase influences their way of using CAS for the given problem.

In total, three studies (da Silva et al., 2021; Reyes-Rodriguez et al., 2017; Santos-Trigo & Reyes-Rodriguez, 2016) use DGE as part of the entry phase. None of the studies using CAS applies it directly in the entry phase. For the three studies, DGE is used for visualization and in an explorative manner. da Silva and colleagues (2021) illustrate a situation where the problem-solvers start by investigating the problem using DGE, GeoGebra, as part of the entry phase, before they move to Microsoft Excel for the attack phase. In this way, the tool influences how they understand the problem and go about it. The visualization and exploration starting in GeoGebra help them enter the question, by which they can navigate on their understanding of what a good answer would demand, which indicate aspects of the mathematical thinking competency.

Letting the students start by working with DGE can be helpful in two ways when considering and solving geometric problems (Reyes-Rodriguez et al., 2017; Santos-Trigo & Reyes-Rodriguez, 2016). (1) The problem solver can use the DGE to explore the given initial conditions for the problem by constructing them geometrically and then from here start working the problem. As for the example above, the DGE makes it easy to construct the given geometric object to explore before starting solving the task. (2) Another way is by relaxing the conditions and then by dragging elements of these constructions in the tool, the problem solver can relate and explore how specifying the conditions can lead to the wanted construction. For these two strategies, as for the example above, using DGE can take part in the entry phase and help to devise a plan for the further process and to focus on the conditions for the problem. In the perspective of the mathematical thinking competency, this is related to the ability to distinguish between different mathematical claims, as they need to understand the role of the conditions, and to relate to the nature of the expected answer to a given question.

The review phase and the use of DGE and CAS for mathematical thinking

Like in the entry phase, the use of DGE or CAS is not necessarily part of the review phase, but rest upon the explorations carried out with the tool. For instance, in the two studies using Maxima (Ismail et al., 2014; Zeynivandnezhad & Bates, 2018), findings indicate that the visualization capabilities of the tool help the students to understand the solution to a given differential equation. Here, Maxima is used in the problem-solving process itself. The students are able to conjecture after experimenting with different provided differential equations in Maxima, and they use the tool for visualization to verify and convince themselves of the solution, as well as a self-checking tool. Thus, as argued for the entry phase, the findings of these two studies illustrate how students show mathematical thinking competency to reflect upon CAS. Also in the study by Fonger (2018), CAS is used in the attack phase, which is emphasized by the principle of predict-act-reflect-reconcile, where act is the activity of using CAS. Here, CAS is used to make the students focus on the review phase, where the students are encouraged to go back to check and reflect upon their results of the prediction and of CAS.

For the studies using DGE as part of the review phase (da Silva et al., 2021; Fonseca & Franchi, 2016; Reyes-Rodriguez et al., 2017; Santos-Trigo & Reyes-Rodriguez, 2016; Turgut, 2019; Yao & Manouchehri, 2019), the features of dragging (including using sliders) and measuring are the most emphasized tools for supporting mathematical thinking in the included literature. Yao and Manouchehri (2019) find that the students' generalizations are shaped by how they use the tool and for what purpose. For instance, they find examinations of empirical cases by dragging and measuring in DGE can facilitate generalizations upon the single cases. Working with dynamic representations controlled by a slider representing a central variable can help students represent and visualize the concept of convergence of sequences (Fonseca & Franchi, 2016). Such visualization and flexibility of switching between different representations, such as algebraic, numeric and various kinds of graphic ones, contributes to the abstraction of concepts and thus, to their transition from elementary to advanced mathematical thinking (Fonseca & Franchi, 2016).

Santos-Trigo and Reyes-Rodriguez (2016) emphasize the review phase by arguing that finding the solution to a given problem should not be the end of the process. Instead, students should be encouraged to look for multiple ways to represent and solve the problem. For this, DGE makes it possible to construct and explore different dynamic models of a problem and to extend the problem from one context to another. Exploring a problem with DGE can support learners in questioning the task and the given conditions, if the learning environment, the teacher and the tasks support it. Such questions often come on the background of extending the conditions or context the original problem is given in (Reyes-Rodriguez et al., 2017). Working with DGE for problem solving in these ways can support aspects of the mathematical thinking competency, such as the ability to pose generic mathematical questions and to relate to the scope of the involved concepts in new contexts. However, it is important to emphasize that working with DGE itself does not make new solution paths and questions emerge. The students need to be encouraged to look for such new paths and questions.

The study of da Silva and colleagues (2021) illustrates how studying different solution paths of estimating π by squaring the circle can lead to further exploration of cubing the sphere. Similarly, Turgut (2019) finds that by dragging and focusing on the specific coordinates of the Cartesian plane of DGE, students can consider co-variation of independent and dependent variables as a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, which can lead them to extend the concept of mapping from being a function to a matrix transformation. Such extensions are important aspects of the review phase, and relates to the aspects of the mathematical thinking competency, such as generalizing and the ability to relate to the scope of a given concept within different contexts.

CONCLUSIVE DISCUSSION

The results of the included literature illustrate that the visual and explorative possibilities of DGE and CAS have potentials relating to different aspects of mathematical thinking. The three phases of entry, attack and review (Mason et al., 2010) cover thinking involved in all kinds of mathematical activities, including problem solving, reasoning and representing. Compared hereto, the mathematical thinking competency (Niss & Højgaard, 2019) is narrower, as the aspects of this competency are primarily connected to the phases of entry and review, why literature was coded from the perspective of these two phases and not the attack phase. By focusing on entry and review and on the mathematical thinking competency, the analysis concentrates on these specific aspects of mathematical thinking.

In relation to the entry and review phases, the use of DGE and CAS is not necessarily directly involved. This holds in particular for CAS, where the entry phase relates to the use of CAS in the forms of which techniques and commands to apply in the attack phase. In the review phase, the reflections are upon the results and visualizations provided by CAS, which then can make it possible

to extend and generalize the question, concepts and processes. On DGE, the literature results indicate that constructing visual representations, dragging and measuring are advantageous for the entry and the review phases. Dynamically exploring a mathematical question, its conditions and the involved concepts and properties can help to understand the question, questioning the conditions and relating to the expected answer. Hence, dragging and measuring in the entry phase may support aspects of the mathematical thinking competency. Similar actions in DGE throughout the attack phase can lead to a review phase including aspects of the mathematical thinking competency, such as questioning the initial situation, the context or domain it is given it, and relating the involved concepts and their varying scope in different contexts. Furthermore, such inquiries can support generalizing and abstracting, reflecting upon the different instances and representations examined in the DGE. Thus, the results of this study indicate that DGE and CAS, respectively, may have different potentials for different aspects of the mathematical thinking competency, which should be investigated with a more thorough literature search in other databases and proceedings of mathematics education conferences.

This paper is based on an initial search as a pilot study to investigate the application of the Thinking Mathematically phases 'entry' and 'review' to capture processes of the mathematical thinking competency of KOM. That the search only provided 16 studies can be the result of the search words. I did not use 'digital technology' or similar general terms, as I wanted to narrow down the results to be about DGE and CAS. Furthermore, studies concerning elements of mathematical thinking or mathematical thinking competency but not explicitly written into this theme may not have been found using this strategy, which should be taken into consideration for the searches of further work. Applying the mathematical thinking competency in more detail for the analysis of the future literature review could provide a categorization of using DGE respectively CAS in relation to specific aspects of the mathematical thinking competency.

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GRAPHING CALCULATOR IN THE CONNECTION BETWEEN GEOMETRY AND FUNCTIONS WITH THE CONTRIBUTION OF SEMIOTIC MEDIATION

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This communication aims to analyze the resolution of a task with the artifact, graphing calculator, where it was intended to make a connection between the domains of Geometry and Function. We sought to understand how the student promoted artifact signs through the development of usage schemes and instrumented action schemes. Later, in the collective discussion, we sought to analyze how the teacher promoted the transition from artifact signs into mathematical signs. Using a qualitative research methodology of an interpretive and descriptive nature, a case study modality was used focusing on the work of two pairs of students. The results show that the graphing calculator worked as an instrument of semiotic mediation and the teacher's orchestration in the collective discussion was essential to the development of the semiotic potential of this artifact, resulting in the construction of mathematical knowledge.

Keywords: Graphing calculator, instrumented action schemes, semiotic mediation, semiotic potential of the artifact, usage schemes.

INTRODUCTION

According to the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2007), calculators and computers influence the way the subject of Mathematics is taught and they improve students' learning. They are devices that enable the visualization of mathematical ideas, the organization and analysis of data, the making of calculations in an efficient and accurate manner and may serve to support students' investigations in any area of Mathematics, namely in Geometry and Measurement, Statistics, Algebra, Numbers and Operations. On the other hand, they enable a softening of some existing boundaries in these areas, allowing students to use their ideas about a certain area to better understand another area of Mathematics. And yet, teaching that values interrelationships between various mathematical ideas, promotes a deeper and lasting understanding, and students, in addition to learning mathematics, also learn to recognize the usefulness of mathematics. According to NCTM (2017), which is a Mathematics program of excellence, it promotes the use of technology as it favors learning, the understanding of mathematical ideas, mathematical reasoning and the communication of reasoning.

In this communication, we analyze the resolution of a task that took place in the 7th grade of elementary education or middle school (Portugal's education system), within the scope of a broader teaching experience, in a public school in the district of Setúbal, in Portugal. An innovative, learning environment of an exploratory nature was promoted, using the Texas Instruments TI-nspire graphing calculator, at a level of education where the use of this artifact is not foreseen, according to the curriculum prescribed in this country (Ministério da Educação e Ciência, 2013). However, in a recent study carried out by the first author, she concluded that the performance of tasks with the graphing

calculator, in elementary education, allows students to reason, reflect, learn and understand mathematical ideas (Pedro, 2020).

We intend to analyze and understand how the integration of the mediating artifact, graphic calculator, promoted the construction of mathematical meanings in the resolution of a task, in which a connection was made between the domains of Geometry and Functions.

The theoretical lines that make up the theoretical frameworks of Instrumental Genesis and the Theory of Semiotic Mediation supported the data analysis. In this sense, we have as goals: 1) to understand how the usage schemes and instrumented action schemes mobilized by the students contributed to the development of the semiotic potential of the artifact, graphing calculator and how 2) in the collective discussion, the teacher guided the evolution of personal meanings, related to the task and the artifact, graphic calculator, to mathematical meanings.

THEORETICAL FRAMEWORK

Instrumental Genesis: Usage Schemes and Instrumental Action Schemes

Instrumental genesis translates into a process through which the subject appropriates an artifact and transforms it into an instrument in solving tasks (Drijvers et al, 2010). The construction of an instrument is characterized by a mixed entity, composed by the appropriation of an artifact, material or symbolic, by the subject, through schemes (Rabardel, 1995). For Drijvers and Trouche (2008) the usage schemes are directed towards the management of the artifact, and the instrumented action schemes are mental schemes whose actions are directed towards the accomplishment of the task. Mental schemes emerge according to the subject's personal meanings and can be spontaneous or mathematical.

Theory of Semiotic Mediation: Semiotic Potential of the Artifact and Didactic Cycle

The Theory of Semiotic Mediation aims to describe and explain the process triggered by a student, which begins with the use of a specific artifact to perform a task and leads him to the appropriation of a specific mathematical content (Mariotti & Maffia, 2018). In carrying out a task, when the transformation of the artifact into an instrument occurs, signs associated with the usage schemes can emerge (Mariotti, 2002). In this sense, the artifact plays a dual role, both as a means of accomplishing a task and as a semiotic mediation tool to fulfill a didactic objective (Mariotti & Maffia, 2018).

From an individual point of view, personal meanings (signs of the artifact) arise that are related to the use of the artifact, namely in what the goal of carrying out the task is concerned. On the other hand, from a specialist's point of view, mathematical signs emerge that may be related to the artifact and its use. In this sense, there is a double semiotic relationship articulated by the artifact, called the semiotic potential of the artifact, which is characterized by the easiness it has in associating mathematical meanings evoked by its use, culturally determined, with personal meanings that each subject develops when using the artifact (instrumented activity) while carrying out specific tasks. The teacher is responsible for the development of the semiotic potential of the artifact. In the collective discussion, the production of signs of the artifact must be promoted, and the evolution of these signs should be guided by mathematical signs, using two pairs of *complementary actions*: *action of returning to the task and the focalization action* and *soliciting a synthesis and offering a synthesis* (Mariotti & Maffia, 2018). This process, called semiotic mediation, develops through the iteration of Didactic Cycles (Figure 1).

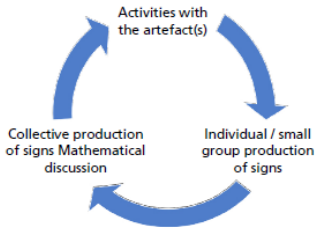


Figure 1. The Didactic Cycle (Mariotti & Maffia, 2018, p. 53)

METHODOLOGY

Using a qualitative research methodology of an interpretive and descriptive nature, the case study modality was adopted, focused on the work of two pairs of students (Creswell, 2012). The fictitious names of Maria and Berta were used for one group and José and Pedro for the other group.

The task focused on a connection between the domains of Geometry and Functions and was carried out with the mediating artifact graphing calculator. The goal was to understand how the semiotic potential of the artifact graphing calculator was developed through the mobilization of usage schemes and instrumented action schemes by the students. And also how the teacher orchestrated the collective discussion promoting the transition from personal meanings related to the task and the artifact into mathematical meanings.

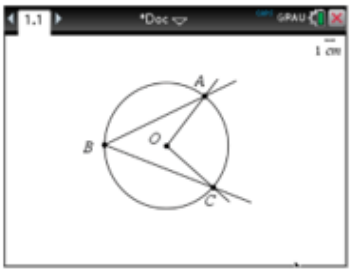
The task was carried out in the natural environment of the class, in a class of the 7th grade of elementary education (Portugal's education system), with 29 students. The students were informed that they had to solve the empirical part of the task in pairs and make an individual report. After analyzing the individual productions of each student, a collective discussion was developed.

The analysis of the data focused on the analysis of the audio recordings recorded, of the images of the graphic representations of the screens of the graphing calculator, of the reports written by the students and the field notes recorded in the logbook. The rules regarding the ethical issues involved in the entire data collection and treatment process were observed (Creswell, 2012). Regarding the *activities with the artifact* and *individual production / small group of signs*, it was analyzed how the students developed the semiotic potential of the artifact, graphing calculator, through the visualization tool and the dragging tool. The students promoted signs of artifact, through the development of usage schemes and instrumented action schemes, in the mobilized communication between peers, with the teacher and in the written reports. In what the *collective production of signs - mathematical discussion,* inherent to the collective discussion is concerned, the data analysis focused on the way the teacher promoted the transition from artifact signs into mathematical signs, according to the two pairs of complementary actions, fostering the construction of mathematical knowledge inherent to the didactic goal of the task (Mariotti & Maffia, 2018).

The Task Presented to Students

The goal of the task (Figure 2) was based on the distinction between inscribed angle and central angle in a circumference. And yet, in the recognition that the measure of the amplitude of an inscribed angle is half the measure of the amplitude of the central angle, which corresponds to it, in a circumference, having been asked for a mathematical model to suit the situation.

1.Using the graphing calculator, build the geometric figure you see on the right image here.



a) What is the difference between an inscribed angle ABC and a central angle AOC in a circumference?

b) What is the relation between the measurement of the amplitude of an inscribed angle ABC and the measurement of the corresponding amplitude of the central angle AOC?

c) What is the mathematical model that applies best to this situation?

Hint:

• On **Page 1**, of the **Geometry** app, define the variable **b** for the **angle** ***ABC*** and the variable **o** for the **angle** ***AOC***.• On **Page 2**, insert the app **Lists and Spreadsheets** and make a data capture of the values of the variables **b** and **o**.• On **Page 3**, insert the app **Data and Statistics** and draw a function taking into account the variables **b** and **o**.

Figure 2. The task presented to the students

According to the theoretical frameworks that supported the analysis of data in this study, in terms of semiotic mediation, the students were expected to**:**

In question a), according to the calculator's semiotic potential regarding the visualization tool and the dragging tool, the students were expected to mobilize the personal meaning of the angle concept and construct the mathematical meaning of inscribed angle and central angle in articulation with the Euclidean Geometry.

In question b), the students were expected to bring out the personal meaning of the concept of measurement of the amplitude of an angle, which aims at encouraging the transition from the geometric meaning of an angle to the meaning of measurement of the amplitude of an angle. When faced with a numerical system, they were expected to establish a possible relationship between the numbers that would appear as a result of the measurement process, which would be automatically performed by the graphing calculator. The personal meaning of division would then emerge, and the students would operationalize this calculation to relate the measurement of the amplitude between the inscribed angle and the measurement of the amplitude of the central angle which corresponds to it. Through the semiotic potential of the graphing calculator, a dragging, optimized by a Dynamic Geometry System (DGS), when dragging the figure on the screen, students were expected to realize that all properties intrinsic to the construction procedures remain invariant, in other circumferences. In this sense, they were expected to conjecture that there is a relationship that applies to any circumference: "In any circumference, the measurement of the width of an inscribed angle is half the measurement of the amplitude of the central angle, which corresponds to it".

In question c), according to the function of displaying on the screen of the graphing calculator, the students were expected to bring out the personal meaning of graphical representation and symbolic representation of a linear function. In this sense, the students were expected to bring out the personal meaning of ~~a~~ linear function of the property conjectured in question b), and were expected to take into account the relationship of dependency between the variables, that is, the abscissa (independent variable) and the ordinate (dependent variable) at each coordinate of the points represented on the Cartesian. And so the students were expected to conclude that the mathematical model that fits this situation is $f(x) = \frac{1}{2}x$ or $f(x) = 2x$.

ANALYSIS OF RESULTS

Regarding question a), students developed usage schemes when they opened **Page 1.1** on the graphing calculator and when building the circumference, they resorted to the **Geometry** app. They clicked:

menu - shapes - circumference. To place points *A*, *B* and *C*, they clicked **menu - points and lines - point on an object** and after, they clicked **menu - actions - text**. To build the *ABC* and *AOC* angles, they clicked **menu - points and lines - semi-straight**.

Later artifact signs emerged through the development of instrumented action schemes evidenced by the visualization tool, by mobilizing the personal meaning of the angle concept. Then, there were instrumented action schemes evidenced by the dragging tool where the students checked their conjectures on other circumferences and proceeded to construct the "definition" of central angle and angle inscribed on a circumference.

However, it was only in the collective discussion that students correctly constructed this definition. José pointed out the personal sign "cut" and the teacher, through a *focalization action*, made the transition to the mathematical signs, "intersect" and "secant".

Teacher: Very well! What about the sides of those angles? What can you conclude?

José: They always "cut" the circumference in two points. One is the vertex, and the other point also belongs to the circumference.

Teacher: (…) What does it mean to cut? I do not know, what is it?
José: Oh, teacher! It means that it "Intersects" the circumference in two points.
Teacher: Ah! OK! Now I understand! And what does it mean to intersect the circumference in two points?
José: Is it secant to the circumference!?

In view of the students'statements, the teacher requested a synthesis, and Berta made the following intervention:

Berta: So teacher: "In the inscribed angle *ABC*, the vertex *B* is over the circumference and the sides of the angle are secant to the circumference. At the central angle *AOC*, the vertex O is the center of the circumference, and the sides of the angle are secant to the circumference".

With the teacher's insistent orchestration, Berta realized that each side of the central angle is constituted by a semi-straight line originating in the center of the circumference, whose extension intersects it at a single point.

In question b) it was intended that students related the measurement of the amplitude of an inscribed angle *ABC* and the measurement of the amplitude of the central angle *AOC*, which correspond**ed** to it. Still on **Page 1.1** of **Geometry** app, students developed signs of the artifact. They evidenced instrumented action schemes when they brought out the personal meaning of the concept of measuring the amplitude of an angle, with the goal of making the transition from the geometric meaning of angle to the meaning of measurement of the angle amplitude. They developed usage schemes by clicking on **menu - measurement - angle** to measure the amplitude of the inscribed angle *ABC* and the central angle *AOC*, to be able to define the variables **o** and **b**. Subsequently, they developed usage schemes inherent to the definition of variable **b**, for the amplitude of the inscribed angle *ABC* and variable o, for the amplitude of the central angle *AOC*. In this sense, they pointed the cursor to the value of the measurement of the angle amplitude and clicked on **var - save var** and wrote **b** and / or **o**, respectively, and then clicked on **enter**. Subsequently, the students were faced with a numerical system resulting from the measurement of the angles'amplitudes, which was done automatically by the calculator. Then, they developed instrumented action schemes based on the personal meaning of the concept of the division operation between 2 variables, **o** and **b** or **b** and **o**, in order to relate the measurement of the amplitude between the inscribed angle and the measurement

of the amplitude of the central angle which corresponds to **it**. In order to operationalize these schemes, they used usage schemes to calculate $\frac{b}{o}$ and / or $\frac{o}{b}$. They clicked on **menu - actions - text** and wrote $\frac{b}{o}$ or $\frac{o}{b}$ and then clicked on **menu - actions - calculate**. Finally, they developed *instrumented action schemes*, based on the dragging tool, when changing the dimensions of the circumference or moving the points *A* and *C*, belonging to the sides of the angles and concluded that the $\frac{b}{o}$ and $\frac{o}{b}$ relationships applied to any circumference and remained constant (Figure 3).

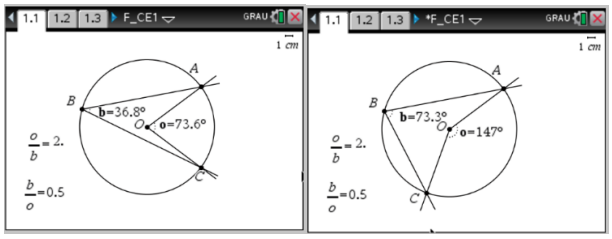
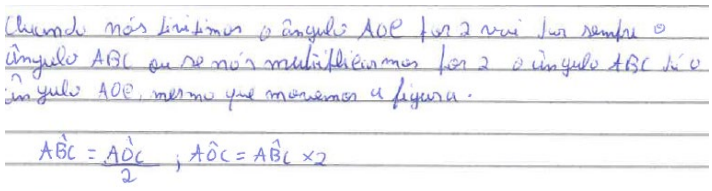


Figure 3. Resolution on the graphing calculator (use of the dragging tool) of item b) of the task, by Pedro

Mathematics signs appeared when the students found that the measurement of the amplitude of an inscribed angle *ABC* is half the measurement of the amplitude of the central angle *AOC*, which corresponds to it. (Figure 4).



[follows English translation of Pedro's Portuguese handwriting: When we divide the AOC angle by two, the result will always be ABC angle; or if we multiply the ABC angle by two, we will obtain the AOC angle, even if we move the geometric figure.]

Figure 4. Pedro's handwritten resolution for question b) of the task

In question c), the students followed the suggestions of the statement and developed usage schemes inherent to the manipulation of the graphing calculator. They built **Page 1.2 - Lists and Spreadsheets** app and captured the values of variables **o** and **b**, designated by **ango** and **angb**, respectively (Figure 5). To build **Page 2,** they clicked **ctrl - + page - Lists and Spreadsheets**. To make a data capture in relation to the values of variables **b** and **o**, they clicked **menu - data - data capture - automatic**. Subsequently, students developed usage schemes when they built **Page 1.3 - Data and Statistics** app (Figure 6) and defined variables b and o as independent or dependent variables. In Maria's, José's and Pedro's resolutions, artifact signs emerged through the development of instrumented action schemes. The students mobilized the personal meaning of the symbolic and graphical representations of a linear function and related question b) (Figure 4) with question c) and defined a linear model that adjusted to the situation.

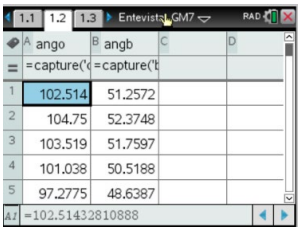


Figure 5. Resolution in the graphing calculator (data capture, in the Lists and Spreadsheet application), in paragraph c), by José

The students outlined the model: $y = 0{,}5\ x$ $(\boldsymbol{b = 0,5\ o})$ and developed usage schemes, when they placed the cursor on the various points of the function and found that the value of the dependent (ordered) variable was always half the value of the independent variable value (Figure 6).

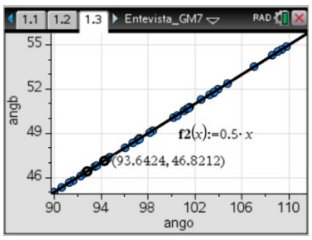


Figure 6: Resolution on the graphing calculator, question c) of the task, by Maria, in the Data and Statistics application

However, Berta randomly placed the variables. When a representation on the graphing calculator screen appeared, similar to the graphical representation, of the analytical expression of a linear function, the student developed the personal meaning of linear function. She established a relationship with the answer given in paragraph b) and defined the linear model. $y = 2x$ $(\mathbf{ango = 2\ angb})$ The student showed an absence of critical thinking. She did not realize that the placement she made of the variables or that the model that adapted to the situation would be $y = 0{,}5\ x$ $(\mathbf{angb = 0,5\ ango})$. The student did not verify the relationship that existed between the abscissa and the ordinate, in the respective points of the graph. She only interpreted the image that appeared on the graphical calculator screen without taking into account the relationship between the variables, which she had highlighted in question b). However, this ambiguity was clarified in the collective discussion, with the teacher's orchestration through a *focalization action.*

Teacher: Berta (…) taking into account the mathematical model that you defined, where are the independent and the dependent variable**s** located? I mean, the variables b and o (…) **ango** and **angb**, respectively**?**

Berta: (…) What I did was y = 2x, because (…) the line passed through the origin of the reference, so it could only be a linear function (…) because in point b) I concluded that the measurement of the angle amplitude to the center is double the measure of the amplitude of the inscribed angle. I should have said that **ango** is the dependent variable and **angb** is the independent variable. But I know I was wrong! In the Data and Statistics application, my mistake was to have put the variables backwards and I didn't notice that the value of the abscissa is always half the value of the ordinate, in this clear situation!

CONCLUSION

The analysis of the data showed that the students, when taking advantage of the graphing calculator, mobilized *usage schemes* and *instrumented action schemes* based on the visualization tool and dragging tool, promoting the development of the semiotic potential of the artifact, graphing calculator. The increase in the collective discussion with the teacher's orchestration was decisive for the construction of the mathematical knowledge (*mathematic signs*). The graphing calculator artifact functioned as a semiotic mediation instrument where students validated the mathematical meanings of "central angle", "inscribed angle", "relation between the measurement of the amplitude of central angle and inscribed angle", "representation graph of a linear function", "dependent variable" and "independent variable". According to the didactic goal of the task, the students showed ease in moving between the various representations: geometric, tabular, graphical and algebraic, having

promoted the connection between the domains of Geometry and Functions. The graphing calculator is not a tool used in elementary education in Portugal, but its DGS facilitated the formulation of conjectures and their verification.The students learned mathematics with an understanding of it.

Acknowledgments
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FINDING THEOREMS AND THEIR PROOFS BY USING A CALCULATOR WITH CAS IN UNIVERSITY-LEVEL MATHEMATICS

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Keywords: Computer algebra system, proof, university-level mathematics.

DISCREPANCY BETWEEN THE ROLE OF PROOFS IN MATHEMATICS CLASSROOMS AND MATHEMATICS AS A SCIENCE

Analyses of curricula and final secondary-school examinations in Germany have shown that the status of proofs in mathematics classrooms changed in the last decades enormously. Whereas in the 1970s and 1980s, proofs had a permanent place in curricula for school mathematics as well as in related examinations, they are only marginally addressed in classroom activities since the Standards for General Certificate of Secondary Education were developed in 2003. They disappeared during the 2000s – except for a few exceptions – from classrooms completely (Brunner, 2014). However, proofs in mathematics as science are one of the most important instruments for gaining and verifying knowledge. Like Rav (1999, p. 6) said: "Proofs, I maintain, are the heart of mathematics, the royal road to creating analytic tools and catalysing growth." Therefore, in order to show learners an authentic view of mathematics, proofs should play a crucial role in mathematics classrooms as a tool for gaining knowledge in mathematics. Proofs should be taught with a view to conviction, explanation, systematisation, discovery as well as communication (de Villiers, 1990). It follows that mathematics teachers in the future need to have first-hand experience with proofs and acquire didactic approaches to teaching them at school. Thus, proofs have to play an important role both in the technical and in the didactic training of mathematics teachers because they need to make experiences in finding and proving theorems to develop ideas for teaching proofs at school themselves.

CAS-ASSISTED PROOFS IN UNIVERSITY-LEVEL MATHEMATICS

Because the usage of computer algebra systems (CAS) is widely recommended in secondary school (it is mandatory in Thuringia), freshmen at university already have digital skills related to solving mathematics problems assisted by CAS. Resorted to their existing skills and knowledge related to digital technology, students can get access to high-level theorems and their proofs because, from a didactical point of view, both above-mentioned processes (finding and proving theorems) can be supported by using calculators with CAS. At the same time, students gain insight into the proving process and develop some didactical approach to assist the proving process by CAS in mathematical classrooms. In this way, the usage of CAS can bridge the gap between school mathematics and university-level mathematics in both directions.

Calculators with CAS can effectively produce many examples of a mathematical situation, and hence, they can help students find theorems inductively. The analysis of the calculations can also suggest proving ideas for the assumed theorem. This process will be demonstrated below on a special property of Fermat numbers as an example, but the poster presented more examples affecting the number theory (divisibility of Fermat numbers, Fermat´s little theorem) and linear algebra (fundamental theorem of algebra, properties of the determinant). A usual topic in number theory at university is not just the definition, but are also various properties of Fermat numbers. Investigating their products

systematically—the calculation can be carried out by using CAS—, one can easily find the hypothesis, each Fermat number equals to the product of the previous Fermat numbers plus 2 (Figure 1).

	F(n)	F(0)F(1)...F(n)
0	3	3
1	5	15
2	17	255
3	257	65535
4	65537	4294967295
5	4294967297	1.84467441E+19
6	1.84467441E+19	3.40282367E+38
7	3.40282367E+38	1.15792089E+77
8	1.15792089E+77	1.3407808E+154
9	1.3407808E+154	1.7976931E+308
10	1.7976931E+308	3.2317006E+616

Figure 1. Comparison of the nth Fermat number with the product of the first n-1 Fermat numbers

Especially because the hypothesis is obviously valid for the first few Fermat numbers, using mathematical induction is suggested. That can also be carried out by using CAS. After defining the values of the Fermat numbers (Fermat(x)) and their products (Produkt(x)) (Figure 2), it should be shown, that Produkt(x)+2 equals Fermat(x+1). Using the induction hypothesis that Produkt(x-1)+2 equals Fermat(x), the factorising of Produkt(x)+2 leads actually to Fermat(x+1).

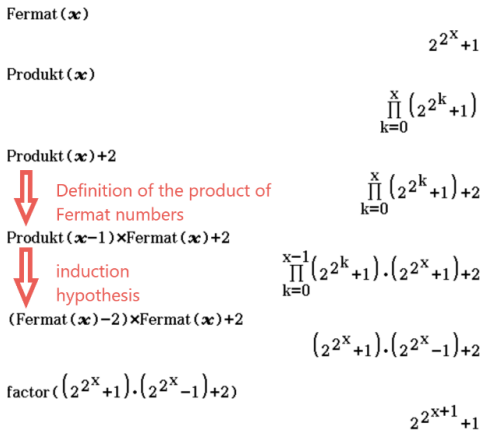

Figure 2. Proof based on mathematical induction, the CAS carries out the induction step

In general, one can conclude that CAS can assist in finding theorems by generating examples for certain mathematics situations at the university level. It can also assist in finding proving ideas in some of those situations. However, CAS has its limits if the corresponding idea is more complex and needs different strategies than exclusively algebraic transformations.

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MATHEMATICAL THINKING IN THE INTERPLAY BETWEEN HISTORICAL ORIGINAL SOURCES AND GEOGEBRA

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The Danish national curriculum, the so-called Common Goals, for mathematics in primary and lower secondary school is based on the Danish mathematics competencies framework (KOM), which describes eight different mathematical competencies. Two of these, the thinking competency and the reasoning competency, were merged together in the Common Goals. From a practice point of view, this is expected to have the consequence that the mathematical thinking competency may be addressed less explicitly as part of everyday mathematics teaching. In this paper, we present two empirical examples from a teaching experiment in 7th grade involving historical primary (so-called original) sources and a dynamic geometry environment (GeoGebra). The interplay between the original source and GeoGebra appeared to create a 'space' for activating and developing students' mathematical thinking competency.

Keywords: Dynamic geometry environments, mathematical reasoning competency, mathematical thinking competency, original sources.

INTRODUCTION

The Danish mathematics competencies framework, the so-called KOM-framework, saw the light of day some twenty years ago (Niss & Jensen, 2002) as part of a ministerial project. Since then, it has entered into the Danish mathematics programs at practically all levels from pre-school through tertiary programs involving mathematics. The KOM-framework defines eight distinct yet mutually related mathematical competencies. Two of these concern mathematical thinking and mathematical reasoning, respectively. Although these two competencies may be more closely related than other of the eight competencies, they are certainly not the same. While mathematical reasoning concerns the production and analyses of arguments, e.g. in the form of chains of statements, supporting a mathematical claim, mathematical thinking concerns the very types of questions and answers characteristic for mathematics, e.g. in the form of propositions, definitions, etc., also involving the scope of these.

Nevertheless, as part of an attempt to make the KOM-framework's competencies descriptions more digestible for practitioners, the 2014 revision of the national mathematics curriculum, the so-called Common Goals, for primary and lower secondary school, reduced the eight competencies to six (Børne- og Undervisningsministeriet, 2019). This was done by clogging competencies together. Hence, mathematical reasoning and mathematical thinking became one competency. Although the official arguments related to this 'merger' of the two competencies were related to elements of simplicity and implementability, it is not far fetched to argue that the embedment of the thinking competency into the reasoning competency may have had the effect that teachers in practice can 'cover' this competency by now focusing on reasoning alone [1].

Acknowledging that the competency of mathematical thinking may be a hard nut to crack as part of one's everyday teaching practice, we offer two empirical examples from a 7th-grade class to *illustrate how elements of mathematical thinking may be put on the agenda in a setting of reading primary*

historical texts by past mathematicians, so-called original sources, in combination with the use of a dynamic geometry environment. The two empirical examples stem from the first author's PhD project, which, at its outset, is concerned with developing students' mathematical reasoning competency through students' work with the content of original sources in GeoGebra. Yet, when analysing data from this project, it became evident that in some situations, mathematical thinking competency became a necessity for mathematical reasoning due to the content of the original sources and the context of GeoGebra. It is not a new finding that historical original sources can play a central role in students' learning of mathematics (e.g., Fauvel & van Maanen, 2000; Clark et al., 2018), nor is the fact that students' work with original sources can do so by contributing to the development of their mathematical competencies (Clark, 2015; Jankvist & Kjeldsen, 2011). Some studies also show the promising potential of the combination of students' work with original sources and the use of digital technologies (e.g., Chorlay, 2015; Jankvist et al., 2019; Thomsen & Olsen, 2019). The contribution of this paper is to exemplify the fostering of mathematical thinking among elementary school students, by having them work with Euclid's postulates and letting them convince each other through geometrical arguments while working with GeoGebra.

THE DANISH COMPETENCY FRAMEWORK

As mentioned, the KOM-framework (Niss & Højgaard, 2019) identifies and defines eight mathematical competencies. Besides the thinking and reasoning competencies, these are the mathematical competencies of: problem handling; modelling; representation; symbols and formalism; communication; and aids and tools. A mathematical competency is defined as "someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations" (Niss & Højgaard, 2019, p. 14). Each competency has both a 'receptive facet', where one, for example, "may think of following and assessing an alleged mathematical proof" and a 'constructive facet', where "the focus is on the individual's ability to independently invoke and activate the competency to put it to use for constructive purposes in given contexts and situations" (p. 19).

As already mentioned, we are, of course, particularly interested in the competencies of mathematical thinking and reasoning. The *mathematical thinking competency* concerns being able to pose and relate various kinds of questions characteristic of mathematics as a discipline, as well as the nature of the answers expected to these questions. This also involves the varying scope of mathematical concepts and terms within different contexts. It involves "distinguishing between different types and roles of mathematical statements (including definitions, if-then claims, universal claims, existence claims, statements concerning singular cases, and conjectures), and navigating with regard to the role of logical connectives and quantifiers in such statements, be they propositions or predicates" (p. 15). Finally, it is also related to the work of proposing abstractions of mathematical concepts, terms, and theories as well as to the generalisation of mathematical claims, theorems, etc. The *mathematical reasoning competency* concerns the analysis and production of arguments—that is, "chains of statements linked by interferences"—written as well as oral, to justify mathematical claims. "The competency deals with a wide spectrum of forms of justification, ranging from reviewing or providing examples (or counter-examples) over heuristics and local deduction to rigorous proof based on logical deduction from certain axioms" (p. 16). Niss and Højgaard (2019) describe wherein the reasoning competency differs from that of thinking:

> In contradistinction to what is the case with the mathematical thinking competency, the reasoning competency deals with the ability to analyse and carry out specific reasoning meant to provide justification for mathematical claims. Whilst such reasoning does indeed make intensive use of logic it goes far beyond logic by also implicating mathematical substance. It is important to stress

that the kinds of claims at issue in this competency are not confined to "theorems" or "formulae" but comprise all sorts of conclusions obtained by mathematical methods and inferences, including solutions to problems. (p. 16)

There is, of course, a relationship between the competencies and mathematical subject matter areas. Yet, it is up to the individual teacher, or task designer, to combine a competency (or several competencies) with a subject matter area (or several areas): "competencies and subject matter areas constitute two independent but interacting dimensions of mastery of mathematics. In the same way, as the competencies cannot be derived from subject matter areas, these cannot be derived from the competencies" (p. 22). The subject area of the following empirical examples is that of Euclidean geometry.

THE EDUCATIONAL SETTING OF AND THE DESIGN BEHIND THE EXAMPLES

Following the KOM framework's matrix structure of combining mathematical competencies with mathematical subject areas, the Danish Common Goals mathematics curriculum combine its (reduced number of) six competencies with three subject matter areas: Numbers and algebra; Geometry and measurement; and Statistic and probability. The examples we present, analyse and discuss in the following stem from a teaching experiment designed to take place in the matrix cell of: (mathematical thinking and reasoning) × (geometry and measurement). They stem from the first quarter of the school year in a 7th-grade Danish mathematics classroom consisting of a teacher and 22 students. The teaching experiment was a combination of students working in pairs with computers, paper and pencil as well as individual work and classroom discussions led by the teacher. The data collection consisted of students' screencast, their written answers to the assignments and video recordings of the classroom. From the data collected during the teaching experiment, we have identified two examples, which we analyse from the perspectives of the mathematical thinking and reasoning competencies. These examples concern students' work with Euclid's second postulate from Book I and Proposition 6 from Book IV: *To inscribe a square in a given circle*.

FIRST EMPIRICAL EXAMPLE: EUCLID'S SECOND POSTULATE

The first example concerns the classroom discussion, which built upon the students' work with understanding and explaining the five Postulates from Book I. Prior to this discussion, the students had used GeoGebra to visually support their explanations and understanding of each of the five postulates. The case concern the discussion about Postulate 2: *Any straight line segment can be extended indefinitely in a straight line*. In the discussion, the teacher used GeoGebra on the classroom screen:

Student 1: [The student reads Postulate 2 aloud] Any straight line segment can be extended indefinitely in a straight line.

Teacher: What does that mean?

Student 1: [Pointing to the line drawn in GeoGebra, while they were talking about Postulate 1] If you have a line, as the one at the bottom, then you can extend it as far as you want to.

Teacher: So, you can make it as long as you want to. How do you do that?

Student 1: Yes, just straight ahead...

Teacher: What means straight ahead?

Student 1: Indefinitely.

Teacher: So, if I should try and show it up here. How could I do it then?

Student 1: You can just drag in one of the points.

Teacher: Drag in one of the points. [Does it until it reaches the end of the screen.] Now we cannot go further here. However, I could zoom in—or rather zoom out. [Zooms out at the screen and drags the line further out.] So, you claim that you can drag into infinity?

Student 1: Ehm…

Teacher: Does someone disagree with that?

Student 2: It might be, for example, that the paper stops at some point.

Teacher: Yes, so we can say that there are some physical limitations. Can we call that theoretical?

Student 2: Yes, you can.

Student 3: Actually, the program also stops at some time. We cannot keep doing it inside the program either.

Teacher: That is correct... I cannot remember; were you the ones who tried to zoom out?

Student 3: Yes, that was us […]

Teacher: So, at one time, the program cannot zoom out any longer. You can say that the program's 'paper' ends.

The students' work with this task can be characterised as supporting their activation and development of the mathematical thinking competency. This competency, in particular, since they discuss if it is possible to extend a straight line into infinity or not, while at the same time touching upon the limitations of different tools (paper and GeoGebra) as opposed to what is going on 'in theory'.

They did not go further into the discussion of infinity as a concept, but this was not the aim of the task either. The aim was to put the students in situations where they, on the one hand, had the possibility to reach an (initial) understanding of the role of Euclid's postulates within Euclidian geometry, which they could use later in their work with reasoning and proving. On the other hand, the aim was to make them familiar with working with an original source in interplay with GeoGebra. More precisely, they were to use GeoGebra to explore the content of the postulates, and hence enhance their understanding of this. In the example above, it is, of course, GeoGebra's dragging functionality that comes into play. The idea behind the design of the teaching experiment was that this activation of the mathematical thinking competency would provide the students with a better foundation for their future work with both the 'receptive facet' and the 'constructive facet' of the reasoning competency.

SECOND EMPIRICAL EXAMPLE: EUCLID'S PROPOSITION 6 FROM BOOK IV

The second example stems from a situation involving parts of Euclid's Proposition 6, Book IV (Figure 1). In pairs, the students had constructed an inscribed square in a given circle in GeoGebra. Next, they were to describe, in their own words, how Euclid proved base *AB* equal to base *AD*, and, following the same line of thought, *BC* equals to *CD* [3]. The students also used GeoGebra to find and formulate arguments, which convinced them of *AB* being equal to *AD*. The task was that the students had to find

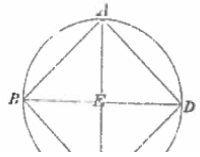


Figure 1: Illustration for Proposition 6, Book IV (Eibe, 1897b, p. 73).

arguments for why *AB* is equal to *AD*, and then why *BC* and *CD* also are equal and equal to *AB* and *AD*. The students did not necessarily have to use Euclid's proof of the proposition. They could, if they wanted to (and understood it). Rather, the aim of the task was to see how the students would handle reasoning and proving in the setting of GeoGebra. We present a downstroke in the work where students 4 and 5 provide their reflections supported by a progression of figures they constructed in GeoGebra (see Figure 2).

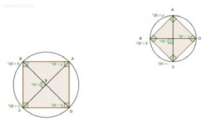
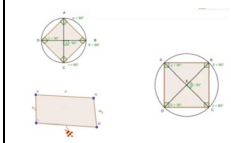
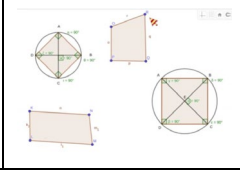
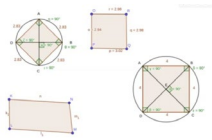
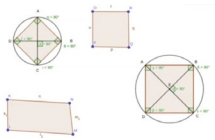
Picture 1	Picture 2	Picture 3	Picture 4	Picture 5
				

Figure 2: GeoGebra screen captures

Picture 1 in Figure 2 is a screenshot from when the students began their discussion. Student 4 begins by reading the task: "So, baseline *AB* equal to baseline *AD*". They then base their arguments on it being a square, so that baseline *AB* has to be equal to baseline *AD*, arguing that if the baselines were not equal, then it would not be a square but a rectangle. Next, they continue to *BC* and *CD* being equal to *AB* and *AD*.

Student 4: See, for instance, this line segment, which goes directly through [the diagonal]. You can see that they are equal at both sides. Otherwise, it would still not be an equilateral square.

Student 5 asked how you could see this and they have a few exchanges on this. Student 4 then added the figure in picture 2 of Figure 2.

Student 5: Why is this a rectangle?

Student 4: Because the sides are not equal.

Student 5: Why did this [the figure] come into our conversation?

Student 4: Because we said that this one… If it was like this. [Begins to draw picture 3 of line segments put together]. […]

They continue discussing. Student 4 mainly argues by referring to the length of the sides in the figure, and whether one can see if they are or are not alike. Student 5 keeps asking how it is possible to be convinced, if one does not 'see' it. Student 5 suggests that they can measure the sides and find the 'measure-button' in GeoGebra and begins to measure the lengths of the different sides (picture 4 in Figure 2).

Student 5: This is not true. [Laughs friendly.]

Student 4: Arhh. It is actually really close. These two are equal. [Points at *RQ* and *RO*. Picture 4, the new figure based on picture 3.]

Student 5: You see? You thought it was a square. How can you be sure then that the one you have constructed in the circle is, in fact, a square? You thought that the one you just measured was a square.

Student 4: See. [Measures the line segments in the circle in the lower right corner of picture 4.]

Student 5: But what if you can not be sure without measuring? How can you then know for sure?

Actually, the two students did not come closer to a conclusion. We interpret this example as focusing on the reasoning competency even though the students ended up discussing and arguing around the concepts of a square and a rectangle. Student 4 reasons by referring to previous knowledge about different types of squares and about the diagonal dividing a square into two equal triangles, although 'triangle' is not mentioned explicitly, and Student 4 does this by using GeoGebra both as a tool to draw and as a tool to measure length.

Student 4 also use different examples of squares to build "chains of statements linked by inferences" by saying and showing that if this is not the case, then it would be like this. This, of course, is not a chain of statements that goes deeper into an actual proof. Yet, it is a chain of similar types of statements, and thereby similar types of inferences. Student 5 activates the reasoning competency by asking: "How can you be sure then that the one you have constructed in the circle is, in fact, a square? You thought that the one you just measured was a square." This question opens possibilities for Student 4 to bring the circle into play, i.e. to unfold and deepen the chain of statements related hereto—even though this did not actually happen. Instead, they ended up using the measuring tool in GeoGebra as a way to attempt to justify and prove [4] that the square had equal lengths of the baselines. Hence, the students did not formulate a strictly deductive proof, nor did they refer to Euclid's proof in Proposition 6, Book IV, connecting that baseline *AB* is equal to baseline *AD*.

DISCUSSION

In the excerpt of the classroom discussion in example 1, there is no direct evidence of an activation and development of the students' reasoning competency. Rather we witness prerequisites for future reasoning by becoming acquainted with and understanding Euclid's postulates, i.e. elements that are more closely connected to mathematical thinking and deductive proving. Even though we cannot directly deduce that the students in the second example draw on knowledge from the first example, Student 4 might draw on this by building up arguments as chains of statements. More precisely, by constructing the new squares and reasoning about what would have been the case, if it was not a square. We also see how Student 5's questions support Student 4's reasoning and argumentation. In other words, this questioning can be seen as a product of Student 5's ability to use the receptive facet of the reasoning competency, and thereby support Student 4 to activate the reasoning comptency's productive facet.

Activation of the thinking competency is a major part of the classroom discussion in example 1. In particular, we see this when: 1) the teacher caused Student 1's explanation and claim that line can be dragged into infinity; 2) Student 2 brings the argument into the discussion that it might not be the case, because the paper stops at some point; and 3) Student 3 states that GeoGebra also will stop at some point. In example 2, one might say that Student 4's arguments on several occasions balanced on the edge between the thinking competency and the reasoning competency. This happens when Student 4 continuingly is circling around the definition of a square as being the main argument, yet finds it difficult to go deeper into this. When Student 5 asks: "Why did this [the figure] come into our conversation?" Student 5 is actually questioning the way Student 4 builds up the argumentation. Recall that the teacher in example 1 actually draws attention to the difference between 'in theory' and 'in reality' when discussing the extension of the line into infinity, where reality also is closely connected to the use of tools. This may be seen as one of the reasons that Student 5 keeps asking how

Student 4 can actually be sure. Both students of example 2 appear to have an awareness that what they are doing cannot be characterised as general proof when they end up measuring in GeoGebra. Hence, the receptive facet of the reasoning competency is in play, although the productive facet of the competency falls short.

CONCLUSION

In our analysis, it appears that the activation of the students' thinking competency acts as a foundation for them to put on a certain kind of 'spectacles' through which to interpret and analyse a given mathematical claim or statement. In this case, the spectacles concern the Euclidian geometry of both the *Elements* and GeoGebra, while working with justifying why the baselines in an inscribed square in a given circle has equal lengths. This also seems to support the students in developing an awareness of the types of reasoning, argumentation and proving—even though they did not fully succeed in carrying out a deductive proof. This conclusion, of course, draws on the full dataset from the teaching experiment, not only the two examples presented in this paper. Still, the two examples presented here made us realise the potential of the designed activities in relation to bringing students' mathematical thinking competency into play. This in particular in relation to how the students' mathematical thinking competency come to support the activation and development of their mathematical reasoning competency. From a strictly mathematical point of view, it may not be so surprising that mathematical proofs, i.e. reasoning, must rest on an understanding of the nature of mathematical axioms and definitions, i.e. mathematical thinking. Still, axioms and definitions, or even theorems and proofs, are not usually on the agenda in Danish lower secondary school. Yet, the work with Euclid's *Elements* put this on the agenda, while the use of GeoGebra provided the students with a familiar and natural setting for addressing the mathematical content of postulates and propositions in play. This is to say, the interplay between the original source, in this case, the *Elements*, and the dynamic geometry environment, in this case, GeoGebra, provided a possibility for the mathematical thinking competency to unfold. Still, this interplay between original sources and digital tools is not sufficient in itself; it requires didactical attention of both teachers and task designers. Nevertheless, the interplay between the history of mathematics and modern-day digital technology offers a fruitful opportunity for nourishing students' mathematical thinking—and in the process, perhaps make the notion of the mathematical thinking competency more 'accessible' to practitioners.

NOTES

1. The other two competencies, which were clogged into one, were the representation competency and the symbols and formalism competency. To some extent, this makes sense since symbols also are mathematical representations, yet the formalism aspect of this competency has probably suffered a similar fate to that of the thinking competency.

2. Students worked with Eibe's (1897a; 1897b) Danish translation of Euclid's *Elements*.

3. Working with one of Euclid's propositions in this way is inspired by Olsen and Thomsen (2017).

4. It is well described in the literature that such use of dynamic geometry environments' functionalities, e.g. measuring and dragging, may cause the students to 'jump to conclusions' at the expense of more formal, deductive reasoning and proving (e.g. Mariotti, 2006; Mason, 1991).

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AN EXAMINATION OF PRESERVICE MATHEMATICS TEACHERS' EXPERIENCES AT AN ONLINE LABORATORY SCHOOL

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The world's first Online Laboratory School (OLS) under the roof of a university was founded during COVID-19 pandemic. The OLS provided high-quality free mathematics courses to hundreds of low socioeconomic status (SES) students and internship opportunities for preservice teachers (PSTs). In this study, we present the structure of OLS and experiences of 43 PSTs (first, third and fourth year) who participated during Fall 2020. Third and fourth-year PSTs planned and taught middle school mathematics lessons under the guidance of supervisors for eight weeks, while first-year PSTs conducted observations. We administered a survey including some open-ended questions to inquire PSTs' views on their experience at the OLS. PSTs gave specific examples related to their professional development, and we were able to track those instances from video recorded teaching sessions. We found that this experience was an effective introduction to the profession for first-year PSTs and all others who learned about online mathematics teaching.

Keywords: Internship, online laboratory school, preservice teachers, teaching experience.

INTRODUCTION

It is widely acknowledged that teachers should develop competencies in order to meet ever changing needs of students in the 21st century. As teachers are recommended to practice student-centered approaches (National Council of Teachers of Mathematics, 2014), teacher education programs must also adjust their programs to meet the needs of preservice teachers (PSTs). In order to support PSTs in student-centered practices, it is recommended that teacher education programs should provide opportunities to integrate theoretical and practical knowledge for PSTs (Grossman et al., 2009). Internship is considered as a fundamental aspect of teacher education in order to bridge the gap between theory and practice, and where teacher candidates have the opportunity to be in the real world of the classroom (Flores, 2016). In line with the research recommendations, the researchers implement a teacher education model with rich internship experiences. Within the scope of the University within School Model (Özcan, 2013), teacher candidates are required to complete 1400-2000 hours of face-to-face internship experience with frequent opportunities for observation, reflection and feedback cycles. These experiences are intended to help PSTs to step into the teaching profession more easily. With the global spread of the COVID-19 pandemic, the Turkish Higher Education Council (2020) made a decision to interrupt practicum and internships in March 2020. There were several approaches both in Turkey and globally to address internship problems for PSTs who were approaching graduation and to maintain the quality of teacher education (Ersin et al., 2020; Vu & Fisher, 2021). For instance, in some cases, PSTs viewed videos of teaching and provided reflections (Vu & Fisher, 2021); others implemented microteaching practices where PSTs taught lesson plans for their peers (Ersin et al., 2020). During the COVID-19 pandemic, it was obvious that the quality of internship practices and lack of cooperation between school mentors and teacher educators became problematic (Özüdoğru, 2020).

In order to solve this problem and to provide teacher candidates with "teaching" experience during the pandemic period, we founded an Online Laboratory School (OLS) under the roof of a university in Turkey. Similar to Laboratory Schools (Mayhew & Edwards, 2007), OLS is founded and directed

by the Faculty of Education, where teacher educators and experienced teachers collaborate as supervisors in order to guide PSTs' practices. OLS provides teaching practice opportunities for PSTs who are expected to act as reflective practitioners (Schön, 1987) and work in collaboration with peers and supervisors in the implementation of new models of teaching. OLS is a virtual school where the teacher candidates' pedagogical and practical knowledge are supported, and classroom management skills are strengthened as they work with real middle school students in simultaneous and interactive teaching. In order to support PSTs' professional development in their online teaching competencies, there is a need to investigate PSTs' experiences and how they view such experiences. In this way, the programs can be improved according to the experiences and perspectives of PSTs.

Situated Learning and Technological Pedagogical Content Knowledge

Internship practices in the context of the University within School model (Özcan, 2013) and the OLS experiences, in particular, are designed by considering situated learning perspectives (McLellan, 1996). Similar to the realistic teacher education perspective (Korthagen, 2010), teacher candidates learn the profession of teaching not by thinking of teaching but by actively engaging in core practices of teaching in a gradual way and by reflection on such practices. PSTs need to interact with authentic contexts and real students in an online mathematics class and experience virtual teaching as a member of a group with similar goals and values in order to grow professionally in virtual teaching of mathematics (Kennedy & Archambault, 2012). In the context of OLS, PSTs share increasing responsibilities of conducting observations, engaging in planning and reflection meetings, acting as a teaching assistant, and as a teacher according to their cohort. In this way, different cohorts of PSTs (first, third or fourth year) have the opportunity not only to observe but also to participate, collaborate and reflect on some of the core practices of teaching, including planning lessons, preparing assessments, using interactive software to enhance student participation and teaching meaningful mathematics by considering student thinking in a synchronous way.

The realistic approach to teacher education (Korthagen, 2010) puts emphasis on interactions between teacher educators and PSTs as well as interactions among PSTs. It is also important that PSTs engage in systematic reflection practices as a group of learners. OLS provided a context for building an online learning community (McLellan, 1996) and fostering reflection during the core practices of teaching as it was easier to plan, observe, teach and reflect as a group of learners as a result of a virtual context without limitations of transportation and place.

An important goal in teacher education, particularly in today's world, is to develop PSTs' technological pedagogical content knowledge (TPACK). Competent teaching with technology requires much more than knowing how to use technological tools (Mishra & Koehler, 2006). TPACK refers to the knowledge needed for teaching with technology which

> requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones (Mishra & Koehler, 2006, p. 1029).

While engaging in collaborative virtual teaching activities has been found to support teacher candidates' technological pedagogical content knowledge (TPACK) (Grandgenett, 2014), there is still much to learn about how to prepare PSTs for online teaching. In order to support PSTs in developing TPACK, researchers and teacher educators need to investigate the effectiveness of different types of experiences provided during teacher education and how PSTs perceive such

experiences (Lantz-Andersson et al., 2018). While this study did not investigate PSTs' TPACK, their answers to the survey questions together with the OLS teaching experience example demonstrate how PSTs' TPACK and beliefs about their knowledge and competency as virtual teachers may have been supported by the OLS experiences. Based on the gap in the literature and considering situated learning perspectives and online learning communities, the specific research questions guiding the study are the following:

How do PSTs view their experiences of internship in the OLS in terms of professional development?

How do PSTs' views of and experiences in the OLS change based on the cohort they belong to (i.e., first, third, and fourth-year PSTs)?

METHODS

Context and Participants

During Spring 2020, we conducted a pilot project of the OLS for five weeks. Realizing its success in providing teacher candidates opportunities to engage in high-quality virtual mathematics teaching experiences, the OLS is continued for the 2020–2021 academic year. For the Fall 2020 semester, the OLS admitted 232 children (4^{th}, 5^{th}, and 6^{th}-grade students) from all over Turkey, and seven university supervisors and 43 PSTs participated in this school for a duration of 8 weeks (see Table 1 for the curriculum in the OLS for 5^{th} grade as an example, the first week was orientation). University supervisors and PSTs decided on the mathematics topics together before OLS began. The PSTs were from different years (first, third, and fourth year), and they did different tasks (first years only obliged to observe classes whereas third and fourth years had to do lesson plans and teach middle school mathematics lessons).

Grade	05.11.20	12.11.20	19.11.20	26.11.20	03.12.20	10.12.20	17.12.20
5^{th}	Numbers and operations	Operations with whole numbers	Operations with whole numbers	Operations with whole numbers	Fractions	Fractions	Fractions

Table 1. The mathematics curriculum of eight-week OLS

There were 15 virtual mathematics classes with 10-15 children in each class. These classes were held as an extracurricular activity outside of the children's formal mathematics class hours. In the OLS, we used Blackboard (BB) Collaborate as a platform where supervisors could support PSTs during PSTs' online teaching using a moderator chatbox communication tool not seen by children. This tool was also helpful for giving feedback to the children. Planning meetings, each took about 1.5 hours, started four weeks ahead of the teaching schedule. All PSTs joined these planning sessions and took the responsibility of planning the lessons under the close guidance of two supervisors. Later all fourth and most of the third year PSTs taught their lessons, and first-year students observed and assisted the lesson implementations. Following the observed lesson, a short reflection meeting took place with the PST, teaching assistants and a supervisor. Every week, a general meeting was also held for all PSTs and supervisors to discuss the implemented lessons in each class and grade level. All of the meetings and classroom sessions were video recorded.

Data Collection and Analysis

At the end of the OLS in Fall 2020, we designed and administrated surveys for all stakeholders: children, parents, PSTs and university supervisors to understand their experiences related to OLS. There were 43 PSTs, including ten fourth-year, 13 third-year, and 20 first-year PSTs participated in

OLS. In this study, we report the survey results of 33 of them who filled the survey (Seven fourth-year, 12 third-year, and 14 first-year). The survey had 15 items, about a third of which were Likert type.

For this study, we focused on the answers to the survey questions given below, which aligned most with our research questions, understanding PSTs views and experiences related to OLS. Also, we were able to track those instances from their video-recorded teaching sessions. Two of us did a content analysis (Cohen et al., 2007) and created themes related to PSTs' answers to the open-ended questions.

The four survey questions used in the evaluation of PST's OLS experience are as follows: a) In what ways did you improve professionally in your OLS experience? b) Provide at least one detailed example that helped you improve during the OLS experience. c) How was your OLS experience different from your traditional/regular internship experience at public or private schools? d) How do you think OLS can play a role in PST education? What would you recommend for the future?

RESULTS

In this section, we report the results of the survey especially focusing on these four questions. The views of the students are supported by experiences from the OLS. For Question a) asking the ways that PSTs improved the most professionally, technology use (f = 26; 79%) and online teaching methods (f = 27; 82%) were the two areas that PSTs think they improved themselves the most through this experience (see Figure 1).

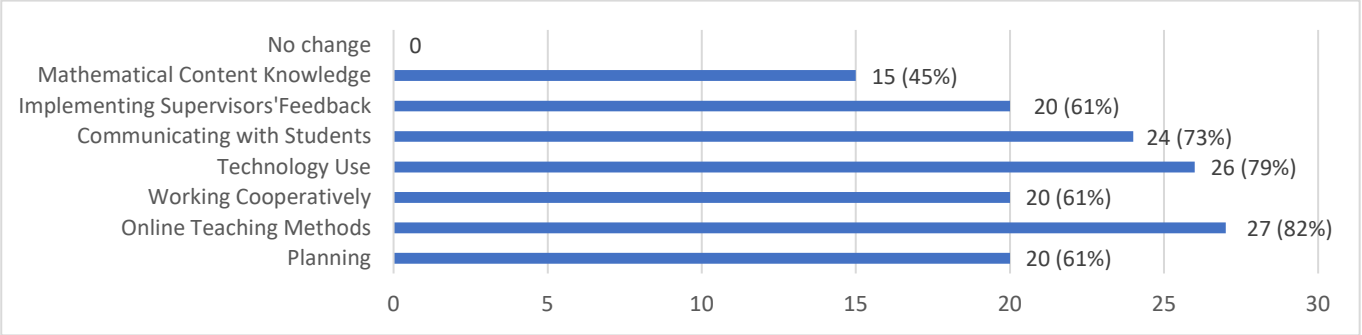

Figure 1. The result of PSTs' answers to a survey question (33 PSTs)

OLS Learning Experiences

For question b) asking PSTs to provide one example that helped them improve during the OLS experience, fourth-year PSTs mostly gave examples related to how they improved in anticipating and managing children's different approaches to problems in class as well as using that information in planning lessons in order to build on children's thinking. As an example, one PST wrote in the open-ended item of the survey:

> For example, I learned a lot when I planned a lesson related to how to calculate a fractional part of a quantity. It was difficult for me to be prepared for children's possible incorrect solutions. But when I created a plan where I thought through this situation, I realized that my teaching went well. I used problems where they calculated unit fractional parts of quantities, and it went as I planned. So I think I improved in teaching fractions (Teacher RK).

On the other hand, third-year PSTs mentioned more general aspects of their learning. For instance, they experienced how to communicate with students or manage a classroom that was usually

accessible to them in theory before OLS. Similarly, first-year PSTs gave general comments, but the foci were how their perceptions related to teaching or teaching profession changed, such as how to communicate with students, how to manage students and how to create lesson plans using technology.

Differences between OLS and Traditional Internship

For Question c) asking how PSTs' OLS experience was different than their traditional internship, generally both fourth and third year PSTs stated that OLS provided them opportunities to be more active such as when preparing lesson plans and being responsible as a classroom teacher compared to their experience at traditional internship schools. For example, one PST wrote in the open-ended item of the survey:

> The biggest difference is that in the traditional internship (face-to-face) you are an assistant teacher or candidate teacher, and you are bound to the mentor teacher. However, you are a teacher in the OLS. And we, as preservice teachers, have the right to make the decisions. We discuss everything from planning to the teaching, and we design them. While we taught once in the face-to-face internship in the whole semester, I taught every week in the OLS, and I was the teacher.

Furthermore, 79% of PSTs mentioned that OLS provided them opportunities for using more technological tools than their regular internship experience. Similarly, some fourth, third, and first-year PSTs responded that OLS was helpful for either receiving feedback from their supervisors during the class or having an opportunity to discuss their observations after the class in a more regular setting, in comparison to their traditional internship experience. For example, one PST wrote in the open-ended item of the survey:

> It was very different in many aspects. For example, the most important one was that I was able to get feedback (via moderator chat boxes) while I was teaching on the spot. By this way, I think the lessons were more effective. This was the most positive and enhancing aspect of the OLS setting.

Considering Future Role of OLS on PST Learning

Finally, for Question d) asking how OLS can play a role in PST education, most PSTs explained that OLS could provide PSTs opportunities to have online teaching experience and to feel like a real classroom teacher, and online internship should be embedded in the 4-year program even after COVID-19. The remaining fourth-year PSTs reported that OLS would be helpful in terms of getting feedback from supervisors about PSTs' teaching performance or evaluating their own teaching performance by watching their teaching videos. The remaining third and first-year PSTs explained that OLS could help PSTs learn how to integrate technology into their classes and support PSTs' lesson planning process, such as how to create and revise lesson plans. For example, one PST wrote, "It helps raising a new generation of teachers who knows how to use technology (for teaching)."

Technology use and Online Teaching Experience in OLS

Based on the analysis of the four survey questions mentioned above and a review of the other questions in the survey, we noticed that PSTs placed high emphasis on technology use and online teaching experience in OLS. As an example of how PSTs used technology in OLS, we present a lesson excerpt on simplifying and expanding fractions in a 5^{th}-grade mathematics class that a fourth-year PST taught. In this video recorded lesson, using Math Playground activity (and Conceptua Math website), the PST showed equivalent fractions on a rectangle and discussed simplifying and expanding with interactive figures.

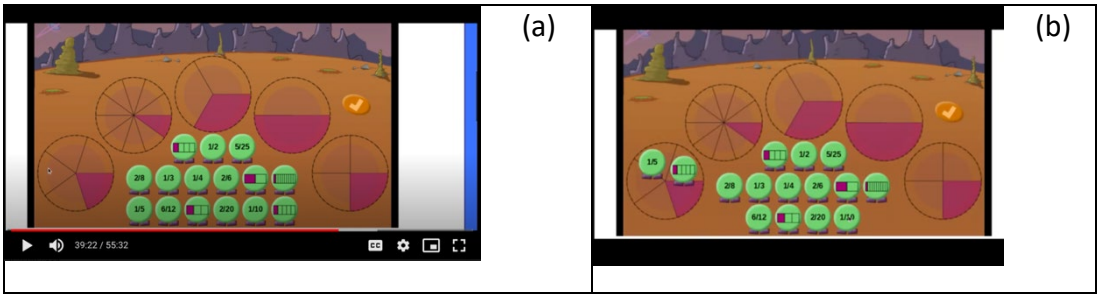

Figure 2. Placing (a) fraction symbols and also other representations, (b) green circles for 1/5.

Later in the video, she used the activity in Figure 2.a, where she presented circles, rectangles and symbols and asked students to place equivalent fractions to the fraction represented with the first circle (meaning 1/5). Students said 1/5 and placed a rectangle with one part pink (see Figure 2.b). The PST asked whether there were other fractions equivalent to 1/5. Some students said 5/25 but could not explain why. The PST moved 5/25 on top of the circle, but since students could not explain it, she put it back to the common area and moved on to the circle with ten pieces with one piece shaded.

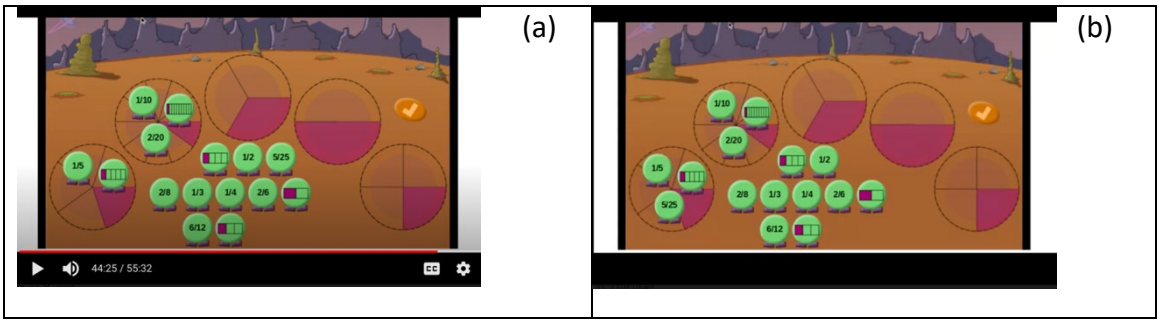

Figure 3. Screenshots (a) of placing green circles for 1/10 (for the second purple figure), and (b) taken at the end of the class after interacting with students

She again asked the children what fractions she should move on top of that circle. As she received the answers, she moved three green circles, 1/10 (first), a rectangle with one part shaded (second) and 2/20 (see Figure 3.a) on top of the 1/10 circle. It was the last moments of the video recorded lesson, and the teacher quickly revisited the figure and students' comments related to 1/5 and asked why she should have placed 5/25. Students talked about expanding the numerator and denominator of 1/5 by 5 (meaning multiplying both by 5). Then after having the students' approval, the teacher placed 5/25 on that circle (see Figure 3.b). PST placed 5/25 in Figure 3.b only after she made the discussion such that she showed 1/10 and 2/20 (by multiplying numerator and denominator by 2) were equivalent as 1/5 was expanded to 5/25 in a similar way. With the affordances of this technological tool, PST could move and place fractions (symbols or representations) as she was conducting the discussion with the students. She used it in an interactive way and could show different representations (symbol, rectangle, circle) in the context of teaching equivalent fractions. This was a common classroom observation in all recorded lesson videos that with PSTs used technological tools, such as Web 2.0 tools, Nearpod, Conceptua Math, GeoGebra, and so on. PST later reflected and mentioned in survey questions that the use of these technological tools in OLS helped them learn how to teach certain topics which were difficult to teach before.

CONCLUSION

Analysis of PSTs' answers to the survey questions indicated that OLS provided a very productive environment for all PSTs, especially for fourth and first-year PSTs. Fourth-year students already did

a year of face-to-face internship before OLS. The third-year PSTs reported professional growth but also seemed to be 'challenged' with the experience because after two years of intensive courses, they were just starting their teaching internship, and it was online with the OLS. This experience provided opportunities to all PSTs to be independent, taking ownership of teaching while planning and implementing the lessons within a teamwork. They improved their confidence in technology use and also online teaching methods. Most of the PSTs benefited from the online support they received through moderator chat boxes. In addition, as indicated from PSTs' answers to the survey questions, different groups of PSTs worked together as a team and were supported by many supervisors. Therefore, the OLS was a great opportunity to build a learning community which was difficult to do in the physical face-to-face internship. Lesson planning, technology integration to plans, and lesson implementations were also great opportunities, and PSTs lived these experiences in more focused ways compared to the face-to-face internship. All in all, PSTs recommended this experience to their peers and also suggested having part of the internship experience online even if things go back to normal. As consistent with PSTs' answers to one survey question explaining that technology use was one area they improved themselves the most in OLS, we observed how technology use increased PSTs' professional development on mathematics content such as fractions with opportunities to use different representations in an interactive way. As PSTs improved their teaching, the OLS represented an authentic context for PSTs to learn about not only technological tools but also pedagogical aspects.

It appears that involving PSTs in the lesson plan design process of OLS motivated them to consider using a wide range of fraction representations, which is necessary for developing fraction understanding. PSTs began to consider how to enhance students' conceptual understanding of fractions with multiple representations. This collaborative experience not only challenged but also encouraged PSTs to reflect on their technological, pedagogical skills and content knowledge (Hansen et al., 2016).

Our results are similar to prior research indicating PSTs' positive experiences and views related to the benefits of virtual internships (Jack & Jones, 2019; Kennedy & Archambault, 2012). We believe that the unique design of OLS, such as learning to implement interactive software and student-centered mathematics lessons as well as supervisor guidance in the online chat, may have contributed to PSTs' professional development. Furthermore, providing such experiences to PSTs in the context of a learning community of peers and supervisors was also found beneficial for PSTs, which confirms previous research recommendations of teacher learning as a group (Lantz-Andersson et al., 2018). This study provided insights into the PSTs' perspectives and experiences related to professional learning in the context of a uniquely designed virtual internship. Future research is needed to focus on and assess the professional learning of PSTs' as evident in their teaching and planning as a result of participating in the OLS.

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A VIDEOGAME AS A TOOL TO ORCHESTRATE PRODUCTIVE MATHEMATICAL DISCUSSIONS

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We structure an educational activity with a videogame starting from the hypothesis that technological tools could support in collecting data on students' achievements and designing mathematical whole-class discussions. In this paper, we present and discuss an example from our case studies, with the aim of analysing the role played by the videogame in supporting productive discussion concerning relational thinking. We conducted a whole-class discussion, and we noticed that the videogame could be a valuable tool for structuring the discussion. Indeed, the log files allow us to follow students' achievements and difficulties in solving tasks, and the dedicated web interface permits students to upload their responses and share them immediately.

Keywords: Educational videogames, web interface, mathematical discussion, relational thinking.

INTRODUCTION

Carpenter et al. (2005) define relational thinking as "looking at expressions and equations in their entirety rather than as a process to be carried out step by step" (Carpenter et al., 2005, p. 54). Furthermore, they sustain that "relational thinking involves using fundamental properties of number and operations to transform mathematical expressions rather than simply calculating an answer following a prescribed sequence of procedures" (Carpenter et al., 2005, p. 54). To develop this kind of thinking, the authors recommend going beyond traditional arithmetic practices and considering elementary arithmetic concepts as a bridge to learn algebra. Involving students in the solution and subsequent discussion of particular tasks seems crucial for focusing them on relations and fundamental properties of arithmetic operations, rather than focusing exclusively on procedures for calculating answers. In tune with this statement, the authors suggest engaging students in solving true/false and open number sentences, which provide a flexible context for representing relations among numbers and operations. As literature states (e.g., Lampert, 2001), involving students in well-designed tasks is not enough, the role of the teachers is central in orchestrating productive discussions. Although Carpenter et al. (2003; 2005) mentioned the importance of teachers' and students' interactions, it seems that the role of teachers and how they could interact with their students are not clearly defined. Since, in most of their papers, the authors present examples of interviews with students and excerpts from discussions with the teacher, we think that classroom discussions are the core of the development of this kind of thinking. Recent research shows that teachers' prompt for relational thinking had an immediate effect on students' relational thinking (Lin et al., 2015).

The aim of this paper is to develop technological tools that could support orchestrating productive class discussions. Our hypothesis is that these tools could help in collecting data on students' achievement and in orchestrating productive discussions. With this aim, we design a set of materials for students (called "unplugged tabs") and a digital tool for those who will orchestrate the mathematical discussion (called "monitoring web interface"). In this paper, after introducing the theoretical frame, we will present an example from our case studies. This example is aimed at investigating the role played by the videogame in supporting productive discussion concerning relational thinking and focusing on how we plan and implement it.

THEORETICAL FRAMEWORK

To reach our goal, we need to define what we mean by designing productive mathematical discussions. Thus, it seems crucial to identify the strategies for orchestrating productive mathematical discussions (Stein et al., 2008).

Stein and colleagues (2008) design a pedagogical model of five practices for discussion facilitation. They constructed this model based on the hypothesis that teachers need to perform a set of practices to prepare themselves for discussions and gradually learn how to become better discussion facilitators over time (Stein et al., 2008). The authors describe the following five key practices:

- *anticipating* likely students' responses to cognitively demanding mathematical tasks,
- *monitoring* students' responses to the tasks,
- *selecting* particular students to present their mathematical responses,
- purposefully *sequencing* the student responses that will be displayed,
- helping the class make mathematical *connections* between different students' responses and between students' responses and the key ideas.

The authors view each of the practices as "drawing on the fruits of the practices that came before it" (Stein et al., 2008, p. 321); together, these practices help to make discussions more likely, and teachers will be able to use students' responses to advance the mathematical understanding of the class as a whole.

Anticipating Students' Mathematical Responses

The first practice consists in trying to imagine how students might mathematically approach the tasks that they will be asked to engage in. Anticipating students' answers involves "developing anticipation about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn." (Stein et al., 2008, pp. 322–323)

In activating this practice, the authors suggest teachers to draw both on their knowledge of particular students' mathematical skills and understandings and on their knowledge of the research literature about typical students' responses to the same or similar tasks. For this reason, the study by Carpenter and colleagues seems to be appropriate to this practice (see, for example, Carpenter et al., 2003): they present lots of examples that illustrate teaching and learning activities focusing on tasks, students and teachers.

Monitoring Student Responses

Monitoring students' responses means paying attention to the mathematical thinking in which students engage as they work on tasks. This practice is commonly done by walking between the stalls while students work. The goal of this practice is to identify the mathematical learning potential of particular strategies or representations used by the students. In tune with this aim, observations and thinking-aloud procedures offer opportunities for gathering knowledge about students' thinking and ways of solving tasks, and these opportunities can be enhanced through technology.

In the literature, the use of computers to follow and register the students' working is often emphasised; for example, software that record audio and screen or produces log files (which consist of a list of events carried out by students) are available for education. For the teacher, however, observing student recordings could be time-consuming, while analysing the log files (or observing the analysis

produced by the software) could be a good compromise to enrich the practice of monitoring (Van den Heuvel-Panhuizen et al., 2011).

Purposefully Selecting and Sequencing Student Responses for Public Display

In these practices, teachers can select and then sequence particular students to share their work with the rest of the class. A typical way to select students' responses could be calling on specific students (or groups of students) or asking for volunteers to share with the class. The purposeful selection of students makes it more likely the mathematical ideas will be discussed by the class. Careful selection of students to present strategies could allow the ideas to be illustrated, highlighted, and then generalised.

After the selection of particular students' responses, teachers can then make decisions about how to sequence the students' presentations with respect to each other. Stein, Engle, Smith and Hughes (2008) present some examples: teachers could

- select the strategy used by most students and then those used by some of them;
- start with a particularly easy-to-understand strategy;
- begin with strategies that are based on common misconceptions or errors;
- relate or contrast right or wrong strategies.

The main goal of these two practices is to lead teachers to present in a particular sequence mathematical ideas to make a discussion more coherent and predictable.

A well-designed software could allow students to upload their responses and send them immediately to the teacher. Teachers can promptly read students' responses, select and then cluster/sequence some of them for sharing and analysis in the whole-class discussion (see, for example, FaSMEd project; e.g., Aldon et al., 2015). In this way, teachers could promote the comparison of different selected solutions.

Connecting Student Responses

Finally, teachers can help students to draw connections between the mathematical ideas that are reflected in the strategies and representations that they use. Stein et al. (2008) stress that having mathematical discussions consists of separate presentations of different ways to solve a particular problem: the main goal is to have student presentations build on each other to develop powerful mathematical ideas.

Research Aim

Starting from the theoretical framework presented by Stein and colleagues, we can better formulate the goal of this paper: exploring whether and how a videogame can facilitate in using the five key practices for designing productive mathematical discussions.

METHODOLOGY

In line with the theoretical framework just presented and with the goal of our research, we structured several activities in order to: involve students in tasks related to relational thinking, monitor students' responses and collect their productions to promote productive mathematical discussions.

Field Trial and Sample

We carry out a field trial divided into three activities: students play the "SuperFlat Math" videogame individually; then they work in small heterogeneous groups on unplugged tabs, and at the end, they participate in a whole-class discussion.

The sample is a grade 4 class from an Italian Primary school in Mantova. The class is composed of 13 students (five females and eight males). The study was carried out in January and February 2021 during school hours. Unfortunately, the field trial was interrupted by the pandemic situation: in February 2021, all Italian schools were mandatorily closed. We conducted the whole-class discussion remotely through the Google Meet platform available to the school.

SuperFlat Math Videogame

We use a skill and drill videogame called "SuperFlat Math", which was designed and developed by Prof. Leonardo Guidoni, from University of L'Aquila, starting from the free and open-source version of Minetest[1]. It is a sandbox videogame that enables primary and lower secondary school students to explore a blocky, procedurally generated 3D world. "SuperFlat Math" presents a list of mathematical tasks, and it provides a message on the correctness of the answers when the player gives the solution of each task. If the answer is correct, players gain points that could be converted into rewards.

The videogame also includes a web interface, which is designed to monitor the classroom's achievements by gathering information like access and play time, scores, number of correct answers, number of wrong answers, tasks and so on. The main goal of this system is to provide teachers/researchers with information that support anticipating and monitoring practices. By examining such web interface, teachers and researchers could immediately identify students' correct and wrong answers as well as the provided solutions to the tasks. Finally, students can upload their mathematical productions, tasks and other files to share them with teachers, researchers or mates. This feature can be useful for selecting and sequencing students' responses.

Proposed Activities

"SuperFlat Math" is divided into several games on different mathematical topics, such as Fractions, Equalities, Operations, Number line, Prime numbers and so on. Each game is composed of several minigames which are composed of a set of short puzzles at increasing levels of difficulty. To access the videogame, each student must have a personal account, which is also used to track his or her progresses in the web interface. Specifically, we consider two games: Parkour (Figure 1) and Swimming pools (Figure 2), on the mathematical topic of Equalities.

Figure 1: screenshot by Parkour minigame

Figure 2: screenshot by Swimming pools minigame

The Parkour minigames is of a perilous uphill path, which presents number sentences or expressions with two possible solutions. As already explained, Parkour minigames are at increasing levels of difficulty: the first half of minigames present number sentences in which students should find the correct solution, whereas the second half contains equivalence between two expressions.

In Swimming pools minigames, there is a pool full of number blocks from 0 to 100: on one side of the pool, there is an open number sentence. The player has to find the correct number block in the

pool to complete the sentence. As in the first activity, these minigames are at increasing levels of difficulty: the first half of the minigames contains open number sentences with two operations, whereas the second half has expressions with parenthesis and two or more different operations.

Unplugged Tabs

Drawing on the work of Carpenter and colleagues (2003), we design unplugged tabs including true/false and open number sentences (as in "SuperFlat Math") to engage students in relational thinking by focusing them on specific properties and ways of thinking about number operations (Table 1). For each task, we ask students to justify their answers so that teachers and researchers could understand their way of reasoning. For example, $38 + 47 = 47 + 38$ focuses on the commutative property of addition. Students might figure out that the number sentence is true by carrying out the addition on each side of the equal sign, but more commonly, they immediately conclude that the sentence is true because the order of the numbers has been changed. This can lead to a discussion of whether this relation generalises to all numbers and whether it is true for other operations.

Examples of true/false sentences	Examples of open number sentences
$8 = 3 + 5$	$25 + 32 = 27 + \cdots$
$3 \times 4 = 3 \times 4 + 3$	$8 \times 3 + 8 = 8 \times ..$
$2 \times 3 \times 5 = 6 \times 5$	$2 \times \ldots \times 7 = 14 \times 5$

Table 1: example of tasks in the unplugged tabs

RESULTS AND DISCUSSION

In the first part of the field trial, all students play the videogame. The web interface shows parameters like play time and scores (Figure 3).

ID	Last access	Task 1: a+b+c=...	Task 2: a+b+...=c	Task 3: a+b=...+c	Task 4: a+...=b+c
S.1	02 Mar 2021 10:05	2/3 66% Score: 60 Detail	1/1 100% Score: 30 Detail	1/1 100% Score: 30 Detail	1/1 100% Score: 30 Detail
S.2	02 Mar 2021 10:41	1/6 16% Score: 30 Detail	1/1 100% Score: 30 Detail	–	–

Figure 3. Screenshot of log files analyses in the web interface

The software allows us to know what and how many tasks students solved, to discover correct and wrong answers and the number of attempts. In Figure 3, we see that student S.1 made three attempts for solving task 1 giving two correct answers; for the same task, student S.2 made six attempts with only one right answer. The web interface also permits to have a detailed list of all the answers given by students.

Such information could be useful for the anticipating and monitoring strategies. The tasks in "SuperFlat Math" are presented in increasing levels of difficulty, and so the first ones could be useful for the anticipating practice then the last ones for the monitoring strategy. However, these collected data does not highlight the mathematical thinking in which students engage as they work on tasks. On the one hand, the web interface gives us an updated snapshot of students' performances; but on the other hand, it does not provide information about their mathematical processes.

The goal of the monitoring and anticipating practices is to identify the mathematical learning potential of particular strategies or representations used by the students, and so their mathematical thinking should not be ignored. For this reason, walking between the stalls while students work seems to be the best strategy for monitoring students, but the web interface snapshot could be useful to rapidly select which students to observe without walking around randomly.

Since students' strategies are not visible from the web interface and circulating between the stalls is not always possible, we administer to small groups of students the unplugged tabs, in which we require them to motivate each answer. We create small heterogeneous groups according to their game scores: in the same group, we include students with high, low and medium scores. We promote teamwork to encourage students to share their strategies and make autonomous mathematical discussions in small groups. Once they solve tabs, students use the web interface to share them with us. The possibility to instantly share the answer allows us to reach a big amount of information without waiting that all students finish to solve their unplugged tabs.

All the answers to tabs are correct, but justifications and group strategies differ. For example, concerning the following task: $\mathbf{25 + 32 = 27 + \cdots}$, some groups reveal the use of relational thinking in the justification: "We have chosen 30 because 25+2=27 and it [27] is what I already have, then on the other side, I took off 2 from 32, and I add it to 25, that results 27 and 32-2 equals 30 and I put it [30] beside 27."

Conversely, other students use computational strategies. For instance, another group writes: "We have chosen 30 because we've computed 25+32=57 and so 27+30=57."

The analysis of these answers allows us to select and then sequence them for designing the whole class discussion. We start by selecting the computational strategies, and then we sequence them with the ones that reveal relational thinking. The web interface permits to rapidly collect students' responses, so we have the possibility to quickly plan the whole-class discussion while students are still engaged with tabs.

We conduct a mathematical discussion based on the anticipating (by using the web interface) and monitoring practices (by using the web interface and the unplugged tabs). According to our data, we select and sequence students' answers following these criteria:

1. we make sure to select at least one response from each group in order to allow all students to participate in the discussion;
2. we choose those correct answers that reveal different strategies by also checking selected groups' achievements in the videogame;
3. we first present the answers that show computational strategies and then those revealing relational thinking so as to reflect on similarities and differences.

Unfortunately, due to the pandemic situation, we interrupt the field trial. For logistical reasons, we conduct the discussion remotely. Nevertheless, all students participated in the whole-class discussion and those who showed computational strategies in the tabs discussed with their mates about relational strategies.

CONCLUSION

We structure activities with a videogame starting from the hypothesis that technological tools could support teachers/researchers in collecting data on students' achievement and in planning and then orchestrating productive discussions. We present and discuss an example from our case studies, with

the aim of analysing the role played by the videogame in supporting the design of a discussion concerning relational thinking.

The goal of this paper is to explore whether and how a videogame can facilitate the planning of mathematical discussions using the five key strategies (Stein et al., 2008). Concerning our research aim, we notice that the videogame could be a valuable tool for the following three reasons. first, the web interface makes the anticipating and monitoring practices possible by uploading the log files, through which we follow students' achievements and wrong/correct answers. Second, the log files do not show students' strategies and ways of thinking, so we need also to walk between the stalls while students work. However, the web interface data are useful to rapidly select which students to observe without walking around randomly. Finally, the web interface also makes the selecting and sequencing practices faster: the ability to retrieve single items speeds up and facilitates the discussion orchestration while students are still working on tabs.

Thus, the web interface seems to be a helpful tool for promoting four of the five key practices described by Stein and colleagues (2008) and for this reason, we suppose that technological tools could provide an added value compared with activities in a paper and pencil environment. All students participate actively in unplugged tabs and the discussion. During the activities, the teacher states that almost all students enjoy playing the videogame and the subsequent activities.

Therefore, it seems that the videogame plays both a motivational and a methodological role. Reflecting on log files and retrieving the tabs is essential, because otherwise, we would carry out the discussion by calling on students randomly, without considering their progress. Without these tools, the collection of data concerning students' performances, strategies or ways of thinking would be more time-consuming. So, the time between play time, tabs activities and the discussion would be too long, because we would need a lot of time to collect and organise students' data.

There are some important aspects not explored in this paper that are crucial for future research. We describe the methodological advantages of a videogame in planning and orchestrating mathematical discussions, but we do not consider its mathematical effectiveness in terms of development of relational thinking. We present a first reflection on our tools and materials by trying them out in small class activities. The future research will be based on successive cycles of design, observation, analysis and redesign of classroom sequences (Design-Based Research Collective, 2003) in order to design a set of materials for students and teachers. The analysis of mathematical effectiveness of such materials and tools needs a longer period and a greater set of activities involving more than one class and one teacher. It will also be interesting to study the usability of these materials and tools by teachers without our support.

Finally, we do not consider the role of feedback provided by the videogame. The scores inform students whether their answers are correct or not; in addition, the message returned by the videogame when students give a wrong answer could allow him or her to reflect on his or her strategy. In Figure 3, students made some attempts in the first task (S1 made three attempts; S2 made six), whereas they gave the correct answer in the next tasks. This suggests that the videogame could be used as a tool to support students in formative assessment practices (e.g., Aldon et al., 2015).

The activities described in this paper is conducted face-to-face, but they could also be carried out remotely. This is another interesting feature of technological tools that could be explored in future research: they offer the possibility to design intriguing activities to support teaching and learning mathematics in different scenarios.

NOTES

[1] https://www.minetest.net/

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Theme 3: Fostering Mathematical Collaborations

with and through technology

TE(A)CHING TO COLLABORATE: AUTOMATIC ASSESSMENT-BASED GROUPING RECOMMENDATIONS AND IMPLICATIONS FOR TEACHING

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Sharing and communicating mathematical ideas is fundamental to learning and teaching mathematics. This paper focuses on fostering mathematical collaborations between students through the lens of topic-specific learning analytics (TSLA). Using example-eliciting tasks, I harness technology to analyze different characteristics of a students' submitted answers and to characterize different types of relationships between various students' mathematical work. I introduce possible schemes for grouping students according to the goal of the task, and demonstrate recommendations for student pairing and how they are used by teachers to improve the collaborations and the teachers' orchestration of group work in an online environment for the benefit of students' learning outcomes.

Keywords: Example-eliciting tasks, online collaboration, student grouping recommendations, topic-specific learning analytics.

COLLABORATION OVER WHAT?

Collaborative learning is considered to be productive for learning (Dillenbourg, 1999). The range of tasks that students collaborate over is wide, from conventional tasks to open inquiry and projects. Students may engage in solving problems together, peer assessing work, devising questions, developing answers, and more. Technology can facilitate many forms of collaboration mentioned above. The ability to communicate and share work efficiently and seamlessly enhances processes that may limit the ability to collaborate. A classic example is peer assessment, when we are asked to exchange our tasks with the people next to us, or discuss some issue with them: this location-based collaboration might be at the expense of other, less constrained forms of grouping students.

Communication and cooperation are not the only ways in which technology can be used to facilitate group work. Computer-supported collaborative learning studies go beyond these functionalities and explore the role of technology also as a mediator, a tool, and an instrument. Technology can be the curator of student work or of a teacher's work, and it may also be the assessor if a platform provides some type of feedback and reports. This paper is aligned with the research branch that uses data and learning analytics to facilitate the process of grouping students and providing teachers with recommendations (Chen, 2005; Mavrikis et al., 2019). To achieve an effective, student-centered practice that provides personal characterizations of student work that can be used for grouping, we need automatically assessed tasks that enable students to express their ideas in an open manner.

Example-eliciting tasks (EETs), which require students to provide examples as answers, are a good starting point. Students' construction and use of examples have been a focus of many studies (Zaslavsky, 2019). I use Zaslavsky's working definition, "mathematical objects are considered to be an example only when the learner and/or teacher perceive it to be an instance of a phenomenon, property, class, or idea" (Zaslavsky, 2019; p. 247). Automatic assessment of learner-generated examples (LGE) has been demonstrated in a digital geometry environment (DGE) (Leung & Lee, 2013) and provided insights about the learners that were verified through interviews. In EETs we harness both examples and exemplifying within the process of mathematical reasoning, therefore LGEs can reveal conceptions of mathematical objects or concept images (Vinner, 1983). These

conceptions manifest in students' example spaces, which are sets of examples provided by students for a certain goal (Watson & Mason, 2006). Example spaces can also inform us about possible difficulties and inadequacies of student work and student knowledge (Zazkis & Leikin, 2007).

One of the challenges we face in EETs is that we need them to be automatically assessed. There are platforms that show that students solve different tasks, in which the students argue about their answers mostly textually or symbolically. The answers are presented to the teacher (Clark-Wilson, 2010; Tabach, 2021), who as a curator, must work with the submissions usually in the moment, but not necessarily. This is a demanding task. In EETs we design tasks that can provide us with automatic assessment of the student work. If we cannot assess it automatically, an EET has less of an advantage over a regular mathematical task that can be assessed manually by a teacher. It does not mean that EETs cannot be accompanied by verbal explanations by students, but this is not the core of the task. To handle different types of mathematical activity that we want to capture within EETs, we developed six design patterns that accompany different mathematical ideas and actions.

(A) **Exemplify existence**: Construct an example of a definition. Given a definition, the student is asked to construct an example that exemplifies that the definition can exist with an object that satisfies the definition (Olsher et al., 2016); (B) **Validate a claim**: Associate the claim with the identification of a pattern of existence, or of a contradiction. A claim could state: "There is a triangle with a right angle." Is this claim true? Initially, students need to choose whether it is true or not based on what they know. In addition, they are asked to provide an example of a right-angled triangle. Such tasks assess not only the student's decision whether the claim is true or not, but also the student's ability to exemplify it, which is a different type of competence (Olsher & Yerushalmy, 2017); (C) **Classify**: Set the conditions that would allow a claim to be true. Students are presented with a set of conditions, not just one, and they need to choose either all of them or a subset of them that they think could coexist, and exemplify them in a mathematical object. These tasks are used to create interaction with the relations between the different characteristics that the teachers aim to address (Yerushalmy, 2020); (D) **Separate**: Articulate the variables that support various examples (to validate or to exemplify). Students are asked to either exemplify and give multiple examples that are as different as possible from one another, or to point out the difference between two given examples. Students must find a characteristic that can distinguish between the two examples (Yerushalmy et al., 2017); (E) **Generalize**: Refer to a mathematical object created by a set of examples. If we ask students to give a large set of examples, they may be able to reach a general rule (Olsher & Lavie, 2021); and finally (F) **Verify** a claim using examples at the beginning of the deductive process or as part of the abductive process. Examples play a key role even if the goal is to generate a deductive proof. The mathematical practice to understand the claim is usually to construct an example where this claim exists, constructing a contradicting example could also be an initial stage of the deductive process (Cusi & Olsher, 2021).

EET EXAMPLE: SEPARATION TASK

The focus of this paper is on the separation design pattern: Submit examples that demonstrate controlled variability. In Figure 1, the student needs to decide whether there exists for this case a triangle that can be constructed by moving the three points. The brackets contain mathematical objects that can be modified. If the answer is positive, students are asked to submit an example to support their answer. But not just one example: three examples that are as different as possible.

1. Decide whether there exists a [TRIANGLE] that [COULD BE CONSTRUCTED BY MOVING THESE 3 POINTS].

2. If your answer is positive, submit an example to support your answer.
3. Submit a total of three examples that are as different as possible.

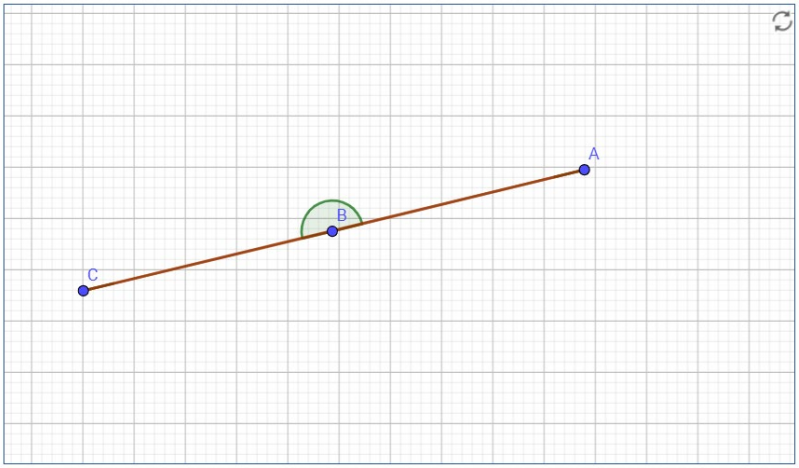

Figure 1: A separation EET

In the applet shown in Figure 1, the student can move any of the points. The starting position is a line. When students work on this type of a task, they must think how to separate the examples: What is the characteristic (or characteristics) of the triangles that the students retain, and what are the characteristics that the students change between examples? This activity gives us some information about the student. A good EET has an infinite number of correct answers because what we try to assess are characteristics presented by a particular student. When having to produce an example, each student plays the role of a curator, which allows us to assess something personal. It is not a task with the answer "42," so that if two students answer 42 we have nothing different to say about them. If we ask for an example, the examples we receive from different students are usually different. If they turn out to be similar, this may tell us something about the teacher's practices and the teacher's choice of examples. The information that we obtain when assessing the students' answers is more student-centered, and we can do it automatically.

EETs can provide evidence of mathematical reasoning and student work. In another task (Figure 2), students received a multi-linked representation (MLR) with a given point. They could press the "new point" button to get a different point that was semi-randomly generated (Bagdadi, 2019). The algorithm placed points on the x-axis, the y-axis, or on neither of the axes. Students were asked to construct three linear functions that pass through the given point.

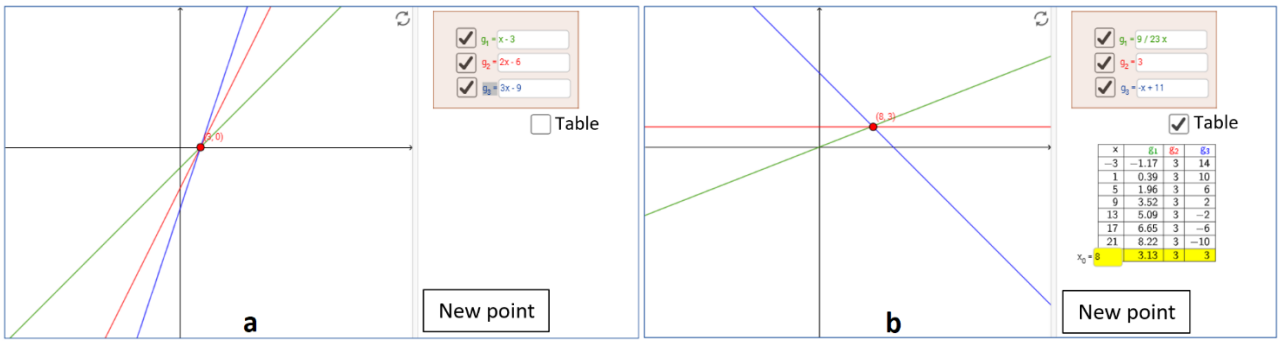

Figure 2: Linear function through a point EET

The different representations in the applet were on-demand. Students entered a symbolic expression, and if they checked the box next to the expression, the graph appeared. On a table of values, students could see a selected set of values for the functions they entered symbolically. They could choose a certain value, which could be the critical value, in this case x = eight, to see the value for each one of the functions they entered. This provided students with tools for verification, but also for argumentation. If they showed these graphs, the graphical representation was for verification. But we do not know whether the graph actually passes through the point; it may be just close to it. If students wanted to exemplify whether the function went through the point, they could use the table of values and plug in the x value of the point to verify, and also to argue about a particular solution.

We can identify work methods of students. Figure 2 shows two different examples. Figure 2a has all correct examples, and Figure 2b has one incorrect example. It is the same task and the student that submitted only correct functions is not necessarily demonstrating more than the student that has one incorrect example. In Figure 2a we can see that all the slopes of the submitted functions are positive, and the functions are of the same type: x-3, 2x-6, 3x-9. Figure 2b shows a wider variety of linear functions in terms of slope, and also in terms of verification tools. Figure 2b shows the use of the table of values to verify the answers, although not in all cases. These characteristics can be analyzed automatically *based on the submitted responses*, without the need to have logged the student actions *during* the solution process.

GROUPING THE STUDENTS

Group learning is a fundamental practice in mathematics education and teachers have general criteria, which they use to group students. Some of these are rooted in interpersonal relations, as noted before, including location and personal relationships between friends or peers (Cohen, 1994). Another type of information considered for grouping purposes is whether a student is high- or low-achieving (Johnson & Johnson, 2002; Maqtary et al., 2019). Homogeneous and heterogeneous are also known types of groupings, describing students who are grouped based on whether they are similar or different in a certain characteristic. There are also encompassing and complementary types of grouping. Encompassing refers to grouping students who know all (or most of) the content the teacher wants them to learn with students who have a wider knowledge gap to fill. In complementary grouping, two students contribute to each other's process of learning (Abdu et al., 2021). When operationalizing the work of students in groups, we use dialogic theory to focus on relationships between interpersonal interactions and individual learning (Bakhtin, 1984). The dialogic view of learning is similar to a sequence of changes in perspective that emerges through interactions with different types of agents (Wegerif, 2011). A dialog consists of voices. A voice is "the intentions […] individual speakers present in each utterance" (Barwell, 2016, p. 335). Voices are situated and always about something, some mathematical object, idea, or feeling. Voices are situated in a setting and are dynamic because of their interaction with the contextual or the social settings and agents that participate in this dialogue (Wegerif & Major, 2019).

A dialogue happens when and where two different voices interact. In a classroom, a teacher, a peer, or a technology can express different voices (Webb, 2009). To describe the setting of the dialog, we use the term dialogic space. A dialogic space (Wegerif, 2011) is a construct that helps us define what makes a certain group better than another group when trying to define two groups that involve a dialogue between learners. When we want to see the different implications of these dialogues, we need to make sense of the dialogic space that exists within this learning. A dialogic space is not like a regular space: it is multi-dimensional and can be expanded to different dimensions; we can define different contextual characteristics for it to happen. We use the dialogic space to bring the existence

of the teacher into the dialog. The teacher's voice stands apart from the two learners who interact and make their voices heard. In the dialog, the fundamental role of the teacher is to coordinate, orchestrate, or facilitate this dynamic dialogic space where the learning happens. How do we widen the dialogic space? How does a teacher open it up? Maybe teachers need to narrow it down somewhat if they want to focus the students? These are the constructs that we use in dialogic learning.

To create a dialogic space and promote dialogic learning, we suggest the use of EETs as a basis for grouping students according to the mathematical characteristics of their submissions. In Figure 3, I present the tasks from Abdu et al. (2021). Task 0 is quite similar to the one presented in Figure 2, but it contains only one function. The aim is to get the students acquainted with the platform and its tools, and promote a process of instrumentation that will assist them in the more complex tasks. Task 1 is for individual work by the students, who are asked to construct a quadratic function that passes through the given points, with the points restricted to the axis.

When students solve this task they provide us with examples we can analyze to determine the relationship between two student submissions. We divide the task characteristics (Figure 4a) into categories of aspects (Abdu et al., 2021). Next, we can define states for each aspect. For this quadratic function task (Figure 4a), we can check whether it has roots. The number of roots is a characteristic of the answer: 0, 1, or 2 roots. The type of extreme values can be minimum or maximum. The symbolic form that the student uses can concern the polynomial, vertex, or intercepts. The number of correct examples can be up to three.

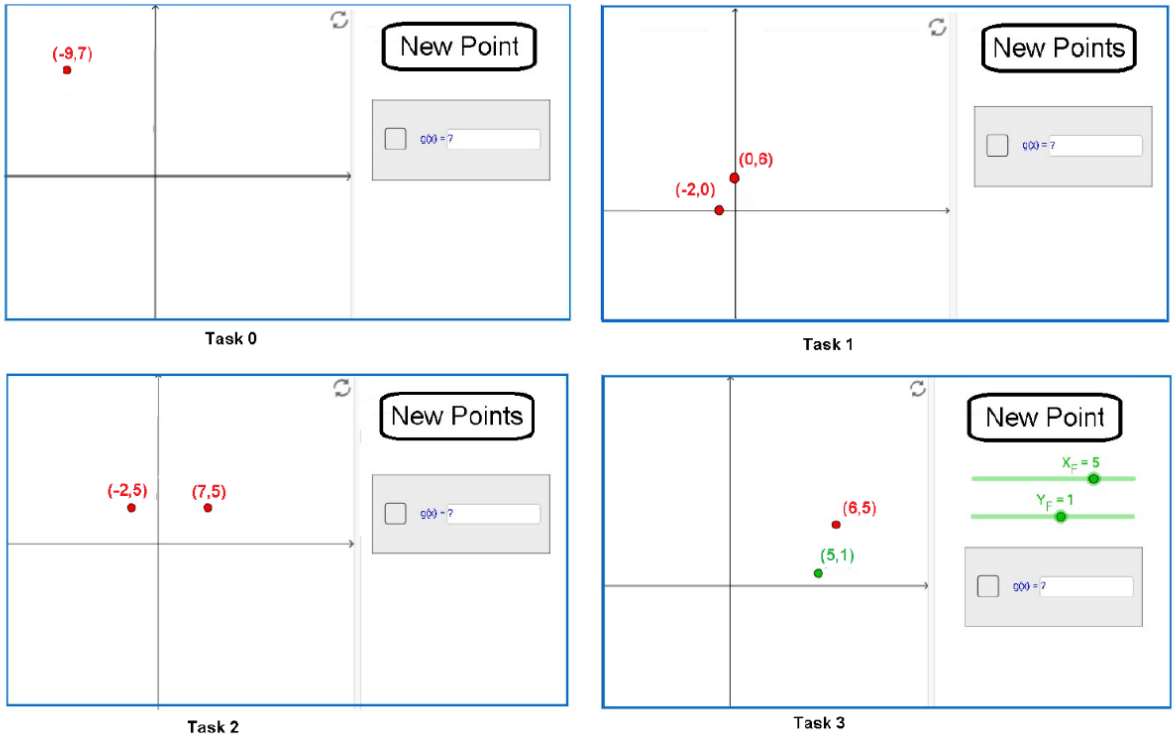

Figure 3: Linear function through a point EET

This mapping might seem specific to the task, but it is not. When using a method similar to that in the triangle task (Figure 1), we can automatically analyze examples submitted by students to see what kind of separation is present in students' work (Figure 4b). Were they separating on angles: acute, right and obtuse? Were they separating on sides: equilateral, isosceles, and no equal sides? We also

learn about their work methods. Did they submit many prototypical examples, having a vertical or horizontal side? And we know the number of correct examples. When we use an EET and define the characteristics that are relevant for the students' work, we can base our grouping on those aspects and characteristics that can be automatically determined.

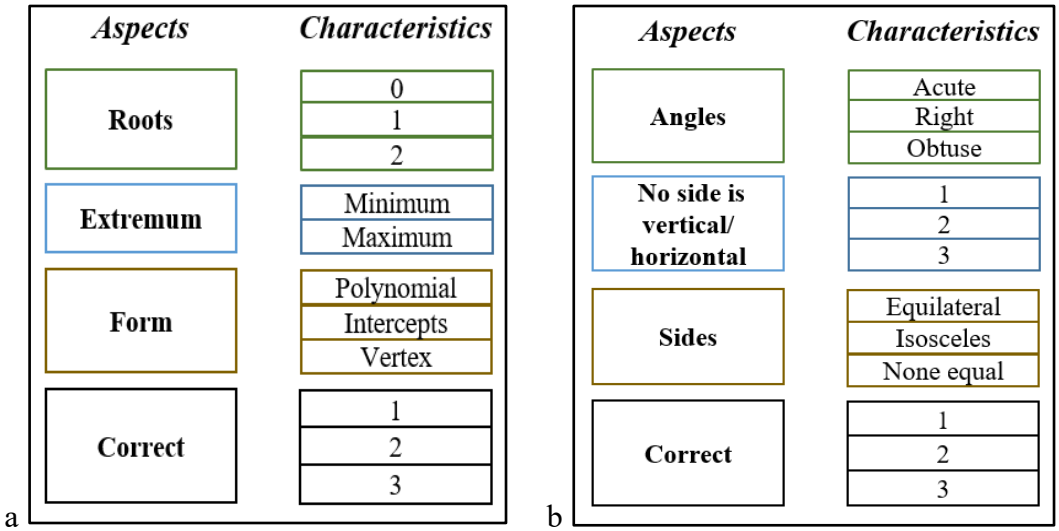

Figure 4: Linear function through a point EET

After characterizing student submissions for Task 1, students were assigned to work in pairs on Task 2 (Figure 3), which had two points with the same y-value, and the students needed to construct three examples of quadratic functions that pass through those points. Task 3, which had a random point and one that could be moved to any location (Figure 3), was carried out later individually. Figure 5 (edited from Abdu et al., 2021) shows examples of different relations between students. The characteristics present in each student submission are colored, and the empty rectangles are characteristics that do not apply to that submission. For example, student A, had all of the characteristics of the peer for Task 1, but also some that the peer did not have (e.g., a function with one root). Thus the relationship between student A and the peer can be defined as encompassing. Similarly, student B's relationship with the peer can be defined as encompassing. Submissions by student C and the peer's have complementing characteristics, that is, some characteristics that appear in the submission of one, do not of the other, and *vice versa*. Thus student C's relationship with the peer can be defined as mutuality. Finally, student D's submission has exactly the same characteristics as that of the peer, therefore the relationship can be defined as similarity.

This was a separation task. The goal was to expand the number of characteristics that students showed in their work. Abdu et al. (2021) shows that when grouping students, mutuality resulted in significantly higher increase in the average number of characteristics than did the encompassing and similarity pairing strategies, which is what we expect in a separation task.

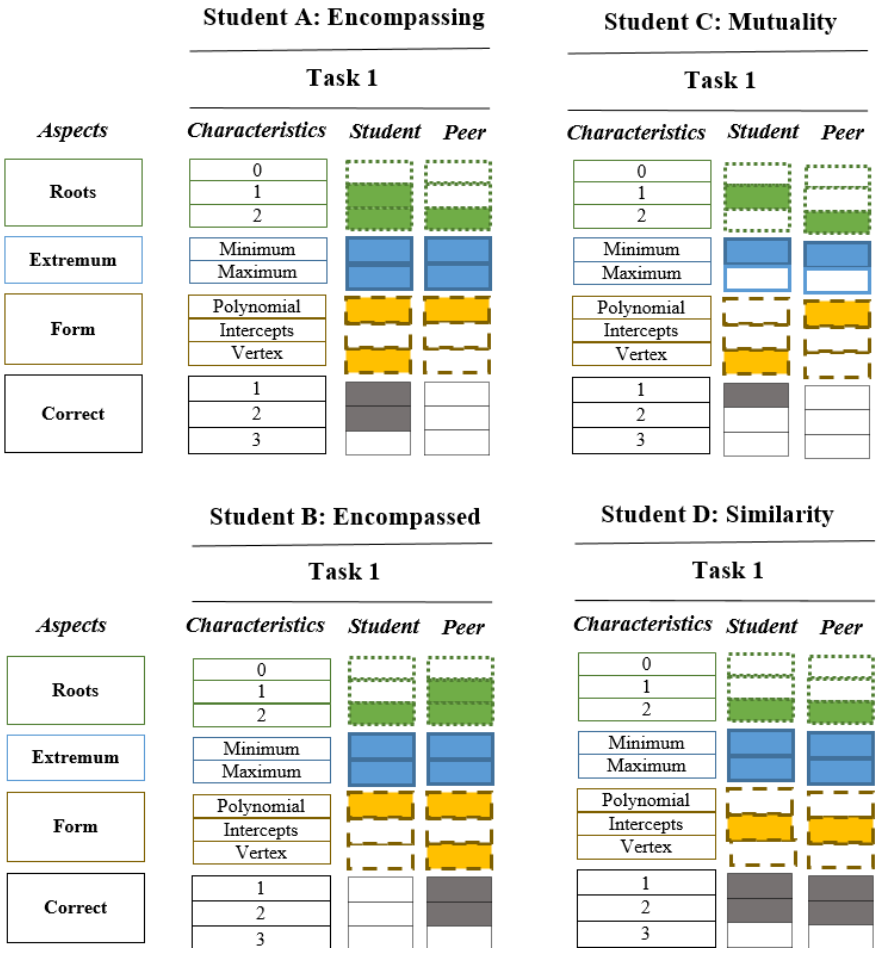

Figure 5: Student comparison (Abdu et al. 2021)

As shown above, not all EETs are separation tasks (or classification and generalization, which we assume can also benefit from a mutuality grouping strategy). In tasks that ask to exemplify existence, I hypothesize that pairing a student who succeeded in demonstrating the existence with one who did not provide an example can lead to peer learning, benefiting from encompassing/encompassed strategy for grouping. In tasks that ask for generalizations, I hypothesize that grouping based on a similarity relationship can be beneficial because group members will face a similar challenge, which is preferable to one partner showing the correct ideas to the other. These hypotheses require validating through research, but the guiding principle is that the goal of the group activity should be a key factor in determining the grouping strategy to be used for an activity.

IMPLICATIONS FOR TEACHERS' WORK

In another study, we examine how teachers facilitate group work in various settings (Shalata, 2021). We investigated how teachers manage the dialogic space between the two voices of students to make it into a productive one? We followed three teachers who carried out the quadratic function activity (Figure 3) in their classrooms. Following the analysis of Task 1 submissions, we grouped the students in each classroom into pairs according to the 3 grouping strategies: encompassing, similarity, and mutuality. Note that we did not give teachers strict pairings but merely recommendations because the teacher's perspective is important, and we wanted them to be able to overrule our recommendation.

The teachers made a few changes in pairings, but these were not based on the students' work but rather on the relationships between students.

We had roughly the same number of groups for each grouping strategy. The lessons were conducted online because of pandemic restrictions on face to face teaching, and the teachers used Zoom software and breakout rooms. When teachers were in a room, everything they did was specific to this group and related to the dialogic space that existed at that moment. The students solved Task 2 in pairs. The interactions within the breakout rooms were coded into four categories: (A) Technical – related to problems with the platform, connectivity; (B) Interpersonal – how students worked together and whether they cooperated; (C) Topic-specific – related to the topic of quadratic function; and (D) General understanding – related to clarifying task requirements and how to exemplify.

Although there were no significant differences between the classrooms with respect to the categories (one class had more technical difficulties), the different grouping types provided some enlightening insights about the teacher's role as a facilitator of the dialogic space. The main finding was that encompassing pairs had significantly more interpersonal interactions than did other pairs. This finding was supported by the teachers:

> "In some pairs, there was good cooperation without me instructing them… others worked separately so it led me to ask them to cooperate… I fit myself to the pairs according to what I saw and in most cases, I had to intervene a lot it was because the weaker student was lost, so I tried to move them towards working together and in that way lead to learning of both of them" (Teacher Taleb, Interview, 19.12.2020).

This finding can be explained by the fact that when grouping students using a mutuality or similarity strategy, both students have voices that can contribute to the dialog in meaningful, content-related, ways. When using an encompassing grouping strategy, the teacher must facilitate the dialogic space and the interaction of voices to help students interact with each other, and not only next to each other. Different grouping strategies can benefit students in different types of activities. Our findings suggest that there is no one general grouping strategy that benefits different activities in the same way. We must keep in mind that the way we group students may implicate teacher actions, and consider what is required from a teacher to facilitate group learning. In the case of the encompassing grouping strategy, teachers must pay attention to the collaboration more than in other grouping strategies. In a similarity grouping strategy, teachers may have to focus on topic-related interaction.

Finally, the data about student work, which includes not only an abundance of information produced by the students but also various analytics and measures, needs to be handled by teachers in their decision making. This may be a lot to handle, but this data can also be handled automatically if we choose the right task design pattern and define the relevant characteristics for our teaching goals. Among these characteristics could be familiar mistakes, but also mathematical characteristics and descriptions of student answers that we want to introduce to the students' mathematical discourse. In these cases, we can use the data to explore different students' voices and to enhance the dialogic space and the group learning process.

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WHEN A DIGITAL TOOL GUIDES MATHEMATICAL COMMUNICATION

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Earlier studies found that the use of digital tools and empractical mathematical communication are deeply related, showing mechanical or random ways of using the tool as well as a lack of conceptual mathematical understanding. However, with respect to Dynamic Geometry Environments (DGE), students may address the tool at least in two different ways, tool-embedded and tool-idling, providing various possibilities of communication. In this paper, we report on a case of two students' communication and their communications competency as they use DGE in these two ways.

Keywords: Communication competency, dragging, empractical talk, GeoGebra, instrumentation.

INTRODUCTION: TOOL-BASED MATHEMATICAL COMMUNICATION

In Denmark, the KOM framework has been introduced to teaching and learning mathematics. This framework includes eight mathematical competencies; among others mathematical communication competency and aids and tools competency, involving digital tools (Niss & Højgaard, 2019). It is expected that students develop both competencies in mathematics, also related to each other. Regarding mathematical communication, two perspectives *communicating to learn* or *communication as a goal in itself* (Erath et al., 2018) are relevant when using digital tools and developing competencies. With respect to this distinction, we aim to investigate in what ways communicating to learn mathematics with a digital tool, and communicating as a goal when using digital tools allow students to show communication competency? (Niss & Højgaard, 2019).

Using digital tools in mathematics classrooms potentially support students' mathematical discussions (Drijvers et al., 2016). Yet, there is a risk that students end up communicating *empractically*, which means that mathematical communication is very context-dependent focusing on the use of the tool rather than the mathematics in play (Jungwirth, 2006). Earlier studies (Bach & Bikner-Ahsbahs, 2020) found that students' use of GeoGebra (a DGE) described from their instrumentation profiles (Guin & Trouche, 1998), is related to two distinct genres of mathematical communication: *Empractical communication* and *conversation* (O'Connell & Kowal, 2012). In a recent study (Bach & Bikner-Ashbahs, accepted), these two genres are empirically related to whether the communication is *tool-idling* (referring to DGE as a static object) or *tool-embedded* (referring to the dynamic features in DGE) resulting in four types of tool-based communication. Our data analyses show that only in tool-embedded mathematical conversation, mathematical communication competency is exercised. However, data on the two ways of empractical communication is scarce, specifically with respect to tool-idling communication. In this paper, we fill this gap by presenting cases of empractical mathematical communication when students use GeoGebra, aiming to answer the following research question: What characterises tool-embedded and tool-idling empractical mathematical communication and how do they relate to mathematical communication competency?

MATHEMATICAL COMMUNICATION

Based on psycholinguistics, we define communication as an activity that "brings persons together somehow by means of spoken discourse" (O'Connell & Kowal, 2012, p. 10). Two roles are here central: *Listener* and *speaker*. Listening is meant to be active as it responds to speaking (O'Connell

& Kowal, 2012). We distinguish between two verbal mathematical communicational genres applied to the field of mathematics (in Bach & Bikner-Ahsbahs, accepted): *mathematical conversation* and e*mpractical mathematical communication.* The former holds certain social qualities such as both listening and speaking, equal participation, turn-taking, open-endedness and verbal integrity. Open-endedness involves bringing the conversation forward and verbal integrity involves an attempt to understand others' speaking, and reviewing it fairly. Empractical communication does not hold the social characteristics of a conversation (O'Connell & Kowal, 2012), as it is a way of communicating embedded in practical acting that refers to itself and its purposes rather than social relatedness.

We call any communication mathematical when it concerns mathematical contents – when discourse is mathematical. Yet, when students communicate empractically, there is a risk that mathematical issues are kept in the background whereas the tool is in the foreground (Jungwirth, 2006; Bach & Bikner-Ahsbahs, 2020). When mathematical communication is addressed as a learning goal, mathematical communication competency comes into play. This competency involves being able to express oneself mathematically and to understand and interpret others' mathematical expressions. Both aspects involve different media (e.g., oral, written, visual and gestural), genres and discourses, related to different people (Niss & Højgaard, 2019). Our previous studies show that students communicating empractically while using digital tools do not show mathematical communication competency, whereas in mathematical conversations, this is more likely to happen (Bach & Bikner-Ahsbahs, 2020; accepted).

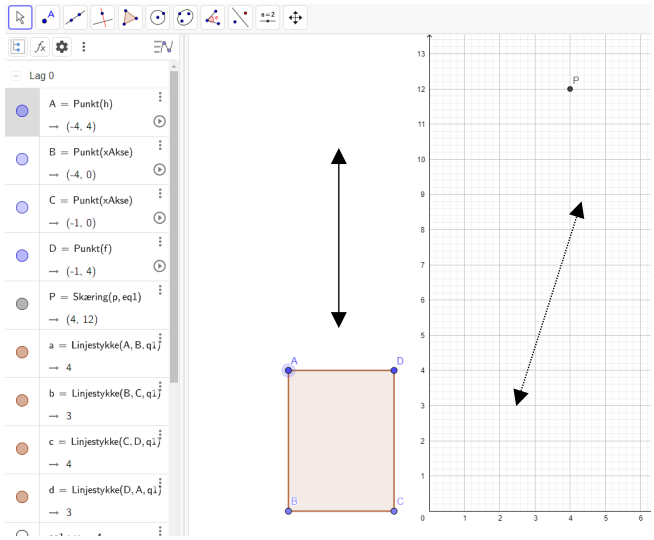
INSTRUMENTAL APPROACH IN MATHEMATICS EDUCATION

We frame the use of digital tools by the instrumental approach involving several concepts: artefact, instrument, instrumental genesis, instrumentation and instrumentalisation. An *artefact* is a material object created by humans, thus, any digital tool is an artefact. *Instrumental genesis* is the process in which an artefact is transformed into an instrument for someone. It consists of two sub-processes: "instrumentation is the process by which the use of the artefact influences the activity of the subject; instrumentalization is the process by which the subject adapts/enriches the artefact to make it more efficient and more suited to their needs" (Artigue & Trouche, 2021, p. 8). In our previous studies (Bach & Bikner-Ahsbahs, 2020; Bach & Bikner-Ahsbahs, accepted), we applied the instrumentation profiles to GeoGebra (DGE), originally developed by Guin and Trouche (1998) involving Computer Algebra Systems (CAS). As the tools at play and instrumental geneses are closely connected to the conceptualization conducted (Trouche & Drijvers, 2010) and CAS and DGE differ, we include the dynamicity of DGE as its main difference in our analyses. Specifically, dragging is relevant as it "reveal[s] cognitive shifts from the perceptual level to the theoretical one and back in students' mathematical activity." (Arzarello et al., 2002, p. 67). Guin and Trouche (1998) find five instrumentation profiles of a tool empirically with respect to three dimensions. *Information tools*: the ability to choose and get information, such as the tool, calculator, theoretical knowledge and peers. *Understanding tools:* the ability to use the information available in the given situation, such as comparisons and semantic interpretations. *Command processes:* The ability to choose and use *understanding tools*. Students with a *random* profile have trouble both using paper-pencil and digital tools. Often, they copy earlier work or perform trial-error strategies without verifying results. Students with a *mechanical* profile primarily use the tool and reason based on results stemming from the tool avoiding mathematical references. Students with a *rational* profile are primarily using paper-pencil and show strong commando processes. Students with a *resourceful* profile explore all information tools available and reason based on comparison and confrontation. Finally, students with a *theoretical* profile do mathematical interpretation and verify mathematically based on the results provided by the tool (Guin & Trouche, 1998). When applying these profiles to DGE, we consider

utilisation schemes for dragging, as through dragging students link their use of a tool with mathematical content (Arzarello et al., 2002).

METHODOLOGY

In this paper, we present data from the second trial of a case study, which is collected in Denmark in an 8thgrade classroom. We reused a task developed for 9th grade, which leans on a study by Johnson and McClintock (2018). It is constructed to exercise communication competency when using digital tools. Conceptually, it focuses on linear functions as covariation, more specifically, to identify, use and conceptualize how two quantities are changing together in a functional relationship. Our task has eight subtasks, requiring separated individual as well as collaborative work. The task is given as a GeoGebra template (Figure 1). It provides exploring of how the area and the height of a rectangle change together when dragging point *A* and how this is translated into a graphical representation.



Subtask 2. *Alone*. Explore the construction/template on your computers by dragging point *A* of the rectangle. Describe the relationship between point *P* and the shown figure (i.e., the rectangle). For point *P*, you have to describe what characterises the *x*- and the *y*-coordinate. Explain why this relationship exists.

Subtask 3. *Together*. Compare and discuss your results from task 2. Summarize here what you agreed upon.

Figure 1. Left: illustration of the template in GeoGebra. The black arrow illustrates movements of point *A*. The dotted arrow shows P's traces when moving *A*. Right: two subtasks.

In the task sequence, students work in pairs having one computer each. Thus, the students' ability to drag and make sense of what is happening with the tool mathematically becomes very important, which does not occur with CAS. As students drag point *A* in the rectangle *ABCD*, point *P* moves. The students are expected to find *P*'s coordinates as *P*=(height *AB*, area of *ABCD*).

Data collection involved screencasts and videos of students working in pairs. Transcription is done in Danish verbatim and translated into English. In collaboration with the teacher, high achieving student pairs were selected based on their national test, hand-ins, and participation in class. The data are analysed by interpreting the data based on the theoretical concepts introduced in the framework.

In the present paper, we focus on one pair solving subtasks 2 and 3. Subtask 2 is an individual part where students' explore the task themselves. Subtask 3 is a collective one reflecting prior work on subtask 2. In subtask 3 we expect the students to show mathematical communication competency.

PRESENTATION AND ANALYSES OF DATA: THE CASE OF ELIN AND DONNA

In this section, we present the data followed by the analyses of the two students' instrumentation processes and their mathematical communication. The data shows a dialogue on the students' explorations related to subtask 2, divided into three smaller excerpts.

When analysing instrumentation profiles, we identify the students' mathematical knowledge in play, how they come to know, the information tools used, and their strategies of solving the task (e.g., copy-paste). Analysing their communication, we identify their roles as listener and speaker; their use of media of communication (i.e., visual, oral, gestural or written); and the communication genre/discourse, including the mathematical language.

Excerpt 1: Exploring the Construction

Prior to the following dialogue, the two students have both been dragging on their individual computers. In the beginning, Donna identifies that *D* is related to *A* (as these are endpoints of one side of the rectangle). Their interaction addresses *P*'s *y*-coordinate, and the height of *AB*. Note that Donna does not interact with her own template in the following excerpt.

1	Elin	Yes but when it stands at 4 [pointing at height of AB]; it is at 12 [pointing at *P*]. And so, this, this one here [*A*, red.] stands on 6, then, it is at 18 [*P*, red.]. And so, when it is at the double, 8, then it is at [she zooms in and out, trying to find *P*, that "disappeared" when she dragged point *A*].
2	Donna	It moves when the figure gets bigger, right? Then you can see the area.
3	Elin	Mmm [means probably maybe]
4	Donna	Isn't it like that?
5	Elin	I want to... I can't… Mmm Yes [responding to Donna's comment].
6	Elin reads the task description again.	
7	Elin	Okay. So *P* is connected to the square and it has a distance.
8	Donna	*P*… *P*.
9	Elin	It has a distance on… So it stands on 4 [*A*, red.] so it stands on (12,4).
10	Donna	But… But it will show what the area is.
11	Elin	6 and then… Wait for a second... *P* related with the square by.. By ... the distance to the square will always be … will always be…
12	Donna	Isn't it just that every time it is drawn longer up, then it is showing the area. It is doing it…

Instrumentation profile. The task aims at identifying the characteristics of the *x*- and the *y*-coordinate for *P*. Looking at the mathematical knowledge in play, Elin and Donna identify that *P* moves when dragging Point *A,* yet, they only focus on the *y*-coordinate for *P*. Donna finds that *P* is "showing the area" of the rectangle (lines 2, 10 & 12), which is the closed they come to the aim. Elin also talks about the *distance* between *P* and the rectangle (lines 9 and 11); it is evident that she identifies the distance from the rectangle to Point *P* in the template which is not the relationship aimed to find. Overall, the excerpt indicates a pre-functional understanding as it primarily focuses on one coordinate (Ellis et al., 2013). Covariation is not addressed, as the dynamic features are not exploited with regard to the characteristics of *P*'s coordinates. Informational tools: The template acts as an information tool, yet particularly from line 2 on it serves as a static representation that the two students do not interact with. Donna only exploits the dynamic features of the tool in relation to the geometrical properties, however prior to this excerpt. In addition, they do not utilise each other as resources. Strategies: The students follow two strategies, dragging at the beginning only and relating numerical features in the template in an unsystematic manner. Summing up, both Elin and Donna show parts of a *mechanical* and a *random work method*. The mechanical work method is identified when dragging, yet their difficulties relating Point *P* to the rectangle, both with or without the tool, indicates a random profile.

Mathematical communication. Listener and speaker: Both students are actively speaking, but they are not listening, as they are not building upon each other's expressions. This shows a lack of turn-taking and verbal integrity. Media of communication: The students communicate orally utilising the template. Communicational genre: Their communication is empractical as it is only understandable by observing their activity, as there is a lack of turn-taking, open-endedness and verbal integrity. Elin utilises the word *square* (lines 7 and 11), which is an incorrect name of the figure although she had earlier identified the figure as a rectangle. Elin's way of using the template is static, visible in her expressions and use of the word "stands" (in line 9), thus indicating *tool-idling empractical mathematical communication.* Donna has embedded the dynamicity from the tool into her communication based on her dragging, and she shows *tool-embedded empractical mathematical communication.* In both ways of communicating, communication competency is not exercised.

Excerpt 2: Negotiation and Further Investigation

This dialogue is the sequel of excerpt 1, meaning that Elin responds to Donna (line 12) in line 13

13	Elin	4... 12... what? [Elin drags point *A*]. Is it right what you are saying? Ups. [Elin drags accidentally point *D*].
14	Donna	You have to go back then [Donna points to the "regret"-button at Elin's computer and the figure comes back to the original one].
15	Elin	Like this. Uh. If you put this one up to 8 [the height of *AB* is then 8]. So it is up on 8… times 3... So it is on [she moves around the coordinate making *P* visible]. 8, 16, 24… It is right! [Responding back to Donna].
16	Donna	Yes. What are we supposed to write then?
17	Elin	The relation is that Point *P* then stands on
18	Donna	On the area
19	Elin	The number there is
20	Donna	The area of the figure
21	Elin	Yes [a bit doubting]

Instrumentation profiles. Mathematical knowledge: In line 13, Elin becomes aware of the relation between the rectangle's area and Point *P*'s *y*-coordinate. *P*'s *x*-coordinate is still not taken into account, but the rectangle's width is implicitly used in the relationship between the height of the rectangle and the *y*-coordinate of *y* (line 15). Covariation is not addressed. Information tools: GeoGebra again serves both static and dynamic purposes. Elin utilises dragging to gain information about *P* and the rectangle (line 15). Then, Elin stops using the tool again and utilizes the template as a static representation again. Elin begins to listen to Donna, making each other into a resource. Strategies: The students identify numbers and compare them; they perform simple calculations and validate their results while dragging. Summing up, particularly Elin shows a *mechanical* work method in this excerpt based on simple investigations in GeoGebra and the accumulation of results stemming from dragging. Donna mainly participates on a practical level, while Donna shows Elin how to use the "regret"-button (line 14), indicating a random profile for Donna due to difficulties.

Mathematical communication. Listener and speaker: In excerpt 2, Elin begins to listen to Donna (line 13), showing turn-taking and verbal integrity. However, there is also unequal participation, since Elin's expression indicates the assumption that Donna understands wrongly. Media of communication: Oral communication and GeoGebra. Their oral communication is only understandable when seeing the use of the tool, as it is embedded in the practical activity. Communicational genre: The students communicate empractically as they are still immersed in the

practical activity. This induces that the communication competency is only exercised in a much-reduced way. In parts of Excerpt 2, the tool is idle and dynamicity is ignored.

Excerpt 3: Characterising the *x*- and the *y*-Coordinate

This excerpt continues as excerpt 2 stops.

22	Donna	[reading from task 3] summarize your results from task 2; write here what you agreed upon.
23	Elin	But we also have to find out for point *P* what characterises the *y*- and *x*-coordinate? [Referring back to subtask 2]
24	Donna	What?
25	Elin	Side length times 3, because it says that it is 4, so 4 -8- 12, so if it is on 6, then it will be at 18 [Elin drags point *A* to get the height of *AB* to be 6, and *P* is at (6, 18)]. If I go to 8, it should be at 24 [she drags and tests again]
26	Elin	Side length for the rectangle. Side length for the rectangle times 3. *n* times 3. *n* times 3 equals 3.
27	Donna	Isn't just like that
28	Elin	Wait a bit. I have an idea for a formula. If you say that *n*, *n* that is the side length. No *x* times 3 equals with. No, wait. Now I am lost. Now I know what the area is. The area is *x*, *y* times 3 equals 3 *y* times 3 equals *x*. *x* squared

Hereafter, the students keep trying to clarify the coordinates, yet without getting closer to the solution.

Instrumentation profiles. Mathematical knowledge: In this excerpt, the students again try to make sense of the x- and the y-coordinate, when they take a numerical view. Yet, understanding is lacking as Elin mixes up the *x*- and *y*-coordinates (line 28). They still show a pre-conceptual understanding of functions. They do not address the concept of correspondence, as they mix up *x* and *y*. In addition, Elin also adds a new variable, *n*. Information tools: They use GeoGebra – again static and dynamic. Then, they bring *n* into the dialogue, which may relate to earlier work. Strategies: Dragging is utilised as a strategy to test conjectures (line 25). However, the students' numerical view is reinforced by introducing the new variable, *n*, which normally is used for a natural number. However, the use of *n* could also be identified as a *copy-paste* strategy from earlier work involving formulas (line 26). Summing up, Elin works on the task showing characteristics of a *mechanical* and a *random* profile again. As Donna is not doing a lot of work, we cannot identify her profile in this excerpt.

Mathematical communication. Listener and speaker: Again, both speak, but without listening to one another: As Donna believes that they are done, and Elin wants to keep on working, they do not share the same goal, hence, their dialogue becomes complicated. Media of communication: They are communicating orally based on the static elements in the template. Communicational genre: Their communication is still empractical. Terms are used as keywords to serve the identified pattern, which emerged during their practical activity: what they see is what they get without conceptualizing. Overall, the students' communication ignores the dynamicity of the template; hence, it is tool-idling.

DISCUSSION: TWO WAYS OF EMPRACTICAL MATHEMATICAL COMMUNICATION

The purpose of this study was to gain a deeper understanding of two ways of empractical mathematical communication, tool-embedded and tool-idling, which we have previously identified (Bach & Bikner-Ahsbahs, accepted). To do so, we have analysed the students' solving of a task focusing on functions as covariation supported by dragging, the key feature for the interactive dynamic environment of GeoGebra (Johnson & McClintock, 2018). The results reconfirm our

previous results showing that both kinds of empractical communication correspond to mechanical and/or random instrumentation profiles (Bach & Bikner-Ahsbahs, 2020; accepted). This seems to have consequences for activating mathematical communication competency and it reflects the students' understanding of functions as covariation, which will now be explained.

Originally, instrumentation profiles were developed for CAS (Guin & Trouche, 1998). As instrumentation processes vary depending on which tool is used (Trouche & Drijvers, 2010; Artigue & Trouche, 2021), the instrumentation profiles had to be adjusted for DGE due to its key feature of dragging (Bach & Bikner-Ahsbahs, accepted). Due to the tight relation of empractical communication to the practical activity with the tool, the two students' communication is closely related to DGE and the instrumentation of dragging. In tool-embedded communication, the dynamic mathematical aspects are exploited by dragging, which makes the students' expressions more dynamic potentially supporting the conceptualizing of functions as covariations. This is different for tool-idling communication as it rather refers to the template in an image-like static manner. Earlier research showed that dragging could support understanding functions as covariation (Johnson & McClintock, 2018). However, in contrast to Johnson and McClintock's result, Donna and Elin do not show an understanding of covariation as *dragging does not serve as a resource for their understanding of functions* (Arzarello et al., 2002). A possible explanation is that understanding what happens mathematically during dragging deeply depends on the type of instrumentation, which determines the level of conceptualizing. A random and mechanical way of using dragging is only superficially related to an in-depth understanding of covariation.

Further, the analyses of the excerpts reveal that the students' communication does not involve active listening. This is evidenced by the fact that the students' oral expressions do not build on each other's ideas, a key characteristic for empractical communication in general (O'Connell & Kowal, 2012). Therefore, their empractical communication cannot serve as a resource for learning, which normally is essential in an instrumentation process (Guin & Trouche, 1998). The students' instrumentation profiles likewise force them to communicate empractically. Thus, we observe a mutual dependence between empractical communication and the instrumentation process, which adheres to the students' conceptualization of functions as covariation on a pre-level.

An essential part of mathematical communication competency is the ability to interpret and understand others' expressions (Niss & Højgaard, 2019), but empractical communication does not support this because therein active listening is not practised (O'Connell & Kowal, 2012). When Elin and Donna show mechanical profiles, they tend to refer to the tool instead of mathematics, and when they show random profiles, they both experience difficulties with the mathematics in play and the tool. Therefore, the two instrumentation profiles do not include a precise mathematical language, hence, they do not exercise mathematical communication competency.

All in all, neither tool-idling nor tool-embedded empractical mathematical communication exercise mathematical communication competency. In addition, our analyses also indicate why mathematical communication competency and the instrumentation of a digital tool cannot well be developed simultaneously. (1) Empractical communication hinders students to exercise communication competency and (2) random and mechanical instrumentation do not include the conceptualizing, such as functions as covariation, which is needed to further develop mathematical language for mathematical communication. Hence, empractical communication and random and mechanical instrumentation mutually impede each other, constraining mathematical communication competency to be developed. This result finally raises the question of what characteristics the design of a teaching-learning arrangement should have to support mathematical communication as a goal in itself as well

as overcoming random and mechanical instrumentation profiles. As our research is limited to some case studies, further research is needed to explore this problem and solve this design issue.

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CONNECTIVITY RELATED ISSUES IN A MODULARISED COURSE INVOLVING MATHEMATICS

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Universities of technology are increasingly looking for ways to facilitate individual student study paths, moving away from a prescribed sequence of courses. In the envisaged modularised curricula, authentic engineering tasks or forms of Challenge-Based learning play an important role (Gallahar & Savage, 2020). Students may need particular mathematical knowledge in order to successfully solve these engineering tasks or problems. Modularisation of mathematics courses is considered a way to provide students with the required knowledge when they need it during their education at the university or even after their graduation. Modularisation involves dividing the learning process into relatively small independent curriculum packages (modules) (Kiliç & Pepin, 2020). Digital technology is used for their provision and access.

In this poster presentation, we report on a study at a university of technology in the Netherlands. We posed the following research question: In which ways can modularised courses involving mathematics for engineering education be enhanced for the benefit of student learning?

In terms of data collection, we observed a modular course requiring mathematical knowledge of statistics and probability. The course consisted of three modules, which were enacted sequentially and assessed separately. The first module addressed mathematical pre-knowledge that was required to understand the subsequent modules. The second and third modules concerned embedded systems and their modelling, and used knowledge from the first module. We conducted interviews with course instructors (N=5), selected students (N=6), and university employees (e.g., deans, education directors, teacher support) (N=4). In the interviews, we asked the course instructors and university staff about the conditions for effective modularisation of courses and their specific experiences with modular courses. We asked the students about their experiences in modular courses and their expectations regarding modular courses in general. We asked how guidance and support in modular courses could help for their learning, in particular how to connect mathematical knowledge to their disciplinary knowledge and skills. Moreover, we asked how modularisation could help students to develop themselves in the engineering profession. We analysed the data using a grounded theory approach (Strauss & Corbin, 1994).

Four main themes emerged from the interview data that are likely to have an impact on modular course design: (1) the importance of connectivity, (2) the role of mathematics knowledge in engineering education, (3) the need for technological support, and (4) practical related issues regarding the sequencing of learning activities and assessment. Out of these themes, specific suggestions were formulated to enhance modularised courses involving mathematics: (a) clear outline of dependencies within and between modules; (b) provision of flexible and adaptive technology-based resources catering for diverse student backgrounds and needs; (c) identification of mathematical pre-knowledge for each module involving mathematics; (d) support for students and instructors to bridge the gap between general and applied mathematics knowledge in engineering modules; (e) support of students (e.g. via technological means) to follow the module flow; and (f) provision of learning activities and assessment to support self-directed student learning. According to the respondents, mathematics modules need to be well-connected to each other and

tailored to each engineering program in which they will be used in order to to make them fit into the learning lines of the curricula and to create 'undisturbed' student learning paths. These connections might be realised by a self-explanatory module structure and by flexible and adaptive resources for learning, allowing for differentiation. To meet these demands, a technologically enhanced learning environment would be required. Such a system might also be used to support students and instructors: guiding students (e.g. with digital self-assessment/feedback) on how to accomplish their goals with the help of the knowledge and skills they develop using the modules and supporting instructors in terms of coherence of their modularised courses. Regarding the assessment procedure, unlike conventional courses, assisting students at particular times before final assessment will be difficult in the constructed and pressured form of modular courses. Therefore, assessment usually takes place at the end of each module (that spread over the semester). Moreover, structured and iterative formative feedback might help students to become autonomous learners. Providing feedback on student progress throughout the module will help students to know where they are in their knowledge development and to develop metacognitive strategies (Pepin & Kock, 2021).

Critical for modular course design and use seems to be the concept of connectivity (Pepin, 2021) that refers to the links and relations made (1) within the mathematical module content (e.g., between different mathematical representations; intra-modular connectivity); (2) between modules and courses (e.g., how the contents of different modules are related; extra-modular connectivity). An appropriate level of extra-modular connectivity is necessary for students to develop their own meaningful study paths and meet curriculum requirements when a prescribed sequence of courses is no longer available. Moreover, connections made between a mathematical module and engineering applications of mathematics will help students give meaning to the mathematical content. A sufficient level of intra-modular connectivity appears essential for students to develop a rich network of mathematical concepts and, in this way, a comprehensive appreciation and understanding of the mathematics itself (Pepin, 2021).

As far as we are aware, the electronic learning environments used at universities of technology do not generally offer the levels of connectivity we have discussed here. Hence, technological developments of these systems need to be considered. Moreover, teachers and educational designers who configure the systems need to be aware of the importance of connectivity to enable the full potential of modular mathematics courses in engineering education.

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THE PURPOSE OF HANDWRITING WITH TABLET-COMPUTERS AND SMARTPENS IN MATHEMATICAL GROUP WORK OVER DISTANCE

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In traditional group work in mathematics, handwriting is a relevant element to enable reasoning, for instance, by supporting the generation of ideas or the storing of information. However, as COVID-19 has forced students to learn mathematics over distance, trad½itional handwriting cannot be used anymore during group work. To address this issue, this exploratory study investigated the question of how students can use handwriting in a mobile-learning setting via Zoom, in which students use tablets and smartpens to collaborate over distance. It was found that, compared to traditional group work with pen-and-paper, the distance collaboration setup allows for handwriting to become a synchronous collaboration tool, for example, enabling the individual development of ideas that can be extended by peers. More research is needed to investigate the particular epistemic role of handwriting and, particularly, the role of handwriting with smartpens in distance collaboration settings.

Keywords: Distance collaboration, mathematical reasoning, mobile learning, writing.

INTRODUCTION

Writing in university mathematics has different functions. Firstly, as a means of documenting and consolidating work (Heintz, 2000), secondly, to communicate with peers, and thirdly, to publish in academic journals (Burton & Morgan, 2000). During collaborative activities where learners negotiate mathematics, writing has a communicative purpose, as it allows learners to realize mathematical objects through symbolic, graphical, or concrete representations (Duval, 2006) and also through vernacular language (Sfard, 2008). For instance, the area of a triangle is not a tangible entity but can be realized as an object through the symbolic representation $A = \frac{1}{2} \cdot a \cdot h$ or by drawing an arbitrary triangle. In this form, writing is mainly happening in the form of handwriting to document mathematical work. Accordingly, handwriting in mathematics is a reasoning tool that allows learners to document their work with mathematical objects, and through this, make this work applicable for negotiations and reasoning.

Traditionally, mathematical handwriting is based on pen-and-paper. As such, it was found that blackboards and paper are very relevant for communicating mathematics in a face-to-face situation, as they provide material links to previous mathematical reasoning, which helps to avoid communicational breakdowns (Misfeldt, 2006). It was also found that computers do not help in this same way in face-to-face communications (Misfeldt, 2006).

With the ubiquity of tablet computers, which provide smart pen functionality, this stance towards computers for facilitating handwriting in communication needs to be revisited. Particularly, with the COVID-19 pandemic, students were forced to collaborate over distance, being relegated to use computers for communicational purposes, even though these might be counterproductive for mathematical communication. In fact, the notion of face-to-face communication changes now that distance collaboration tools such as MS Teams or Zoom are being used frequently, which allows face-to-face communication over distance. Yet, there is little research on how writing functions in these "new" face-to-face collaborations over distance in a university mathematics setting.

THEORY

With respect to the functions of handwriting in mathematics, Misfeldt (2006) provides a categorization of the purposes of traditional handwriting in mathematics, of which the following are relevant for the here investigated issue:

1. *Heuristic treatment:* Learners heuristically come up with ideas, try them out and make connections between them.
2. *Control treatment*: Learners engage in a deeper investigation of their heuristic ideas. Control treatment can take the form of an investigation of a proposition or an open-ended investigation, for example, by means of performing a calculation.
3. *Information storage*: Learners write in order to save information for later access and use.
4. *Communication:* Learners use handwriting for communication in various forms, such as annotating existing text or commenting on ideas (Misfeldt, 2006, p. 27).

In particular, Misfeldt found that the communicative function of handwriting can come in the form of public or private communication. The public function of handwriting consists of students using written signs to communicate an idea or previously written signs as a deictic or gestural reference in oral communication. The private function of handwriting consists of learners using writing to create a private space for developing new ideas on their own, without immediately making these ideas public to the rest of the group (Misfeldt, 2006).

With respect to hybrid collaboration, the incongruence between computer code for writing formulas/diagrams and the conventionalized mathematical formulas that can be easily used during handwriting can lead to a breakdown of communication, hindering learners from using computer writing for public and private communication (Misfeldt, 2006). In other words, hybrid collaborations with traditional computers without touch functionality can hinder students' collaboration in mathematics, as it does not allow for convenient use of writing for the above-described four functions of mathematical writing, and in particular, writing for public communication purposes.

Yet, mobile technology has changed the notion of computers, allowing for new forms of collaborative learning (Schuck et al., 2017). Mobile technology enables hybrid forms of communication, where students can collaborate over distance, as learners can see each other, screens can be shared (Pegrum et al., 2013), material can be distributed, or questions can be discussed via social media (Simonova, 2016) or tools such as MS Teams/Zoom. This form of hybrid collaboration is further supported by having immediate access to digital resources through a mobile network connection. As a result, learning can occur at any place or at any time, in collaboration with peers or even experts all over the world (El-Hussein & Cronje, 2010; Pegrum et al., 2013).

Thus, with tablet computers and smartpens, the problem of the inconvenience of using writing for public communication purposes could be alleviated, as it provides a convenient way for students to use handwriting in distance collaboration setups. However, there is a lack of research that addresses the issue of how university students use handwriting during distance collaboration with tablet computers and smartpens.

Therefore, this paper discussed the following research question:

With what purposes do students use handwriting in distance collaboration settings, where the distance collaboration is implemented with tablet computers and smartpens?

METHODOLOGY

In an exploratory case study, five groups of two university students (Groups A–E) volunteered to take part in the study presented here. The students were asked to work collaboratively on a proof in vector geometry, which was a familiar topic for students. They were recruited from a first-year course on linear algebra, which was taught in English as Medium of Instruction at a technical university in the Netherlands. The students were mostly Bachelor students of applied mathematics or computer science. They were asked to collaborate over distance in a Zoom meeting, using iPads and smartpens. Students were asked to work in English, but Dutch-speaking students sometimes used Dutch during their work. The students reported that they experienced this setup as highly relevant because at the time of the study, they experienced a hard lockdown with limited opportunities for collaboration.

For the purpose of this study, a variety of tools to enable Distance Collaboration was used:

- *Web conferencing tool:* Zoom (iPad app).
- *Tablet:* Apple iPad tablets.
- *Smart Pen:* Apple Pencils.
- *Online whiteboard:* Students worked on a shared PDF-file, which acted as a whiteboard for enabling handwriting.
- *Keyboard:* Apple smart keyboards (to enable proof-writing).
- *Further resources:* The lecture script from the linear algebra course was given to students in the form of a PDF that they could access on the iPad.

The distance collaboration was simulated in a laboratory setting by asking students to collaborate from different rooms on campus. The data collection was realized with the video recording function of the Zoom app, resulting in videos where students' conversations and their writing on the white-board were captured simultaneously. Zoom was chosen because it provided the functionality of screen recording for data analysis purposes, the possibility to use a PDF as a whiteboard so that the task could be displayed on the students' writing space, and the integration of handwriting via a smartpen.

The students' collaborations were supervised by an interviewer, who provided students with help for using Zoom as well as content-related hints to ensure students' continued engagement with the proving task. The students were asked to prove the theorem shown in Figure 1.

Definition: A medial triangle of a triangle ABC is the triangle with vertices at the midpoints of the triangle's sides AB, AC and BC.
Theorem: The orthocenter of the medial triangle DFE of an arbitrary triangle ABC is the circumcenter of triangle ABC.

Figure 1. Theorem that students were asked to prove

Afterwards, the video data was transcribed. The transcripts were analyzed using inductive content analytical methods (Mayring, 2015). Firstly, in a segmentation step, instances of use of handwritings were identified, and the associated episodes inventoried. Secondly, by constant comparing and contrasting of these episodes with respect to the writing purpose (who is being addressed in the text), two main categories were found (general vs. mathematical purpose). Thirdly, all episodes falling under one category were further analyzed, resulting in a typology of episodes with similar usages of handwriting during distance collaboration. In this step, the above-described list of purposes of writing in mathematics was used as sensitizing concepts. Accordingly, the episodes below represent

idealisations of purposes of handwriting, with some groups utilizing multiple purposes at different times in their work. The different types of uses of handwriting are presented in the following.

RESULTS

Overall, the analysis found two different purposes for handwriting during distance collaboration on mathematical proving. Firstly, students used handwriting with smartpens for the general purpose to enable or manage the process of collaboration (see General Purposes 1 and 2). Secondly, students used handwriting with smartpens for a decidedly mathematical purpose, namely the purpose of mathematical problem solving, in line with notions that conceptualize proving as a problem-solving process (see Mathematical Purposes 1 to 3).

4.1. General Purpose 1: Explicit Reference Tool to Enable Communication in the Collaboration Process

It was found that handwriting can be a general reference tool to capture ideas in the process of collaboration, that is, with a general purpose. During this use of handwriting, students simultaneously use handwriting and oral communication, verbalizing their writing while they write. This use of handwriting seems to support information storage and public communication at the same time, allowing the other student to follow the thinking process of the other student. However, there was only one instance of this use in the data.

Group B, Minutes 38:13 – 39:44

Dirk: Yeah, the green ones are easier, so [refers to green lines in a previously drawn triangle]

Carl: [writes more and reads aloud what he writes]

Dirk: Good point. How do you find CDE?

Carl: You have to find the intersection point of AF and then … let's take BD [carries on writing and reading loud]. I have to make the line BD, right?

In the episode, a student represents the vertices of a triangle, using handwriting to express vertices in their vector representation. Here, the use of handwriting as a reference tool allows the student Dirk to follow the other student Carl. This enables both students to ask each other questions and to answer them. Here, students also refer to their drawing, in this case as "the green ones" with which he refers to green elements in their drawing.

4.2. General Purpose 2: Structuring the Collaboration Process

In this category, students use handwriting with the explicit general purpose of structuring their collaboration process. In these episodes, the students talk about how to best capture their work. In other words, the students work on the metalevel of how to best use handwriting during their proof writing. As the transcript below illustrates, the student Esha explicitly names the function of handwriting for accomplishing a "more concrete idea of what we are doing".

Group C: Minutes 43:59 – 44:09

Esha: Yeah, I think you can start to write it down, so we have a more concrete idea of what we are doing.

Faiza: Wait, I want to first draw [erases]. I am going to draw a non-conventional triangle. [starts drawing and writing]

This episode reminds of the function of handwriting for heuristic treatment, as the student Esha explicitly mentions that they intend to form a more concrete idea.

4.3. Mathematical Purpose 1: Explanation of Mathematical Thinking to Partner

With a specific mathematic purpose, handwriting was used to explain a mathematical concept to the other student. This purpose seems to be a form of public communication. The case below highlights this purpose on the example of the definition of the orthocenter. In the episode, the student Hendrik makes use handwriting to make a drawing of the geometrical situation in the given proving task, in order to explain his thinking process to his partner Gemma.

Group D: Minutes 19:24 – 20:56

Hendrik: Yeah, I could not find orthocenter either. Ah, yes, okay. I found it. The orthocentre is the point … wait, I will just draw it. [draws]

Hendrik: Yeah, the altitudes of the triangle passing through a common point. So it is… [draws] and, that one [draws]. The orthocenter of the medial triangle, the circumcentre of the triangle ABC… the medial triangle. Circumcenter. Perpendicular bisector. The line should draw the perpendicular bisector, right?

Notably, in this episode, the student also uses the digital environment to connect different sources for meaning-making, namely the lecture script to look up a definition of orthocenter and handwriting to realize the definition in their drawing of a triangle. Hence, handwriting and the digital environment contribute to each other for the benefit of the students' collaborative mathematical reasoning.

The presented function reminds of the purpose of handwriting for public communication, as proposed by Misfeldt (2006). Here, this purpose of handwriting is tightly ingrained into the overall reasoning process of using established knowledge to generate ideas, enabled by a digital resource (the lecture script).

4.4. Mathematical Purpose 2: Visualization of Mathematical Processes

In the second mathematical purpose, students use handwriting to generate a drawing that represents the situation described in the proving task. As can be seen below, initiated by talk to structure the reasoning process (Marc in turn 1), the student Marc begins to draw a triangle. During this process, the students try to understand the concepts in the task description (orthocenter, medial triangle, circumcenter). As the student Marc is thinking aloud, his partner Lisa can contribute to this process (Turn 4).

Group E, Minutes 16:14 – 16:59

Marc: First, let's draw a triangle? [draws]

Lisa: Yeah.

Marc: The definition here. [writes] I think. The orthocenter of the medial triangle DFE of an arbitrary [reads]. Erm, what is an orthocenter?

Lisa: Orthocenter, I think it's orthogonal.

Marc: Ah! Of the medial triangle DFE. So the medial triangle is so [draws]. Is the circumcenter of triangle…

Similar to the previous episode, the students use handwriting for public communication and for storing information. Particularly, the students use the drawing process as a means to understand the given task. This public communication ensures that the partner who is not writing can contribute to

the process and possibly check whether the concepts in the task have been adequately realized in the other student's writing.

4.5. Mathematical Purpose 3: Semi-Private Reasoning

It was expected that in the distance collaboration setting, there would be few opportunities for students to use handwriting to create private spaces. Yet, contrary to this expectation, handwriting in the distance collaboration setup was also used for creating room for individual reasoning. The following episode highlights how the student Esha developed her reasoning, supported by handwriting. The other student, Faiza, gave Esha room to develop her thoughts. At the same time, as Esha's reasoning was not private in the actual sense (that is, the other student can see the writing), Faiza can build on Esha's reasoning afterwards.

Group C: Minutes 18:47–21–57

Esha: You want to compare the angles? I don't [think it] will actually work. So, basically, if we take the center to be O… So far, let's just assume that this is the circumcenter. So we have to [incomprehensible] the orthocenter is also the circumcenter. The orthocenter of DFE is the circumcenter of ABC. So, we know that [starts writing] OA = OB. It's also obvious if you take the triangle OAB because OA = OB because it is the midpoint. So we have that, but how do we prove that the definition of the orthocenter is the perpendicular bisector? How do we prove that? Can we prove [incomprehensible] Yes, OK.

Faiza: It would be easier to [incomprehensible] the perpendicular bisector if we, for some reason, know it is an equilateral triangle. Because we know… let's say it was not in any case … it would not go to the same point for all of the…

Thus, similar to creating private spaces in traditional group work settings, handwriting can support the creation of individual lines of reasoning in distance settings. Possibly, handwriting functions as a signal to the other student to give some room for developing such a line of reasoning. Interestingly, in contrast to a traditional setup, the distance collaboration ensures that handwriting cannot be completely private, giving the other student the opportunity to extend or build on the student's line of reasoning. Therefore, in distance collaboration setups, handwriting does not support the creation of actual private spaces, but the creation of semi-private spaces for individual thought that, at that moment, is independent of the partner but can easily be taken up by the partner later on.

CONCLUSION AND DISCUSSION

This paper investigated the question of with what purposes do students use handwriting in distance collaboration settings, where the distance collaboration is implemented with tablet computers and smartpens. Overall, it was found that one can distinguish between two different purposes: a general purpose to structure the process of collaboration in the distance setting (see section 4.1. and 4.2.), and a specific mathematical purpose to support mathematical reasoning and fulfilling the mathematical task at hand (see section 4.3. to 4.5).

The second purpose (section 4.3–4.5), which highlight particularly mathematical functions of handwriting, does remind of the heuristic functions found by Misfeldt (2006), but also showcase differences due to the distance setting (for example, private space in contrast with semi-private space).

Overall, the main finding presented in this paper is that, in contrast to the traditional use of handwriting in pen-and-paper setups, the distance collaboration setup allows for handwriting to become a synchronous collaboration tool. This collaborative function is probably enabled by the fact

that handwriting is always public, that is, visible to the other group members. If implemented over a longer time, handwriting could become a fully utilized collaborative tool in distance collaboration, possibly fulfilling similar functions as oral communication. However, compared to traditional setups, handwriting has some limitations here, as deixis or gestures cannot be used to reference previously written elements. This limitation could well be an advantage, as it forces students to make implicit connections explicit in their reasoning process, e.g., by highlighting written elements or by color-coding elements (as Mathematical Purpose 1).

Following research in writing didactics, mathematical writing can be understood as a problem-solving process requiring writers to make decisions about how to represent mathematical objects and their manipulations (Kruse & Ruhmann, 2006). The mathematical purposes of writing found here support such a conceptualization of mathematical writing as a reasoning process. Accordingly, similar to findings in the secondary school context where mathematical writing was found to be beneficial for consolidating and reviewing knowledge (e.g., Colonnese et al., 2018), it can be suspected that mathematical writing, and particularly 'forced' public writing, can have similar benefits for collaboratively consolidating or reviewing knowledge in the process of proving. Hence, there is a further need to investigate the epistemic role of handwriting in learning mathematics. Such research could also address the question of whether handwriting on tablet computers could have further benefits compared to traditional handwriting in mathematics, as tablet computers allow students to integrate other resources into the writing process, such as the lecture script or online searches.

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AFFORDANCES OF UNIVERSITY-BASED ONLINE LABORATORY SCHOOL: TYPES OF FEEDBACK

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The purpose of this research is to investigate the types of feedback given by supervisors to preservice mathematics teachers (PSMTs) in a University-based Online Laboratory School (OLS). The OLS was founded during the Covid-19 pandemic aiming to provide online internship experiences. In the mathematics program for Spring 2020, there were 124 students (4^{th}–7^{th} grade) who participated in the school, and there were seven supervisors who gave feedback to 23 PSMTs. The feedback was gathered from three different recorded sources, moderator chat boxes, short reflection, and general meetings. Content analysis was used as the method of analysis. Feedback given during moderator chat boxes and short lesson meetings showed similarities and were mostly about 'teaching process', whereas the feedback pieces given during general meetings was about 'planning'. We will discuss the benefits and disadvantages of the feedback given during OLS and its contribution to the field.

Keywords: Feedback, mathematics, online teacher education, preservice teacher education.

INTRODUCTION

The primary role of the supervisor in clinical supervision is to strengthen the prospective mathematics teacher's (PSMT) ability to assess his or her own teaching (Kent, 2001). Supervisors should move beyond the internship model to a critical, dialogical approach where PSMTs and supervisors work together to transform teaching and learning (Beck & Kosnik, 2010). PSMTs often depend on their supervisors' feedback in helping them improve their pedagogical and personal skills (Ali & Al-Adawi, 2013).

Due to the Covid-19 pandemic, field experiences and internships for PSMTs became harder to create. Schools were on a break of face-to-face education, and PSMTs were not able to continue their internships and field experiences face-to-face. Hence, to eliminate this unexpected outcome, an Online Laboratory School (OLS) was founded in order to provide PSMTs with an online internship for five weeks for Spring 2020. OLS used a Learning management system, and PSMTs had their own virtual classrooms and real students. While PSMTs was teaching, there were supervisors observing and guiding them in the classrooms and provided feedback during (instant), after and in the general meeting.

Online Teaching Internships

With the evolution of online learning, it is imperative that teacher preparation programs offer not only online courses but also prepare preservice teachers to teach online (Feher & Graziano, 2016). Virtual internships can provide preservice teachers with the skills required to teach online, and according to Theele et al. (2019), it can also engage PSMTs in a course about teaching strategies to make PSMTs familiar with the teaching context without a real-life internship. Future teachers must have the skills and knowledge to teach effectively in online environments, as well as in traditional environments (Duncan & Barnett, 2009). Kennedy et al. (2013) state that the key in the online teaching experience is communication with the supervising teachers. Their findings show that communication needs to be constant and deep to advance PSMTs' professional development. Quality mentors should provide

PSMTs with regular, timely, critical, and actionable feedback which relates to practice (Hounsell, McCune, Hounsell, & Litjens, 2008).

There is a lack of research about how to supervise and how to provide well-designed feedback as a formative assessment in online internship settings. We believe that the supervising procedures of OLS can provide research with alternative ideas on how to support/assess the PSMTs in an online setting.

OLS provides PSMTs with continuous support/feedback from supervisors for their professional development from three different sources. The first one is during the lesson implementation, from the moderator chat boxes, supervisors can give instant feedback, which is actionable. This kind of support enables PSMTs to take immediate action during their teaching. The other two are from the short reflection meetings and the general meetings enabling PSMTs to continue receiving detailed feedback and conduct a discussion enhancing their development. All these sources provide PSMTs with constant feedback/support from the supervisors and advance communication.

This study investigates the new ways within the 'new normal' in teacher education and the different feedback types given for the development of PSMTs. The research question is: What kinds of feedback is given by the supervisors during and after the implementation of mathematics lessons in an OLS?

METHODOLOGY

Context

The University within School Model

The university applies a model called 'University within School Model' (Özcan, 2013). Within this model, PSMTs are in close contact with the schools from the beginning of their first year until they graduate by performing some volunteer work, as well as fulfilling field experiences requirements for some of their courses. PSMTs are required to complete four semesters of internships within the last two years of their program. During their internships, PSMTs are assigned as teacher assistants to one of the 13 schools in Istanbul that the department carefully chose to work with. Each PSMT has a mentor teacher and a university supervisor during this internship experience.

During their internships, PSMTs work closely with their mentor teachers and receive constant feedback for their contributions and lesson implementations. In the first two semesters of their internship, PSMTs usually observe and assist the classroom teacher. In the last two semesters of their internship, they take the responsibility of planning and teaching the whole class at least ten times. The total internship hours range between 1,400–2,000. Right after the PSMT implements a lesson, the mentor teacher, supervisor, and the PSMT discuss the lesson in a three-way meeting. This was how the internship was designed and feedback was given before the Covid-19 pandemic.

Online Laboratory School

During the Covid-19 pandemic, the university took action to continue providing PSMTs with the experience they need no matter what, and the OLS was created in Spring 2020 (Tunç-Pekkan et al., 2020). This experience helped PSMTs experience teaching under different circumstances before they graduated and prepared them for the online teaching environment.

As Lave and Wenger (1991) stated, membership is an important condition for learning, and learning is relational. This relation and how such relations might be constructed in online laboratory school is a new area for research. In OLS, PSMTs and supervisors work closely together during lesson planning as well as implementation and reflection of their experiences. This is a new community and a different

situation than a physical one, so the membership construction and what is learned is different and needs to be investigated. Supervisors' feedback is a foundational component in building such relationships; therefore, we investigated the nature of the feedback and its role in PSMTs' development.

OLS lasted five weeks, PSMTs taught 4^{th}–7^{th} grade mathematics. Blackboard Collaborate Ultra was used as the learning management system. There were three virtual classes for 4^{th}-grade level, four virtual classes for 5^{th}-grade level, two virtual classes for 6^{th}-grade level, and three virtual classes for 7^{th}-grade level. Seven university supervisors guided 23 PSMTs throughout this online preparation program. In each virtual classroom, there were three PSMTs and at least one supervisor on duty: one main teacher, one PSMT as an assistant, and one PSMT as a substitute. We had three PSMTs in each lesson; in case, a technical problem or a connection problem occurred with the main teacher, the other PSMTs could help.

Blackboard Collaborate Ultra enabled PSMTs and the supervisors with a private chat section. The learning management system included the 'Everyone Chat' and the 'Moderator Chat'.

Moderator Chat Boxes. This feature was only available to those attending as moderators, i.e., the PSMTs and the supervisors. Moderator chat boxes came in handy because PSMTs were able to receive immediate feedback while teaching. The lessons were 35–40 minutes.

Short Reflection Meetings. After the implementation of the lessons, the PSMTs and the supervisor conducted a meeting in order to discuss the lesson and provide feedback. These meetings were called short reflection meetings and usually lasted about 10-15 minutes.

General Meetings. At the end of the day, all PSMTs and all supervisors met, and the general discussion of the lessons took place. These meetings were called general meetings and usually lasted about 1-1.5 hours.

These three sources of feedback were recorded for each lesson implementation. Moderator chat boxes were recorded in written form, whereas the short reflection and the general meetings were video recorded.

Data and Analysis

For this study, we focused on a smaller part of the project; feedback was given to 4^{th} and 6^{th}-grade classroom teachers since the researchers were also teachers in those grades. Because of the convenience, knowing the data by heart, it is decided that the research team's foci would be these grades. There were 11 moderator chat recordings and 12 short reflection meeting recordings for 4^{th} grade, and there were seven moderator chat recordings and eight short reflection meetings for 6^{th}-grade levels. In addition to those, there were four general meeting recordings conducted from OLS.

We used content analysis (Cohen et al., 2007) when analyzing the written chat box messages, recordings of the lessons and the meetings. The study is a qualitative and an interpretive study. In our study, we have analyzed the written chat box records and recordings of the meetings according to a coding scheme: this scheme was adapted using the formative assessment form that supervisors used for observing lessons (Yüksek Öğretim Kurumu, 1998; Bulunuz & Gürsoy, 2018). In that evaluation form, there were five categories: mathematical knowledge, planning, teaching process, classroom management, and communication.

During the analysis, we have eliminated some of the feedback instances based on their quality and effectiveness to the PSMT. For instance, feedback statements like 'the lesson went well', was not

included in our analysis. The reason is that they are ineffective with how the PSMTs leads the lesson; they are said to express opinions, not to provide further assistance to the PSMT.

An example from data is presented below to discuss how we categorized the feedback.

Mathematical Knowledge: The example lesson implementation (April 30, 2020, 4th grade) was about equivalent fractions. This data was taken from moderator chat box records. The PSMT (PSMT 1) opened an online application (Conceptua Math) to be used as an instructional tool. In the first question of the application (see Figure 1), there were two same size shapes, but they were divided into different pieces. The first shape was shaded, and 1/2 was written underneath.

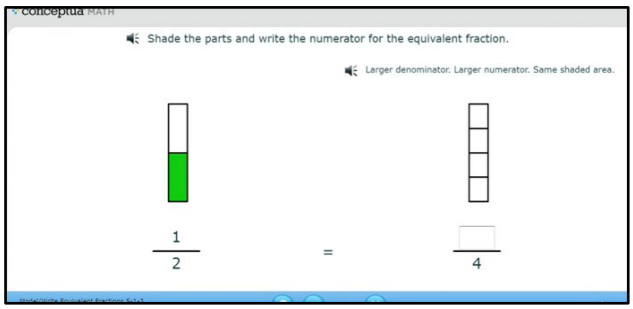


Figure 1. PSMT's presentation of the problem related to finding an area and equivalent fraction of 1/2

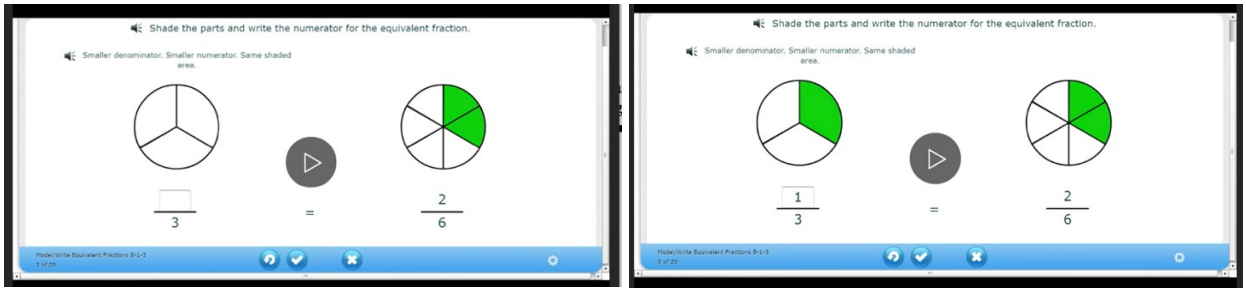
PSMT 1 presented the application and said:

PSMT 1: The one on the right is divided into four pieces; how many of them have to be painted? UE (specific student) you tell me. (PSMT actually does not give a full instruction)

Student: Is it because they need to be equal?

PSMT 1: Yes, they will be exactly the same.

Student: You need to paint two of them (PSMT 1 asks students how many were colored and gets '2' as an answer and types '2' in the numerator. She presents another problem; see Figure 2 (a))



(a) **(b)**

Figure 2 (a) PSMT's presentation of a problem related to finding an equivalent fraction of 2/6. (b) Screenshot after PSMT made the discussion related to painting and writing fraction of 1/3.

PSMT 1 : Who is going to answer in this question? İE should answer. 10 sec passed.

İE: For them being equal right?

PSMT 1: Yes.

Supervisor A wrote a question through the moderator chat box:

Supervisor A: What do you mean by saying they have to be equal.

In the meantime, in the main class, PSMT 1 talks:

PSMT 1: I want them to look like the same; how many pieces do I need to paint for them to look the same?

Student: One piece.

PSMT 1: This is one piece (clicking on one piece). Does it look like the same?

Student: It looks alike.

PSMT 1: So, what should I write to the numerator?

Student: You need to write one to the numerator.

PSMT 1: Ok. Let's look at this part, on one side, there is 1/3, and on the other side, there is 2/6. When I represent these two figures, using equal shapes, don't they look like same? (see Figure 2 (b)).

Students: … yes

PSMT 1: Have you heard of expansions of fractions?

Some Students: … yes

Supervisor A writes in the moderator chat box: What are equivalent fractions?

PSMT 1: So, you heard of it…1/3, when we expand it, it becomes 2/6. 1/3 and 2/6 look like the same as a model. Are we all ok? Until this point? Because of this, I call them equivalent fractions… one over three is equivalent to two over six. Let's keep going… (she moves to another question).

Supervisor A writes in the moderator chat box:

Of the same whole, if the parts show the same quantities of the whole, then they are equivalent fractions… You need to emphasize the equal wholes… We divide equal wholes into different parts, but those parts represent the same quantity, and we call that fraction (as in equivalent fraction).

After posing and helping students to solve another similar problem, PSMT made a closing comment with using the feedback from the Supervisor A:

PSMT 1: When we model equivalent fractions, we need to pay attention to some points, these models, they all have to be in equal size; is that ok? Secondly, the pieces that I made, for example, in the previous problem it was 1/3, I divided them into three pieces; these pieces have to be equal to each other.

The feedback statement given by Supervisor A was counted as one instance since they were related to PSMTs' knowledge related to equivalent fractions. Through the classroom instruction, PSMT 1 seemed to give incomplete directions and to make not enough connections to the figures and goals of the lesson. She had the goal of teaching 'equivalent fractions' but this goal seemed to be implicit. Supervisor A tried to guide the PSMT 1 by asking her to make her problem statements complete and making her provide explanations related to 'equivalent fractions'. Supervisor A seemed not to be satisfied as she guided through the lesson, which was also observed in her discussion in the short reflection meeting and in the general meeting.

Planning: This data was taken from the general meeting's records. Feedback from Supervisor B made for the 4th-grade lesson plan about equivalent fractions:

> Models alone do not work on a fraction concept that they just learned. I gave them this suggestion; the best concept is half because students grasp the concept of half really good, and it's real-life. Daily examples are also good and can be provided with visuals. We used to give videos to 5th graders, and it could also work with 4th graders as well. A pizza gets divided into two, and one-piece is eaten or served, gets divided into four equal pieces, and two pieces are eaten or served. In the end, when they see they are all the same amount, students can make really good inferences here.

This feedback is placed under the category of planning because it suggests an alternative approach to develop their lesson planning and an integration of appropriate materials into the lesson plan.

For the remaining three categories, we will provide descriptors of the situations where supervisors gave feedback to PSMTs. Because of the limited space, we cannot give detailed examples.

Teaching Process: There were many other situations that we coded as teaching process such as if a PSMT: a) Has used different teaching approaches in an efficient and flexible way during implementation, b) Has known when to directly answer students' questions and when to turn it into a thinking opportunity for all, c) Has tried to expose different understanding types and used them in her explaining, d) Has differentiated whether the students were following or not and adapted the flow of the lesson accordingly, e) Has used computer technology or other supplying materials in an efficient way, and the supervisor gave feedback based on them.

Classroom Management: Included feedback based on situations such as PSMTs using time efficiently, efforts for the participation of all students, creating efficient routines, using positive and negative feedback in an efficient way

Communication: Included feedback based on situations such as PSMTs' voice tone, using understandable instructions and language, terminology etc. during teaching, using technology in an interactive way to make communication better

RESULTS

Different *feedback types* given by the supervisors, a) During lessons through moderator chat boxes (written), b) in Short Reflection Meetings videos, and c) General Meetings videos were coded using the framework explained earlier and presented in Table 1. There were three virtual 4th-grade classes and two virtual 6th-grade classes.

There were 11 moderator chat box recordings of 4th-grade classes and seven moderator chat box recordings of 6th graders. There were 20 Short Reflection Meetings videos (12 from 4th-grade classes, and eight from 6th-grade classes). There were four General Meetings videos used for analysis. In Table 1, the source, type of feedback instances and frequencies are given.

The most amount of feedback statements were placed under the category of 'teaching process' through moderator chat boxes, and these were especially given in the 4th-grade classrooms (see yellow highlighted in Table 1). The reason for this situation might be due to the nature of the teaching process, which is more open to immediate intervention.

In short reflection meetings, the most amount of feedback statements were given under the category of 'teaching process'. This situation can be explained due to these meetings being able to provide

more time and reflection opportunities to discuss the feedback given through moderator chat boxes in a deep manner.

The most amount of feedback statements were given under the category of 'planning' through general meetings. The reason for this situation might be that during general meetings after each PSMT experienced teaching the subject in parallel classes, supervisors could also see the differences in implementations and could make a general analysis of what worked well or not in the plans.

Feedback Source	Moderator Chat Boxes		Short Reflection Meetings		General Meetings		Total		
Grade	4th	6th	4th	6th	4th	6th	4th	6th	
Mathematical Knowledge	10	2	6	2	4	2	20	6	26
Planning	3	2	9	9	14	9	26	20	46
Teaching Process	34	11	16	12	3	3	53	26	79
Classroom Management	19	4	6	10	2	7	27	21	48
Communication	4	3	4	9	0	4	8	16	24
Total	70	22	41	42	23	25	134	89	
	92		83		48				

Table 1. Frequencies of different feedback statements, sources, and grade levels in OLS

CONCLUSION AND DISCUSSION

Generally, moderator chat boxes were used more for the 4th-grade level. This might be due to the fact that the 4th graders are younger in age and require more management, thus leading PSMTs to be more in need of feedback/guidance. Along with that, most of the PSMT did not teach 4th grade before. Therefore, PSMTs needed further support, and they received the most feedback through this source.

The quality of the feedback given through the categories of classroom management and communication can be discussed from different perspectives. In our study, we witnessed that the feedback statements placed in classroom management categories are mostly about how PSMTs manage the participation of the students and the timing. In addition, feedback statements placed in the category of communication is mostly about the technical issues and the use of voice. If the internship was face-to-face, there is a probability that we would see more feedback given on these categories because, in a face-to-face classroom environment, feedback given could concentrate more on these categories. However, this case is different in an online setting. Classroom management and communication are considerably easier to deal with by PSMTs. Also, supervisors may have preferred to focus more on the other three categories since PSMTs needed more support for their improvement.

In addition to these findings, PSMTs are able to make immediate changes in their actions while they teach and face the consequences of the actions immediately via moderator chat boxes. Besides moderator chat boxes, short reflection meetings and general meetings were opportunities for PSMTs

to receive deeper feedback/support. All of these qualities of the OLS makes the internship meaningful, and it strengthens the University Within School Model.

We can also conclude that in the online teacher education process, PSMTs are eligible to receive more support than they used to when teacher education was face-to-face.

Furthermore, the accuracy and the reliability of the findings from this study is not guaranteed to reflect generalizability. Analysis from a larger group should be examined in order to create more general outcomes. Our results are aimed to provide the literature with the alternative approaches to online internship and to online PSMT education.

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EXAMINING HEURISTIC WORKED EXAMPLE VIDEOS IN A COLLABORATIVE SETTING: THE CONCEPTION OF THE PROJECT MOVIE

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Keywords: ICAP hypothesis, interactive video, mathematical modeling, secondary education.

THEORETICAL BACKGROUND

Videos are gaining increasing importance in the educational setting. As video production has become easier, many videos can be found online. A frequent video format used is presenting a problem and a step-by-step solution ("worked example videos") such as those found on the KhanAcademy (https://khanacademy.org/). Studies have shown that (paper-based) worked examples are beneficial in algorithmic domains (e.g., Sweller & Cooper, 1985) as well as in less-structured domains like modeling using heuristic worked examples (e.g., Zöttl et al., 2010). There has been little research on worked example videos in mathematics, but Kay and Edwards (2012) provide first results that this video format is beneficial in a middle school algorithmic domain as those kinds of videos have had a significant impact on short-term learning in their study. Moreover, there might be particular advantages for cooperative work, as shown, for example, in the subject of physics for dyads (Hausmann et al., 2008, 2009). This result is in line with the ICAP hypothesis (Chi & Wylie, 2014), which suggests that students' learning outcome will increase with the mode of engagement (interactive > constructive > active > passive). When watching a video, a constructive mode of engagement would, for example, include making sense of the concepts displayed in the video to oneself (i.e., creating self-explanations). The interactive mode of engagement also involves the discussion with a peer about those concepts and thus constructing knowledge collaboratively (Chi & Wylie, 2014).

The project "MoVie – Modeling with Videos to Enhance Students' Competencies" explores to what extent videos can be used to foster heuristic skills in mathematics in a collaborative setting. It provides a framework for creating "heuristic worked example videos", analyzes students' behavior while working with those videos, and studies how they affect strategic knowledge and solution processes in the domain of modeling.

RESEARCH QUESTIONS

When students work with heuristic worked example videos in the domain of modeling, the following research questions emerge:

RQ1: Which patterns do students show while working with the heuristic worked example videos?

RQ2: How and to what extent do students articulate self-explanations while working with the integrated self-explanation prompts?

RQ3: Which changes in terms of modeling-related strategic knowledge can be observed after working with the heuristic worked example videos?

RQ4: Which changes in the solution process of modeling tasks can be observed after working with the heuristic worked example videos?

METHOD AND OUTLOOK

In a laboratory study, dyads of 12th-grade students will work with heuristic worked example videos. The production of those worked example videos is based on guidelines from heuristic example research (Reiss & Renkl, 2002; Renkl, 2017) and is being combined with Mayer's (2020) cognitive theory of multimedia learning. The videos are segmented based on a solution plan, with each segment displaying one step of the modeling process. Furthermore, the videos include an integrated break between each segment. They include self-explanation prompts and explicate used heuristics. The dyads will be videotaped in order to analyze the usage pattern and the communication (RQ1 and RQ2). A pretest and posttest containing a modeling-related strategic knowledge test and different modeling tasks are used to address RQ3 and RQ4. Stimulated recall interviews will be conducted to gain a deeper insight into example processing and the usage of heuristics.

The goal of this study is to generate recommendations on how videos can be implemented in a collaborative and thus communicative context to help students develop modeling competency.

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Theme 4: Innovating with Technologies

for mathematical learning

AN INTERACTIVE DIGITAL ENVIRONMENT FOR TEACHING AND LEARNING DEDUCTIVE GEOMETRY (FULLPROOF): DESIGN PRINCIPLES, FUNCTIONALITY, PEDAGOGY AND IMPLEMENTATION RESULTS

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Keywords: Deductive geometry, FullProof, interactive digital environment.

INTRODUCTION

Learning and teaching geometry in schools and universities is considered a serious challenge (Jones & Tzekaki, 2016). Learning and teaching deductive geometry is considered even more challenging (Duval, 1998; Hartshorne, 2000). For decades, the professional community has been looking at how to tackle this challenge. In modern mathematics education, in particular, during the Covid-19 era, the use of digital technology seems to be relevant more than ever. The professional literature indicates that teaching mathematics using technological tools helps in the process of constructing an abstract knowledge of mathematics, and geometry, in particular (Lagrange et al., 2003).

The FullProof platform has been designed to address the complex challenges of deductive geometric proofs, combining a smart algorithm with advanced pedagogical approaches, providing the users with an effective teaching and learning environment.

THE FULLPROOF PLATFORM

The FullProof platform supports various possible solutions of geometric proofs and enables pedagogical scaffolding such as interactive diagrams and smart clues. The platform provides the students with immediate, detailed, and personalized feedback on their solutions. The teacher receives automatically a set of reports that tracks the class progress, with options to zoom in on every individual in the class. During the past three years, the platform has been implemented successfully in dozens of middle schools, high schools and colleges. The platform is integrated naturally in all teaching and learning phases, including frontal teaching, remote (online) teaching, hybrid teaching, and self-guided practice.

AIMS OF THE WORKSHOP

The participants were introduced to FullProof's capabilities through its interface, functionality and reports, followed by a discussion of its potential and consequences of its implementations. The workshop focused on both the students' and teachers' points of view as a base for a professional discussion.

The workshop was in two parts. The first workshop focused on student options within the platform. The participants accessed and experienced the user side while solving questions using the platform. The second workshop focused on teacher options and on pedagogy, which was followed by a presentation of the results of research done with FullProof.

Each session concluded with a group brainstorming session. The main outcomes of the reflective discussions regarded the platform's ability to accept any possible solution, provide personalized feedback and support the users throughout the learning process stages.

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UNDERSTANDING LINEAR FUNCTIONS IN AN INTERACTIVE DIGITAL LEARNING ENVIRONMENT

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Linear functions are the first important example of mathematical models that students face; crucial to their understanding is the role of the slope, which is a complex concept due to its many different conceptualizations. Problems in understanding the slope are often caused by difficulties in connecting its different meanings. This paper presents an interactive task developed in a Digital Learning Environment aimed at introducing linear functions in grade 8 and approaching an interconnected concept of the slope. The task was proposed to 299 Italian students in a classroom-based context. Through the analysis of a collective class discussion that occurred while solving this task, we show how the emergence of different conceptualizations of the slope can be elicited and supported by interactive technologies in a Digital Learning Environment.

Keywords: Formative assessment, interactive digital learning environment, linear functions, mathematics education, slope.

INTRODUCTION AND THEORETICAL FRAMEWORK

Linear Functions and the Slope

Linear functions are one of the first mathematical models students face in their studies. They emerge within algebra, since they involve simple operations among numbers and variables, and they offer many prompts for reflecting on and understanding mathematical models. The first hurdle that students have to overcome when dealing with linear models is the concept of variable, which is often not well defined in school Mathematics. It can create confusion among the terms variable, unknown, parameter and their relations with numbers and constants (Schoenfeld & Arcavi, 1988). The second hurdle is the dependence between "x" and "y", the variables through which argument and value are usually expressed. The concept of joint variation is one of the most problematic at school teaching. It seems that many difficulties with Mathematics, even at the university level, can be attributed to an underlying misunderstanding of this concept (Carlson et al., 2002). Joint variation is recurrent in secondary school Mathematics since several functions are studied with their properties and representation forms; linear functions are the first example through which this concept is approached. The third point that needs attention is the relationship between "m", the slope, and "c", the intercept, in the standard equation "$y=m\,x+c$" , which determines the trend of the line. A study by Bardini and Stacey (2006), focused on the understanding of m and c in linear functions, shows that, as expected, the slope is a more complex concept than the intercept. However, students tend to omit c as if it is not part of the function, maybe due to the little attention dedicated to the intercept compared to the slope in the classroom activities. Conversions among different semiotic registers (numeric, symbolic, graphic, and real-world context) seem to influence the students' interpretation and understanding of these elements (Bardini & Stacey, 2006).

The concept of slope is crucial for understanding linear functions and for the development of following important mathematical concepts, such as the derivative and differential equations (Rasmussen & King, 2000); its complexity and difficulty is probably due to its many conceptualizations, which could not be properly connected in the students' mind (Stump, 1999).

Based on Stump's work (1999), Moore-Russo and colleagues (2011) distinguished 11 different categories of the conceptualization of slope: geometric ratio (rise above run), algebraic ratio (change in y over x), physical property (steepness), functional property (constant rate of change), parametric coefficient (m), trigonometric conception (tangent of an angle), calculus conception (derivative), real-word situation (static physical situations such as a ramp or dynamic functional situations), determining property (property that determines if the lines are parallel or perpendicular), behavior indicator (real number which indicates the increasing, decreasing, or horizontal trends of a line) and linear constant (the property which shows the lack of curvature on a line). Above all, it seems that developing the concept of slope as rate of change at an early stage is crucial to make sense of the algebraic and graphic-related meanings (Deniz & Kabael, 2017). An interesting vertical study by Gambini et al. (2020) in the Italian context tries to analyze how the understanding (and misunderstanding) of the concept of slope changes from grade 8 to 14 (from lower secondary school to university), using data from national standardized tests and university entry tests. From the results, they observe that students have trouble integrating algebraic thinking and meaning. By grade 8, mainly reasoning numerically, they should have acquired the concept of variation (functional and physical properties) associated with the slope of a linear structure. By grade 10, students should have associated the symbolic and graphic aspects (geometric and algebraic ratio), but it seems that they have abandoned the quantitative reasoning, which helps confer meaning to the involved objects. This split between the different interpretations of the slope continues and is consolidated in grades 13 and 14 when the comprehension of the derivative concept would require integrating the different aspects, joining the mathematical formalism to the numeric, symbolic, and graphic aspects. What too often remains is the algebraic computations, disconnected by their meaning (Gambini et al., 2020).

An Interactive Digital Learning Environment for Mathematics

Interactive technologies are promising to be helpful in understanding dynamical concepts as linear functions and the slope. In this paper, the term "interactive" is intended in the Moreno and Mayer's (2007) meaning, which is a property of the technology through which the student's action is encouraged and where what happens next depends on this action. In previous work, we defined a Digital Learning Environment (DLE) as a learning ecosystem composed of a human part (the learning community), a technological part (constituted by a Learning Management System integrated with tools for doing and assessing Mathematics, populated by activities and resources, and by the devices to access the learning materials) and all the interrelationships among the components (interactions, methodologies, learning and teaching processes) (Barana & Marchisio, 2021). A DLE can enable learning and teaching in classroom-based, online, blended or hybrid modalities. The design of interactive activities in a DLE for this study follows Grabinger and Dunlap's (1995) model, such that they: evolve from and are consistent with constructivist theories; promote study and investigation within authentic (i.e., realistic, meaningful, relevant, complex, and information-rich) contexts; encourage the growth of student responsibility, initiative, decision-making, and intentional learning; cultivate an atmosphere of knowledge-building learning communities that utilize collaborative learning among students and teachers; utilize dynamic, interdisciplinary, generative learning activities that promote high-level thinking processes to help students integrate new knowledge with old knowledge; and, assess student progress in content and learning-to-learn through realistic tasks and performances.

This paper aims at investigating the following research question (RQ): how can interactive activities in a DLE help 8th-grade students approach linear functions and build an integrated concept of the slope? Based on the theoretical framework discussed above, in the following paragraphs, a teaching experiment is presented and discussed, with the purpose of providing an answer to this RQ.

METHODOLOGY

To answer the RQ, we designed an interactive task aimed at approaching linear functions, and in particular the role of the intercept and the slope, using different interacting conceptualizations of the slope. It was implemented in an interactive worksheet using an Advanced Computing Environment (Maple) in an integrated Moodle platform. We proposed the task to 13 8th grade classes (299 students) in Turin (Italy). The experimentation took place in 2018, before the pandemic, in a classroom-based context; the researcher—author of this paper—helped the teachers manage the activities. The task was included in a wider path on formulas and functions (Barana & Marchisio, 2019). The activities were videotaped and successively selected, transcripted and analyzed according to the Moore-Russo and colleagues' (2011) framework in order to identify how the different conceptualizations of the slope emerge in the DLE.

The interactive task on which we focus in this paper is shown in Figure 1. The problem leads to exploring three different linear functions: one passing through the origin, one intersecting the x-axis, and one intersecting the y-axis. Students are asked to explore the numerical representation of the problem filling in the tables in the interactive file, which are initially empty. In the box below, graphs are interactively generated with points and lines using data from the tables. The tables and the interactive graphs help students reason and visualize the trend of the reading of a book by the three friends. The activity engages the learners asking them to insert the graph of their reading, envisaging the speed they would read the book with; thus, it opens up to explorations, comparisons, and discussions. A set of automatically graded questions completes the activity, focused on the graphs' analysis, leading to writing the formulas through which it is possible to express the mathematical models. The questions are: (1) After how many days from the beginning of the Holyday will they end the book? (2) Marco is advantaged because he has already read 30 pages. After how many days will Valentina reach Marco? (3) Who reads faster from the day when they start reading? (4) How many days does it take Luca to read the book? (5) How fast should we read the book to have a vertical line? (6) Write three formulas that express the number of pages read by the three friends as a function of the Holiday days. (7) If the book was 300 pages long and the three friends would keep reading at the same pace, after how many days from the beginning of the holiday would Luca reach Valentina?

This activity was carried out in the classroom: the task was displayed at the Interactive White Board (IWB), and the students worked on one task at the time in small groups, with paper and pen. Each step was discussed with the teacher and the researcher through the IWB, using the interactive worksheet and the automatically assessed questions to drive and support the discussion. Since the experiment was also taken in schools in disadvantaged socio-economic contexts, it was not possible to make students access the activities through digital devices in the classroom; however, the activity, together with other similar ones, was available in the DLE and accessible from home.

RESULTS

In all the classes, the activity started with a verbal description of the real context. Students had to translate it into a numeric register filling in some tables. The students moved from the tables to the graphic register, and drew the points and the lines on a cartesian plane. As the last step, they had to deduce the algebraic formulas for the models. We selected an excerpt, a part of a collective discussion that followed the graphs' drawings, under the input of imagining how students themselves would read the book and add the trend of their reading to the other graphs. We present it in the following lines since it is meaningful to show the students' understanding process of linear functions and slope. The discussion occurred in a school mainly attended by students from lower social classes.

The Holiday Book

For the Summer Holidays the Italian teacher has asked students to read the book. The book is 180 pages long.

- Valentina starts reading it on the first day of Holidays and she reads 15 pages per day.
- Marco has already started to read the book and he has already read 30 pages, so he starts from page 31 on the first day of Holidays. He reads 10 pages per day.
- Luca rests for the first 4 days, then he starts reading the book and he reads 20 pages per day.

Who will be the first to finish the book?

Fill in the following table writing, in the "Pages" column, the book's point that the three friends reach at the end of each day.
Use the "You" table to indicate how you would read this book.
Then observe the graphs below and answer the following questions.

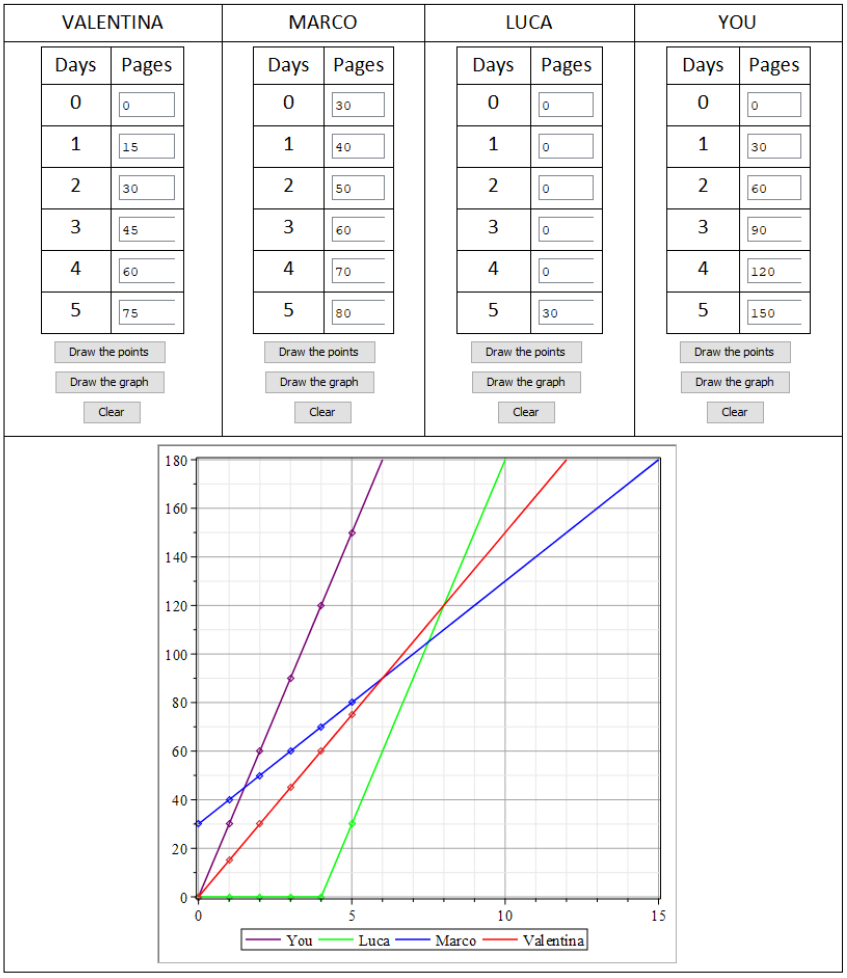

Figure 4. Part of the interactive activity "The Holiday Book" on linear models. The activity, originally in Italian, has been translated into English for the comprehension of the paper.

1	Researcher:	Well, how would you read this book?
2	Luigi:	I would read one page per year.
3	Researcher:	One page per year? Let's say one page per day. How would Luigi's graph be if he reads one page per day?
4	Camilla:	Very little inclined.
5	Researcher:	Yes, how many days does he need to read the whole book?
6	Class:	180 days.

7	Researcher:	180 days. Look how little its values increase from the horizontal axis. [She displays the line through the interactive worksheet at the IWB]. Ok. Is there someone who reads the book a little faster?
8	Simone:	I would read 20 pages per day.
9	Researcher:	So, 20 the first day, 40 the second… [filling in the table at the IWB and displaying the line]. This is the graph. How much time will he take to complete the book?
10	Class:	9 days.
11	Researcher:	Ok. Anyone else?
12	Cecilia:	I would rest for two days, then start the book and read 20 pages per day.
13	Researcher:	[Filling in the table] Two days of rest, so we start from 0 and have 0 for the first two days. Then we reach 20 at the end of the third day, 40, 60, … This is the graph [displaying the graph of the function]. Cecilia, are you faster than Simone?
14	Cecilia:	Yes, I am the fastest one.
15	Researcher:	Are you sure? Indeed you are the first one to end the book.
16	Cecilia:	Yes, I meant that I finished the book before everyone else.
17	Researcher:	Yes, you finish the book one day before Valentina, but what about Simone? Who reads faster?
18	Gianluca:	They are the same.
19	Researcher	How do you understand it from the graph?
20	Gianluca:	Because they are parallel lines.
21	Researcher:	Exactly. The lines are parallel. Even if the book was very much longer, Cecilia would never reach Simone. They increase by the same number of pages each day, but Cecilia started later. Ok, is there anyone else who reads even faster than Cecilia?
22	Mattias:	If I work hard, I think I could read even 35 pages a day.
23	Researcher:	[After filling in the table and displaying the graph]. Ok, you can see that Mattias is faster than the others. How long does he take?
24	Mattias:	[Observing the graph] Less than 5 days.
25	Researcher:	Ok. Would anyone read even faster?
26	Biagio:	One time, I read a whole book in a day.
27	Researcher:	Ok. Let's say that Biagio rests four days and then reads the whole book in one day. How would his graph be?
28	Alessia:	[miming an L] Horizontal until 4, and then vertical.
29	Mattias:	Parallel to the y-axis.
30	Biagio:	Yes, it's like that [miming a vertical line with his hand].
31	Camilla:	No, it's not vertical!
32	Andrea:	She's right. It cannot be vertical. The fourth and fifth points should be connected.
33	Camilla:	Yes, you have to connect the fourth day [pointing at the point (4,0) on the plane] to 180 [pointing at (5,180)].

34	Andrea:	Yes, on the fifth day, he reads 180 pages.
35	Alessia:	But it's more or less vertical.
36	Andrea:	Almost, but it's not vertical.
37	Researcher:	[displaying the graph at the IWB] It's very, very steep. At the end of the fourth day, he was at 0 pages, but at the end of the fifth day, he was at 180. If we imagine that he reads the same amount of pages each hour, we have a very steep line.
38	Samuele:	So, if he takes one second, would it be vertical?
39	Researcher:	How should he read the book to obtain a vertical graph?
40	Luigi:	He should have already read the book.
41	Researcher:	But if he had already read the book, he would start from 180, not from 0.
42	Gianluca:	He should take one second.
43	Simone:	Yes, but the graph would be inclined of the space of one second.
44	Researcher:	Exactly, there should be a little time which makes the line to be inclined.
45	Samuele:	That's right. If there is a bit of time, there is a bit of inclination.
46	Cecilia:	There should not be any time at all.
47	Camilla:	Right, the time should be zero.

Through this dialogue, we can notice how different conceptualizations of the slope emerge. We start with a real-word conceptualization ("one page a day", lines 2 and 3) which translates to a physical property ("very little inclined", line 4) through a functional property (the constant rate of change of which students had experienced while completing the tables). The students' intuitions (lines 4 and 6) are confirmed by the interactive graph at the IWB. Thus they can experience the correspondence between a numeric approach to a graphic approach in studying linear functions. The same observations can be repeated in the discussion about Simone's line. Here the researcher stresses the functional property (line 9) filling the table to build the graph. After that, she elicits a reference to the geometric ratio of the slope, asking students how much time he needs (horizontal shift) to complete the book (vertical shift). Cecilia's line gives a prompt to speak about parallel lines and see the slope as an invariant for parallel lines (linear constant conceptualization). From here, the discussion focuses on increasing velocity in the real-world situation and seeing what happens to the line, with particular reference to the reduction of the horizontal shift to the limit case. In the end, the impossibility of having a vertical linear function emerges from the impossibility to reduce time to zero. The real-world conceptualization helps attribute a meaning to the functional and geometric properties and to connect different conceptualizations in a unique concept. From the discussion, we can also observe other prompts for analyzing other aspects of linear functions, such as horizontal ones (having zero slope, line 28), intercept (line 41 and previously analyzed drawing Marco's read), and intersection with the x-axis (line 28 and, previously, Luca's read). The following analysis, driven by the questions in the worksheet, aimed at also introducing the algebraic relations among variables and coefficients in a linear function, thus leading to the parametric coefficient conceptualization of the slope, and useful to connect also symbolic aspects to the numeric and graphic ones. Similar discussions took place in all the classes when solving this problem. In all the classes, all the students actively participated with interest in the discussion. The problem was comprehensible for everyone. The discussion about how they would have read the book actively engaged even the less interested students, such as Luigi, who usually disturbed his classmates. Thanks to the well-designed contextualization, even Luigi's provocative answer could become a very interesting prompt for mathematical discussion: lines with

a low slope. As a result, Luigi kept concentrated until the end of the discussion, when he proposed a new intuition, this time incorrect.

The interactive worksheet supported the students' discussion, conjectures and argumentations. The possibility to fill the interactive table and immediately generate lines in the graph below helped them visualize the correspondence between different registers and connect different conceptualizations of the slope. The worksheet also supported the teacher and the researcher in orchestrating the discussion and driving the class towards the creation of shared knowledge. Through this discussion, we could observe an interactive DLE composed of the class with the teacher and the researcher; the interactive task displayed at the IWB; interactions among the learning community, mainly consisting in dialogues, and between the community and the technologies. The discussion itself is part of the interactive DLE. The task follows Grabinger and Dunlap's (1995) model. In particular: it promotes active learning; the context is relevant and meaningful; it engages students with their experience; it promotes collaboration and discussion; the activity is dynamic and supports the generation of understanding; the interactivity supports self-assessment. Similar tasks were repeated in the classroom during the following lessons and as online homework to facilitate students to generalize the acquired knowledge and transfer it to new cases. The interactivity and the automatic assessment helped students explore the other problematic situations and check their understanding step-by-step.

CONCLUSION

In conclusion, we can answer the RQ: "how can interactive activities in a DLE help 8th-grade students approach linear functions and build an integrated concept of the slope?". The interactive task presented in this paper allowed students to examine the linear models identified by reading a book at a constant speed. Through the interactive worksheets, they could compare different graphs corresponding to reading with different speeds and observe how the graphs change when the book is started before or after the beginning of the holidays. The classroom discussion selected and shown in this paper allowed us to observe how different conceptualizations of the slope emerge while discussing collectively in a DLE. The interactive activities elicited the emergence of many of the different conceptualizations of the slope identified by Moore-Russo and colleagues (2011), in particular: real-world, functional property, physical property, geometric ratio, and linear constant. Through the following activities, the parametric coefficient and algebraic ratio were also introduced. They are the main conceptualizations accessible at grade 8; developing robust connections among these conceptualizations can open the path to an interconnected understanding of the slope at higher grades. Above all, the stress was posed on the constant rate of change of the function, which, in the literature, is indicated as crucial for the development of a unified concept of the slope. However, in order not to lose the numeric and graphic understanding achieved with this task, similar activities should be repeated in higher grades of instruction and adapted to encompass also the algebraic conceptualization. This is a big challenge, since in Italy, after grade 8, students change school and start their upper secondary path. This discontinuity in the students' school life (change of teacher, class, friends, subjects, and methods) can reflect on the disconnection in building some fundamental concepts as the slope. Thus vertical projects, aimed at sharing vertical learning paths and using similar methodologies at different stages of education, and a focused teacher training, could be helpful to save these achievements and reinforce them to build solid mathematical knowledge. All the materials developed in this experimentation were shared with all the Italian teachers through the national Problem Posing and Solving Project (Brancaccio et al., 2015) with this goal. We aim at developing further this research, on the one hand, also examining the students' results in the final tests, compared with that of a control group; on the other hand, developing similar activities for upper secondary school in order to include other more advanced conceptualizations of the slope.

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VIRTUAL REALITY IN MATHEMATICS EDUCATION: DESIGN OF AN APPLICATION FOR MULTIVIEW PROJECTIONS

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Virtual reality (VR) is a new and innovative technology with potential for mathematics education. However, there has been little development and research in the area of VR and mathematics to date. In this paper, the opportunities and challenges related to VR technology in mathematics education are discussed using the example of a multiview projection VR application developed by the authors.

Keywords: Geometry education, mixed reality, multiview projection, orthographic projection, virtual reality.

INTRODUCTION

The term virtual reality (VR) refers to an artificial reality created by special hardware and software that allows a user to interact relatively naturally with digital objects. This new form of human–machine interface enables the development of innovative learning scenarios. The potential of such scenarios is explained in this paper using the example of a VR application for multiview projections developed by the authors. For this purpose, the basics of VR technology are discussed in the context of mathematics education. This is followed by a description and presentation of various approaches to the topic of multiview projections using haptic and digital educational resources. Subsequently, the developed VR application, Dreitafelprojektion-VR, is described and reflected upon from a teaching perspective. The conclusion discusses proposed directions for future research related to the VR app. The research on VR described in this paper is part of the project DigiMath4Edu at the University of Siegen [1].

VIRTUAL REALITY TECHNOLOGY IN MATHEMATICS EDUCATION

VR technology is a form of computer graphics that creates a three-dimensional (3D) virtual environment with which a user can interact according to certain built-in rules. A frequently used definition of this technology is provided by Bryson (1996):

> Virtual reality, also called virtual environments, is a new interface paradigm that uses computers and human-computer interfaces to create the effect of a three-dimensional world in which the user interacts directly with virtual objects. (p. 62)

Unlike traditional 3D computer graphics, VR systems do not offer a purely visual presentation but instead aim to provide a multisensory perception (visual, acoustic, haptic) in real time. Specific 3D displays are used to visually mediate the 3D content, typically using stereoscopic methods that present a different image to the left and right eye. Within the computer simulation, the user can interact with virtual objects in real time. For this purpose, 3D input devices are provided that (for example) recognize body movements or gestures and translate them into interactions (cf. Dörner, Broll, Jung, Grimm & Göbel, 2019).

The main feature that distinguishes VR from other human–machine interfaces is so-called "immersion." In a technical sense, this can be understood as the requirement that a user's sensory impressions shall be addressed as comprehensively as possible by the output devices (cf. Dörner et

al., 2019). This can be achieved by isolating the user from the real environment as much as possible, while addressing as many of the user's senses as possible through VR. VR targets to have the output device surround the user to the possible extent and to offer a vivid representation, rather than providing only a small field of view (cf. Slater & Wilbur, 1997).

These new possibilities of VR technology have led to changed interactions between humans and machines:

> The promise of immersive virtual environments is one of a three-dimensional environment in which a user can directly perceive and interact with three-dimensional virtual objects. The underlying belief motivating most virtual reality (VR) research is that this will lead to more natural and effective human-computer interfaces. (Mine et al., 1997, p. 19)

Compared to traditional human–machine interfaces, VR results in a particularly natural and intuitive interaction with the virtual 3D environment. While a completely natural interaction is not yet possible with current technology, interaction with the virtual world and the representation of this world via various sensory channels is becoming increasingly realistic. Consequently, VR systems give users the opportunity to gain experience in a virtual world. The term "presence" (sometimes also called "immersion") is often used to describe these mental experiences:

> However, presence as discussed in literature related to immersive VR can most often be characterized by the concept of presence as transportation: people are usually considered "present" in an immersive VR when they report a sensation of being in the virtual world ("you are there"). (Schuemie et al., 2001, p. 184)

The presence of a VR experience comprises three aspects. The location illusion suggests to the user that he or she is actually at the location represented by VR; the plausibility illusion makes the user perceive simulated events as if they were really happening; and involvement expresses how involved a user feels in a virtual environment (cf. Dörner et al., 2019).

The most common form of VR systems are head-mounted displays (HMDs) — that is, displays positioned on the head directly in front of the user's eyes. In this paper, we used the Oculus Quest, a standalone VR HMD. No computer is necessary to use the Oculus Quest, as calculations are performed by a system integrated into the HMD. Optical and acceleration sensors within the HMD and its controllers are used as input devices, which enables the precise detection of head movements, movements in space, and movements of the controllers. Further inputs can be made using buttons on the controllers.

There is a long tradition of using VR technology for educational purposes. As far back as the 1960s, the U.S. Air Force had begun developing VR flight simulators to train pilots (cf. Kavanagh, Luxton-Reilly, Wuensch & Plimmer, 2017). VR systems are especially widely used in the field of vocational training — for example, to simulate large technical systems, such as airplanes, trains, or industrial plants (cf. Köhler et al., 2013).

VR technology also has great potential for education in typical schools. Learning environments within VR can surpass the limits of the real world and can thus represent innovative learning aids:

> VR offers teachers and students unique experiences that are consistent with successful instructional strategies: hands-on learning, group projects and discussions, field trips, simulations, and concept visualization. Within the limits of system functionality, we can create anything imaginable and then become part of it. The VR learning environment is experiential and intuitive. (Bricken, 1991, p. 178)

This makes VR learning environments particularly appropriate for experience-based learning (cf. Hellriegel & Cubela, 2018). However, there are also various challenges associated with augmented reality and VR technology in the classroom. These include uniquely high financial costs, a lack of realism, and the possibility of incurring health impairments (e.g., cybersickness) (cf. Cristou, 2010).

Several VR applications are available that are specifically designed for teaching mathematics. Applications for teaching mathematics at the middle and high school levels often relate to the fields of geometry (e.g., VR Math) and analytic geometry (e.g., edVR). In higher education, the area of multidimensional calculus is a particular focus (e.g., Calcflow).

However, there is still a lack of research on the use of VR technology for learning mathematics. An empirical study by Kang et al. (2020) investigated the impact of a VR multidimensional calculus app on engineering students' learning of mathematics. Although the individuals who used the VR app reported that they were better able to imagine the concepts after using the app, they did not perform better on average than the comparison group on a subsequent test. Dilling (2022) examined high school students' mathematics learning with the app Calcflow in the context of orthogonal projections of vectors in a case study. He observed that students situated their mathematical knowledge at the 3D representations of vectors, planes, and lines and used these representations to develop and justify mathematical hypotheses. In this sense, they learned about mathematics as an empirical science in which mathematical concepts are tied to the real world (cf. Burscheid & Struve, 2020; Dilling & Witzke, 2020; Dilling et al., 2020). In summary, VR applications indicate a trend toward a more illustrative approach to mathematics wherein students learn mathematics based on (virtual) empirical applications.

As it has been mentioned above, the aim of this paper is to present a VR app developed by the authors. The basis for this development has been the approach of subject-matter didactics. Subject-matter didactics origins in German-speaking countries (German: Stoffdidaktik) and focuses on the mathematical content taught at school. The aim is to provide students and teachers with accessible approaches to mathematical content knowledge. For this purpose, the mathematical content and "essential concepts, procedures and relationships including appropriate formulations, illustrations and arrangements for teaching" are analyzed (Hefendehl-Hebeker, Vom Hofe, Büchter, Humenberger, Schulz & Wartha, 2019, p. 26). In the following, a detailed description of the topic multiview projections from a subject-matter didactic point of view, including existing teaching approaches, will be given. When presenting the VR app on this topic in the subsequent section, connections to this analysis will be explicitly sought.

MULTIVIEW PROJECTIONS IN GEOMETRY EDUCATION

Projections are an important topic in mathematics education in schools. They serve as an intersection of the topics of spatial geometry and plane geometry and are thus crucial for the interconnectedness of students' knowledge. Moreover, projections are of particular importance in the context of computer representations, and dealing with them is an important competence in the digital world.

In German curricula, dealing with solids and two-dimensional (2D) representations of them is already required in elementary school. For example, students are supposed to "relate two- and three-dimensional representations of buildings (e.g., cube buildings)" (Conference of the Ministers of Education and Cultural Affairs, 2004, p. 10). Orthographic projections are a particularly accessible entry point to the topic of projections.

Orthographic projection is a special form of parallel projection wherein the projection rays are parallel to each other and perpendicular to the projective plane. In this paper, we consider special orthogonal

projections on three perpendicular projection planes. This type of projection, called multiview projection (German: Dreitafelprojektion), is often used in technical applications or in architecture and is also important in mathematics education.

In the Multiview projection, the *xy*-plane is referred to as top view, the *yz*-plane as front view, and the *xz*-plane as right view (see Figure 1a). If a section of the projected body is not parallel to the corresponding projection plane, it is represented shortened in the multiview projection. To create a sustainable perception of the multiview projection, it is helpful to illustrate it with a coordinate corner — for example, made of cardboard (see Figure 1b). By changing the angle of observation, students can adopt different perspectives (from above, from the front, from the side) on the object placed in the coordinate corner and transfer different views of the object to the surfaces behind or below the object.

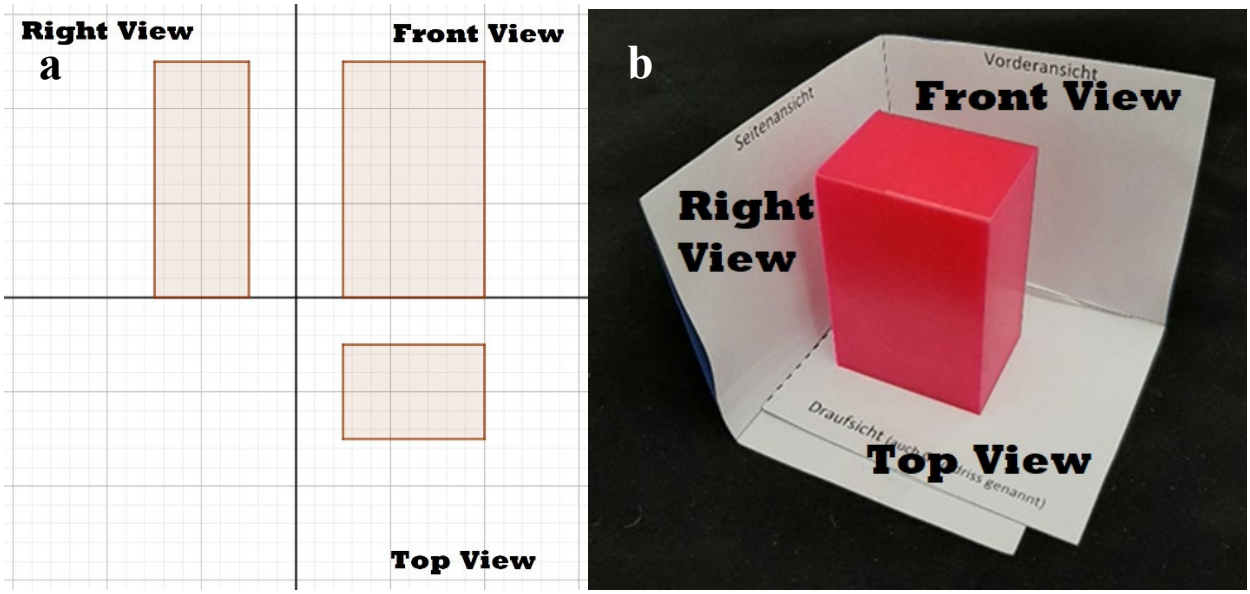

Figure 1: Planes of a multiview projection (a) and a three-dimensional object in a coordinate corner (b)

Several educational materials have been developed for teaching multiview projections. For example, the game Schattenbauen [2] ("Building Shadows") from the educational publisher Dusyma offers an illustrative and playful approach. In the game, orthogonal projections of composite solids (called shadows of buildings in the game) are provided on sheets. Students must build the corresponding structures from basic solids directly on the top of the "top-view shadow" (see Figure 2a). For this purpose, they are provided with cubes, cuboids of different lengths, and triangular prisms. By adopting different perspectives, the matching of the building and the shadows can be checked.

Multiview projections are also addressed in various digital resources. One example is the app Klötzchen [3] by Heiko Etzold (cf. Etzold & Jahnke, 2019). The app focuses on dealing with cube buildings in elementary school. It enables the construction of buildings from unit cubes in different views. For this purpose, the app is divided into two screens on which different views of the same cube building can be displayed. Figure 2b shows a simple building composed of four-unit cubes: on the one hand as a perspective view with corresponding top, front, and right views, and on the other as a building map with number values. By using the app, it is possible to practice transferring among different representations of cubes and solids composed of cubes (cf. Rahn & Dilling, 2020).

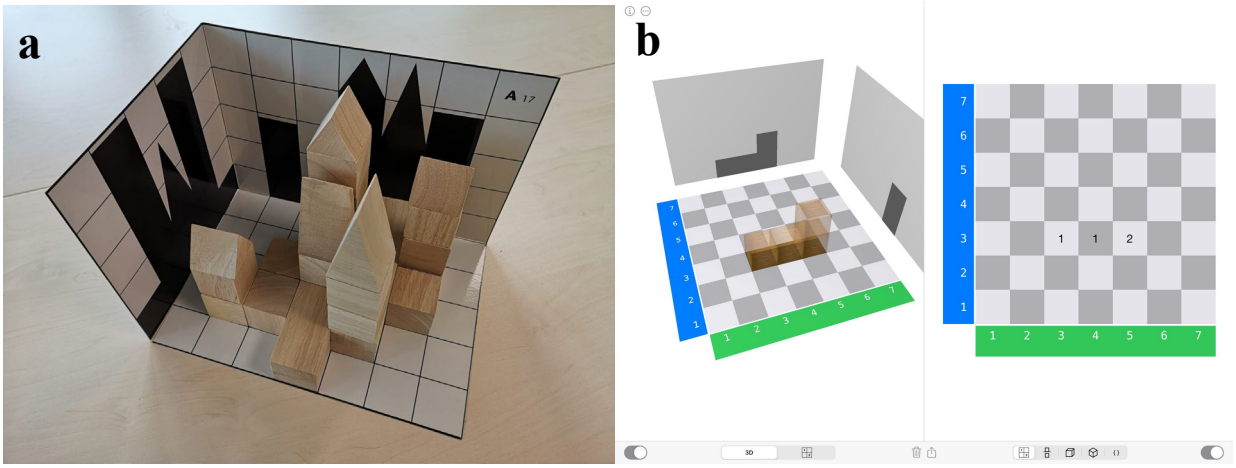

Figure 2: The game Schattenbauen (a) and the app Klötzchen (b)

THE APPLICATION: DREITAFELPROJEKTION-VR

Dreitafelprojektion-VR is a VR application optimized for the VR headset Oculus Quest that was developed by Frederik Dilling and Julian Sommer for learning and practicing the use of multiview projections. In the center of the virtual environment stands a table on which four 2D drawings are presented (see Figure 3a). In the lower right corner, there is a drawing of the top view of a 3D object, as it is standard for multiview projections. This drawing is colored blue and represents the surface on which the single basic solids can be positioned. The black drawings above and to the left of the blue drawing represent the front view and a representation of the right view rotated by 90 degrees. The right view is rotated in order to enable the user to easily control the solution visually. In the upper left corner, the top view is again shown, this time in black so that the top view can still be considered after the blue area has been filled with basic solids.

To the right of the user, there is another table with three basic solids on it: a cube, a cylinder, and a three-sided prism in the shape of a roof (see Figure 3b). The solid can be picked up and moved by moving the controller close to one of the solids and pressing the Index Trigger. By releasing the Index Trigger, the solid can be dropped or placed in a certain position on the blue drawing of the top view or on top of other basic solids. Any number of basic solids can be generated and deleted again by throwing them into a garbage can to the left of the user.

If a solid is placed on the table in front of the user, parts of the previously black drawings of the top, front, and right views turn green or red, depending on whether the object is correct (see Figure 3c) or incorrect (see Figure 3d) with respect to the presented projection. This nontrivial formative assessment and feedback can help students without presenting the solution. It is possible to have several correct solutions (e.g., in the scenario shown in Figure 3, up to two of the three stacked cubes can be replaced by cylinders). This demonstrates that not all properties of the original 3D object are preserved in a projection. If a complete object matching the projections is assembled on the blue drawing, a firework appears as feedback that the task has been completed successfully.

The development of the VR app for multiview projections was intended to combine the advantages of classic haptic and digital materials. The aim was to retain the playful approach of the haptic material, such as the game Schattenbauen. Students can pick up various basic solids in the app, rotate them, and place them in certain positions in the virtual room. The intuitive handling of the solids enables students to test different approaches and to view the buildings in a non-distorted manner (unlike 3D representations on a 2D screen).

VR technology also provides some improvements on the classic approach. For example, buildings that would not be possible in reality (such as those containing overhanging components) can be created without any additional effort (see Figure 3c/d). Furthermore, by making small changes to the program, a large number of new tasks can be created so that many examples can be experienced in a short amount of time and with little effort. Finally, the application includes a powerful automatic formative feedback system (see, e.g., Fahlgren et al., 2021). If parts in all three orthogonal projections are displayed green, objects have been positioned correctly (a correct partial solution does not necessarily lead to a correct total solution). In contrast, if a part is colored red, this indicates that a mistake has been made at the corresponding position, and the students can adjust their constellation accordingly.

In addition to the many opportunities presented by this modified approach to the topic of multiview projections, various challenges can arise as well. These include the handling of the app and the fact that the objects lack haptic feedback, which could be problematic, particularly for elementary school students.

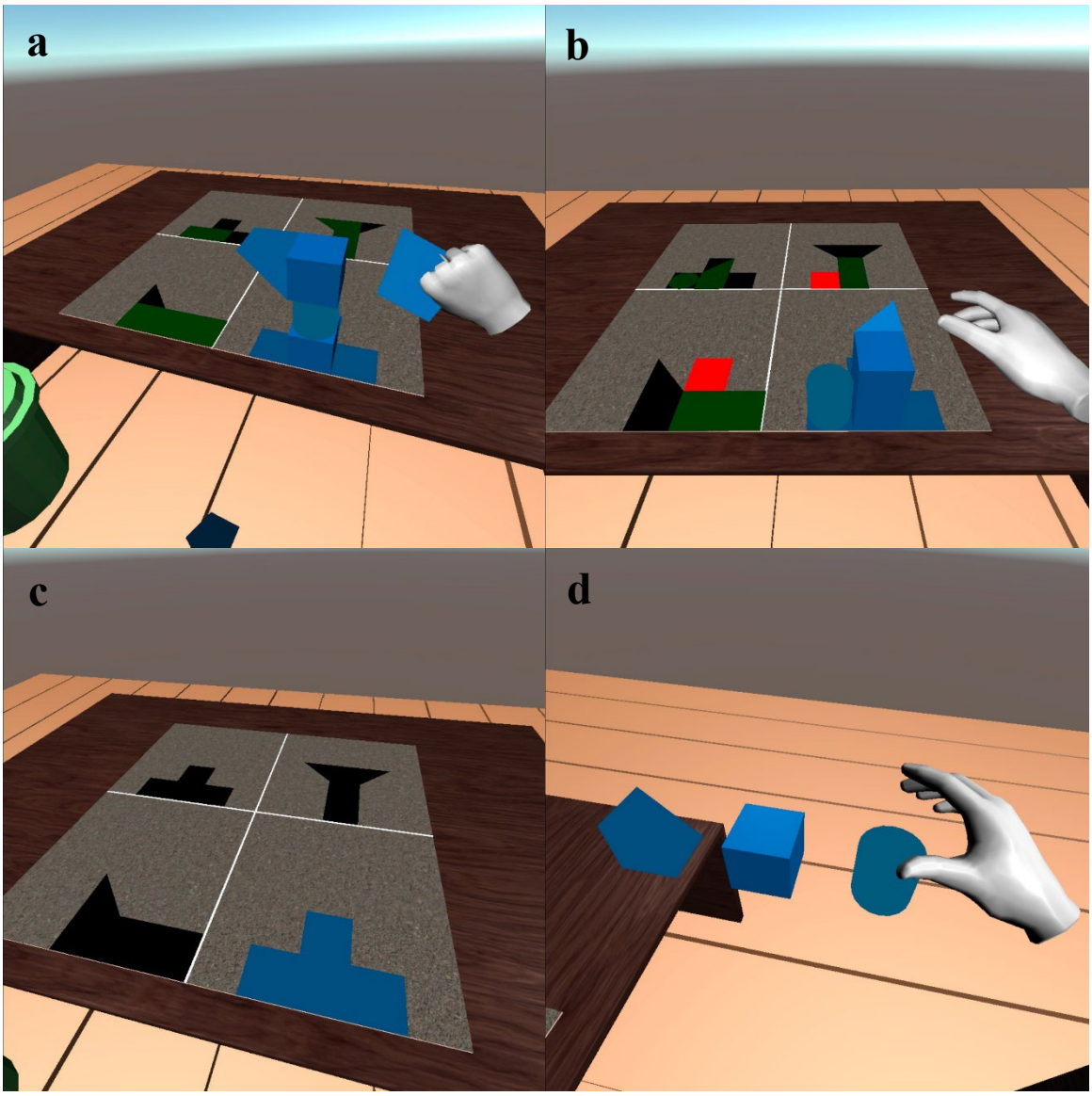

Figure 3: Screenshots of the VR application Dreitafelprojektion-VR

CONCLUSION AND OUTLOOK

The theoretical description of the app Dreitafelprojektion-VR already demonstrates some opportunities and challenges with regard to virtual learning environments in mathematics education. VR technology is particularly suitable for three-dimensional representations. However, 2D content can also be inserted. In this way, Dreitafelprojektion-VR enables an action-oriented approach to link two- and three-dimensional representations, similar to the game Schattenbauen described in the third section. Students can assemble solids and intuitively check the projections by looking from different perspectives. Furthermore, classical approaches (e.g., Schattenbauen) can be extended, for example, through automated formative assessment and feedback.

Using a design-based research approach, the authors of this paper intend to further develop the application for multiview projections together with other VR applications, as well as establish general design principles for VR apps, for example, based on subject-matter didactic approaches. This design-based research will be supported by concrete case studies on teaching and learning with VR technology — in particular, identifying characteristics of such processes (e.g., situatedness of knowledge) with regard to empirical and application-oriented mathematics teaching (cf. Dilling, 2022).

NOTES

1. The supporters of the project can be found at the following link: www.digimath4edu.de

2. https://www.dusyma.com/de/Schatten-Bauspiel-zid102753

3. https://apps.apple.com/de/app/kl%C3%B6tzchen/id1027746349

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MIXED REALITY IN MATHEMATICS EDUCATION

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Keywords: Augmented reality, virtual learning, virtual reality.

This workshop focused on augmented and virtual reality technology and its use for mathematical learning scenarios. The term virtual reality (VR) refers to an artificial reality created by special hardware and software, in which a user can interact in a comparatively natural way with digital objects. Augmented reality (AR), in contrast, is not a completely virtual environment but an integration of virtual objects in the physical reality. Both technologies can be arranged in the mixed reality continuum (MR) (cf. Milgram et al., 1994).

The use of VR and AR for educational purposes already has a long tradition. In the 1960s, the US Airforce started the development of a VR flight simulator for use in pilot training (cf. Kavanagh et al., 2017). AR and VR systems are particularly widespread in the field of vocational training, for example, to simulate large technical systems such as airplanes and trains (cf. Köhler et al., 2013). However, AR and VR technology also offers great potential for education in elementary, middle, and high schools.

Currently, there are some AR and VR applications that have been designed for teaching and learning mathematics at schools or universities. However, empirical studies on the impact of AR and VR technology on students' mathematics learning are yet to be conducted. There is also a lack of reliable findings on the design of AR and VR applications.

The workshop began with a short introductory talk on the principles and the history of AR and VR technology in education. This was followed by the introduction and testing of selected AR and VR applications (prototypes) developed by the workshop organizers for mathematics education. Finally, a collaborative discussion about the opportunities and challenges of AR and VR for mathematical learning processes as well as the design criteria for applications was facilitated. The key points of discussion were:

- How will students respond to an immersive virtual world?
- How might AR and VR technology change our lives in future society?
- What is special about learning in an immersive virtual environment?
- Is it in terms of cost and availability of apps realistic that VR and AR technology will be widely used in education in the near future?

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DESIGNING TASKS AND FEEDBACK UTILIZING A COMBINATION OF A DYNAMIC MATHEMATICS SOFTWARE AND A COMPUTER-AIDED ASSESSMENT SYSTEM

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This paper reports on the planning of a design-based research (DBR) study, where the main aim is to develop principles in designing technology-enhanced learning environments utilizing a combination of a dynamic mathematics software (DMS) and a computer-aided assessment (CAA) system. The focus is on the design of tasks and automated feedback of high quality so as to enhance first-year engineering students' engagement in and conceptual understanding of mathematical contents. The paper outlines the rationale for the project and highlights theoretical aspects that will be considered in the study. Moreover, some findings from a pilot study that will inform the first cycle of the DBR study are presented.

Keywords: Computer-aided assessment, dynamic mathematics software, formative feedback, task design, university mathematics.

INTRODUCTION

It is well established that the transition from secondary school mathematics to university mathematics is challenging for many students. The literature highlights several reasons behind this challenge; at university, students meet a new teaching practice, e.g., lecture format (instead of lesson format), larger student groups, less teacher contact, new requirements of learning habits and study organisation (Jablonka et al., 2017). Besides the wide variety in background, interest and prerequisite knowledge among students (Rønning, 2017), many students enter mathematics courses in higher education with insufficient basic mathematical skills (Abdulwahed et al., 2012). This, in turn, leads to unsuccessful study results for many students (Jablonka et al., 2017), which might cause problems, not only in subsequent mathematical courses, but within other applied subjects, e.g., mechanics and electronics, as well (Harris et al., 2015).

To tackle the 'transition problem', many educators in higher mathematics education have introduced continuous assignments to increase students' engagement early during a course, and prevent students from waiting to work on course material until shortly before the final exam (Rønning, 2017). To ensure that students give time to these frequent assignments, they are (most often) graded and constitute part of the course examination. This, in turn, requires a major effort from the teacher in terms of correction work (Rønning, 2017). However, the past decade has seen the rapid development of technology that supports teachers in this time-consuming work by offering automated correction of student responses. A common notion for these types of technology is computer-aided assessment (CAA) systems. Today, many first year mathematics courses in higher education utilize mathematically sophisticated CAA systems, such as STACK and Möbius (e.g., Rasila et al., 2015).

The literature reports several important affordances provided by CAA systems. For example the possibility of randomizing values for variables, parameters and formulas (Rønning, 2017), and the opportunity of providing students immediate feedback on their progress (Rasila et al., 2015), which, in turn, provides support for more independent study by students (Barana et al., 2018). At the same time, researchers in the field of technology-enhanced assessment point out the potential risk of such

assessment focusing on lower-order skills of mathematics (Attali & van der Kleij, 2017; Hoogland & Tout, 2018) and solely on the correctness of a final answer (Rønning, 2017) because such types of task and feedback are most straightforward to implement in CAA systems. Consequently, there is scope for designing CAA tasks that address higher-order skills in mathematics as well as for designing feedback that goes beyond categorizing a final answer as being right or wrong (Rønning, 2017).

One possibility to increase the learning potential when using a CAA system is to embed another type of technology: dynamic mathematics software (DMS) (Rasila et al., 2015; Sangwin, 2013). This type of technology is widely recognized as a tool that can promote inquiry and foster students' conceptual understanding in mathematics (Fahlgren & Brunström, 2014; Jaworski & Matthews, 2011). It is the instant feedback on students' action that makes it possible to use a DMS environment as an arena for exploration, conjecturing, verification, and reflection. Even if DMS feedback does not explicitly provide hints on how to proceed, it provides information that could be used in a productive way by the user (Moreno-Armella et al., 2008; Olsson, 2018). However, there is a need for novel types of task to utilize the opportunities provided by DMS environments (Fahlgren & Brunström, 2014; Joubert, 2017).

Although DMS and CAA systems are both in widespread use on their own, there are few studies that have investigated the integration of these two types of technology (Luz & Yerushalmy, 2019). This paper reports on the preparation for a design-based research (DBR) project, which aims to develop principles to guide the design of a technology-enhanced learning environment in which DMS tasks are embedded in a CAA system that (automatically) provides elaborated feedback based on students' responses. It is the cyclic nature of progressive trial and refinement of design principles that makes a DBR approach suitable for this project. Each cycle consists of three main phases: (a) preparation and design, (b) implementation, and (c) analysis and refinement (Bakker, 2018; Cobb et al., 2003). The focus of this paper concerns the first phase, preparation and design, of the first cycle of the planned DBR study. To inform this first phase, a pilot study was conducted in autumn 2020. In the following, we first describe the planned DBR study, including methods for data collection and analysis. Then, we introduce the pilot study and illustrate by an example how the pilot can inform the main DBR study.

PROJECT DESCRIPTION

In the planned DBR project, the intervention will consist of computer-based mandatory small group activities involving extended task sequences that form part of a calculus course for first-year engineering students (from various programs). The primary outcome of a DBR study is a deeper understanding of *how* and *why* certain instructional interventions work (or do not work), leading to experimentally grounded design principles: in this case, principles to guide the design of a technology-enhanced learning environment in which DMS tasks are embedded in a CAA system that (automatically) provides elaborated feedback (EF) based on students' responses.

Theories Guiding the Design

In total, the planned study will involve three cycles which will progressively trial and refine the design principles. Each cycle will be guided by a *hypothetical learning trajectory* (HLT), which besides the designed learning activities, includes the intended learning goal of the tasks as well as hypotheses about students' learning processes (Simon, 1995). In the development of the HLTs, including (re)designing of tasks and related feedback, several theoretical perspectives will provide guidance. Since the main focus of the proposed DBR study concerns formative feedback, theories related to

different types of feedback will be central, specifically in guiding the development of elaborated feedback provided by the CAA system.

Shute (2008) uses the notion of 'formative feedback' and defines it "…as information communicated to the learner that is intended to modify his or her thinking or behaviour for the purpose of improving learning" (p. 154). Broadly, the feedback information provided to a learner can be of two main types: verification or elaboration (Shute, 2008). The simplest example of verification feedback is whether the student response is correct or incorrect (Narciss, 2008). Verification feedback that also provides the learner with the correct answer to the task is termed 'knowledge of the correct response' (Narciss, 2008). In addition to these types of verification feedback, the literature refers to 'try-again feedback' (Shute, 2008). Elaborated feedback provides the learner with additional information, besides correctness, in various ways. One type of elaborated feedback, suggested by Barana et al. (2018), is to provide hints to guide students towards a solution. In their model of formative automatic assessment in mathematics, they suggest 'interactive feedback' in terms of step-by-step guidance throughout a possible solution process. By asking students to solve simpler tasks, they encourage them to recall previous knowledge and then gradually acquire the knowledge necessary to solve the problem. However, Rønning (2017) argues that there is a risk that this will result in a simpler and less interesting problem. Besides offering conceptual hints or guidance necessary for solving a task, elaborated feedback can provide an explanation for why a particular response is incorrect, or it can consist of a worked-out example (Shute, 2008). Furthermore, the format and timing of feedback presentation can vary. The literature distinguishes between immediate and delayed feedback (Narciss, 2008; Shute, 2008), and according to Vasilyeva et al. (2007), the feedback can be of one or several of the following forms: text, graph, animation, audio, or video. Besides the elaborated feedback provided by the CAA system, the DMS will provide students with feedback based on their interaction with the DMS. This type of feedback is regarded as implicit rather than explicit (Shute, 2008).

Moreover, theoretical aspects related to the design of different types of task utilizing the affordances provided by a CAA system will be important in the DBR study, e.g., example-eliciting tasks (Harel et al., 2020) and other types of task as discussed in the section describing the pilot study (see below). To prompt students to generate examples is not a novel idea – it has been proposed as a way to engage students actively in their development of conceptual mathematical understanding (e.g., Watson & Mason, 2002). Besides these more generic theories, also topic-specific theories will be needed, e.g., learning theories related to functional understanding in mathematics (e.g., Oehrtman et al., 2008).

Altogether, the planned project will imply many important design choices at various levels. To articulate the theoretical rationale for the choices and to analyse them after empirical testing, the design tool of *didactical variables* (Ruthven et al., 2009) will be employed. Put simply, a didactical variable is any aspect of the task (and related feedback), or the task environment, which may influence the unfolding of the expected trajectory of student learning. Next, we will elaborate on the three phases of each DBR cycle:

(a) *Preparation and design.* Except for the first cycle, which will be guided by the pilot study, this phase concerns the revision of the HLT in light of the knowledge gained from the previous cycle(s) and the emerging generic principles. This, in turn, involves (re)designing of the learning activities, i.e. tasks and related elaborated feedback. Crucial in this phase is the identification and articulation of didactical variables attached to the characteristics of the learning activities. Related to these characteristics, hypotheses on student performance, including utilization of the elaborated feedback, are formulated as part of the HLT.

(b) *Implementation.* This is the conduct of the activities, including data collection from students. Mainly, there will be four types of data sources: (i) CAA responses, (ii) surveys, (iii) focus group interviews, and (iv) recordings of student screens. As in the pilot study (described below), the CAA responses will consist of both short (most often individual) answers that will be analysed automatically and group answers to open-ended tasks (e.g., explanation tasks) that need to be analysed manually. In close connection to the implementation of the activities, a survey will be performed to capture students' overall perception, particularly on the feedback provided. To better understand students' perception of various types of feedback (indicated in the survey), we also plan to perform focus group interviews. These will be audio-recorded, and notes will be made to indicate instances related to the HLT (and corresponding didactical variables). However, as van der Kleij and Lipnevich (2020) point out in a recent review "…research provide[s] very limited insights into how student perceptions of feedback relate to engagement with feedback and subsequent meaningful outcomes." (p. 23). Accordingly, to receive information about students' actual utilization of the feedback provided, we plan to collect screen recordings (including audio) from four groups while working on the activities.

(c) *Analysis and refinement.* In this phase, the data analysis process takes place. Data analysis will compare the HLT with the "actual learning trajectory (ALT)" (Bakker, 2018, p. 61), focusing on key didactical variables. Further, it will involve both quantitative and qualitative methods. The preparation for the data analyses will depend on the type of data collected as follows:

(i) *CAA responses.* The CAA system automatically provides descriptive statistics on the degree to which the students have succeeded in performing certain tasks as well as to what extent they have utilized the various types of elaborated feedback provided. The responses to the open-ended questions, on the other hand, need to be prepared manually, as was done in the pilot study.

(ii) *Surveys*. The surveys will primarily consist of closed questions delivered by an online survey tool enabling quantitative data analysis, e.g., descriptive statistics and cross-tabulation.

(iii) *Focus group interviews*. Guided by the notes taken during the focus group interviews, a selection of relevant instances of the audio recordings will be transcribed verbatim. Next, in preparation for a thematic analysis (Braun & Clarke, 2006), the data will be organized into initial codes related to student perceptions of different types of feedback.

(iv) *Screen recordings*. The screen recordings will generate an extensive data set; hence, we need to identify episodes related to the HLT. In these episodes, students' actions will be described and their reasoning will be transcribed verbatim. These episodes will then be organized into initial codes.

Next, in the data analysis process, themes will be generated based on patterns in the initial codes from the screen recordings and interviews (Braun & Clarke, 2006). These themes will then be used to generate conjectures about students' performance as well as their perception and utilization of various elaborated feedback. These conjectures, in turn, could be tested against the other data material (i.e. CAA responses and surveys), looking for confirmation and counter-examples. Altogether, the analysis process will generate the ALT. Finally, the findings (ALT) will be compared to expectations formulated in the HLT. Reasons behind any differences will be discussed within the research team, providing input to the revision of the HLT in the next cycle as well as development and refinement of more generic design principles.

When the three cycles are completed, a retrospective analysis aiming at the finalisation of generic design principles, grounded in their empirical testing in each of the cycles will be made. In contrast to the ongoing analyses (described above), retrospective analysis seeks "…to place the design

experiment in a broader theoretical context, thereby framing it as a paradigm case of the more encompassing phenomena specified at the outset" (Cobb et al., 2003, p. 13).

THE PILOT STUDY

The pilot study involved 256 first-year engineering students taking a first course in calculus. As part of the course, the students were asked to perform two computer-based mandatory small group activities designed for a DMS environment (in this case GeoGebra) embedded in a CAA system (in this case Möbius). The activities involved sequences of various types of task with a focus on the understanding of the function concept. To encourage student collaborations, students were divided into (101) small groups. However, to ensure active involvement by each student, there was a need to embed individual elements. Accordingly, the activities contained both tasks that require a group answer and tasks that require an individual answer.

Primarily, the focus was to trial the applicability of different types of task in this 'new' environment as well as to get a deeper understanding of student strategies when performing these tasks. In this way, the pilot provides useful insights into the design of tasks as well as elaborated feedback in the upcoming DBR study. Mainly, three types of task were designed. Firstly, tasks where students were requested to provide examples of functions satisfying specific conditions, i.e. example- eliciting tasks (Harel et al., 2020). In this type of tasks, a design principle was to ask students to provide two examples in order to encourage them to reflect on which parts of the function formula that are possible to vary without affecting the given conditions. Secondly, we constructed tasks where students were asked to determine a function formula for a given graph, e.g., a rational function graph. For both of these types of task, a design principle was to promote students to use the DMS to verify their conjectures before submitting their answer into the CAA system. Finally, we trialed tasks in which exploratory activities in the DMS were central, and where the students were encouraged to explain their empirical findings. In this case, a design principle was to ask students to provide a jointly agreed response to encourage communication and reasoning. Besides the DMS feedback, the CAA system (automatically) provided verificative feedback as well as delayed feedback in terms of worked-out examples illustrating anticipated solution strategies.

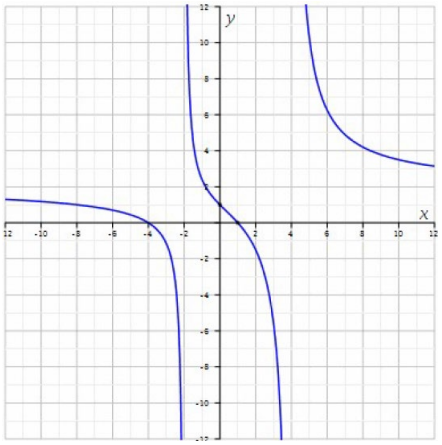
The pilot study generated two types of data: student responses to the tasks (generated by the CAA system) and data from an online survey capturing students' overall perception. The findings from the survey indicate that students found the various types of task instructive, and that they found the DMS feedback useful. In contrast, the elaborated feedback in terms of worked-out examples was utilized to a much lesser degree. This finding highlights a need to focus on the development of elaborated feedback that engages students. The data generated by the CAA system offered information about student strategies when performing the tasks, which will provide useful guidance in the (re)design of the tasks and related elaborated feedback. Furthermore, the pilot study provides useful information about methods for data collection and analysis. In the following, we give an example of how findings from the pilot study will inform the first cycle of the main DBR study.

An Example

The detailed analysis on a sequence of tasks addressing rational functions revealed some unexpected student strategies, i.e. the ALT differed from the HLT. For example, in the task presented in Figure 1, we hypothesized that students should first realize that it must be a rational function with one horizontal and two vertical asymptotes, and then utilize the vertical asymptotes to construct the (factorized) denominator and the horizontal asymptote to conclude that the numerator should be of grade two with the coefficient 2 in front the x^2 term. Finally, we expected them to realize that they

could utilize the zeros or two other points to finalize the function formula. The analysis of student responses to task ii) (in Figure 1) revealed that almost all students realized that it must be a rational function, and they also utilized the vertical asymptotes to construct the denominator. However, almost half of the students did not utilize the horizontal asymptote as expected. Instead, most of them, utilized the zeros together with one further point, e.g., (0,1) to construct the numerator.

Below is the graph of the function ***g****.*



i) Use the graph to determine the function formula.

Check your suggestion in GeoGebra before submitting it as an answer to the task.

Group answer: $g(x)$ = _____

ii) Explain how you used the graph to determine the function formula.

Group answer: _____

Figure 1. Task as it is presented in Möbius

This prompted the research team to discuss various options to tackle this particular issue as well as some general principles, both in relation to task design and to the design of elaborated feedback. For example,

- Should tasks be designed so that they cannot be solved without making use of certain key ideas? In the present task, it was the obvious zeros that made it straightforward to find the function formula without using the horizontal asymptote. However, the possibility to use different approaches based on various graph features may promote instructive student discussions.
- Should tasks be designed so that the key ideas are explicit? In this case, it might be an option to indicate the asymptotes in the graph. However, to be able to identify asymptotic behaviour in a graph is a central part of understanding rational functions. Consequently, this kind of scaffolding might simplify the task too much.

Concerning feedback, we discussed the following: when students solve a task without using some key idea, should they then be presented with a question probing that idea, or with a further task that cannot be solved without using that idea, or with a similar task and with feedback asking them to come up with a solution which does use the key idea?

When discussing these options within the research team, both pros and cons were identified. For the task in Figure 1, we decided to develop automated and adapted feedback, which in turn required a redesign of task ii). Instead of asking for an explanation, students were prompted to declare the features (of the graph) used to determine the function formula by choosing among various options (identified in the pilot study). Those students who have not used the horizontal asymptote, were given a new similar task in which they were urged to use the horizontal asymptote.

This example illustrates the complexity of designing tasks and related elaborated feedback. It also shows how information about the ALT could inform the (re)design of tasks to better utilize the affordances provided by a CAA system, i.e. automated correction and adapted feedback.

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TEACHING MACHINE LEARNING WITH DECISION TREES IN MIDDLE SCHOOL USING CODAP

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Keywords: CODAP, data science, decision trees, statistical projects.

Automated decision-making based on machine learning is relevant in many societal applications. Students' everyday experiences include such data-driven decision models (e.g., personalized advertising) that they encounter as black boxes. With increasing societal relevance, there is a growing demand for data-driven procedures to be taken up in school education (Engel, 2017; Engel et al., 2019; Ridgway, Ridgway, & Nicholson, 2018). The decision tree method is a highly transparent machine learning method, which allows students to understand the final decision model and the algorithm used to build it. Engel et al. (2018) showed an approach for the teaching of decision trees using the free and web-based Common Online Data Analysis Platform CODAP (codap.concord.org).

During the workshop, we presented an innovative series of lessons using digital technologies for a data science project in middle school. The context of the data project is personalized advertising on online platforms. We use self-reported survey data from 492 adolescents concerning their media behaviour. We address the topics of data exploration and decision trees in machine learning. The platform CODAP allows a quick entry into data science with drag-and-drop handling. This makes it easy to investigate relationships between different variables and manually create decision trees. The final objective of the data science project is to predict personal interests from media behaviour using decision trees.

The workshop aimed to introduce a data science project using the digital platform CODAP. It used hands-on activities to introduce the data and how CODAP can be used for analyzing data and for manually creating decision trees. We provide insight into our series of lessons and discuss special opportunities and limitations of the software. The main goal of the series of lessons is that students understand how a decision tree is constructed, how it can be interpreted and used, and how a decision tree can be evaluated in terms of uncertainties.

The workshop participants had inputs on data exploration and decision trees. Building on this, the participants themselves carried out selected tasks from our series of lessons designed for middle school. Some basic tasks on data explorations supported participants to get to know the data, followed by tasks on manually creating a decision tree as a predictive model. The workshop concluded with a lively discussion about our approach that encouraged us to proceed with our approach.

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SUCCESSFUL MODELING PROCESSES IN A COMPUTER-BASED LEARNING ENVIRONMENT

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The implementation of a computer-based learning environment for mathematical modeling can be valuable in many ways. For example, it is possible to combine different digital media and tools to present tasks in a more realistic way. In addition, the stored log data also offers new research approaches. The project "Modi - Modeling digitally" combines the above considerations. This paper reports on a study in which 42 students were asked to work independently during a two-week period. Since many difficulties could occur, the four most successful students were identified and analyzed from a qualitative point of view, however, by also adding variables from the log data. It can be derived that these students deal with hurdles in such a way that they try to rely on instructional videos, but generally work in a linear way. Nevertheless, differences in the use of tools and the built mathematical models, as well as other modeling-specific sub-processes, can be identified.

Keywords: Computer-based learning environment, dynamic geometry system, log data, modeling.

INTRODUCTION

The use of various digital media and tools to promote modeling competence seems promising, as contexts can be represented more realistically and mathematical models can also be used more authentically. This issue is approached in the project *Modi – Modeling digitally,* where a computer-based learning environment (CBLE) on mathematical modeling was created. There, we focus on the dynamic geometry software GeoGebra, but also a supportive structure as well as further possibilities for enhancing self-regulated learning. These three aspects are analyzed primarily on the basis of log data based on the learners' interactions with the CBLE. This offers the possibility of new evaluation methods. In this paper, we aim at identifying properties of successful learning processes within a CBLE. First, the theoretical framework including relevant studies is presented to derive the research questions for this paper. Then, the survey with the digital learning environment is explained in order to be able to describe successful modeling processes with the help of qualitative criteria and quantitative variables from the process data. In the following section, we first summarize properties of and findings on CBLEs. Subsequently, theory and findings regarding modeling are described.

THEORETICAL BACKGROUND

Computer-Based Learning Environments

With regard to a *conceptual clarification,* the most characteristic aspect of CBLEs is the use of digital devices. For example, this involves the opportunity to provide three-dimensional illustrations or dynamic geometry systems (DGS) (Drijvers et al., 2010; Jones et al., 2010; Lichti & Roth, 2018). Furthermore, the concept CBLE functions as a generic term for a computer- or web-based delivery of learning materials in a pre-structured way (Baker et al., 2010; Isaacs & Senge, 1992; Jedtke & Greefrath, 2019). In this contribution, a CBLE is a medium, which offers the possibility of combining other digital media and tools for learners, is pre-structured and web-based. This can also result in new ways of interacting with mathematical tasks and acquiring mathematical skills or knowledge (Engelbrecht et al., 2020). Concerning the *theoretical properties,* CBLEs enable teachers to provide

open-ended learning environments in which students not only learn through a (digital) tool, but can also investigate mathematical contexts by referring to differentiated materials. Mobile as well as flexible learning processes - based on the teaching and learning scenario - can also be promoted by the use of digital learning environments, as learning then takes place "across multiple contexts, through social and content interactions, using personal electronic devices" (Crompton, 2013, p. 4). Regarding *empirical findings,* self-regulated learning can be stimulated in digital learning environments and thus, they have a great potential as cognitive and metacognitive tools to support this kind of learning (Greene et al., 2011). Veenman (2007) gives an overview of studies that concentrated on self-regulated learning in CBLEs. Most of these studies focus on metacognitive skills, but some also take aspects such as motivation or different types of knowledge into account. Summing up these studies, a link between self-regulated learning and metacognition was found.

We conclude that CBLEs need to combine these properties that we implement in our modeling environment, as presented later.

Mathematical Modeling in a CBLE

Mathematical modeling focuses on the translation of a real-world problem into mathematics and back again (Niss et al., 2007). Blomhøj and Jensen (2003) describe mathematical modeling as processing a whole modeling task in a certain context, whereby six the mathematical action related sub-processes are named.

A model used for analyzing and describing students' modeling processes with digital technologies is depicted in Figure 1. The sub-processes of mathematical modeling are also presented there.

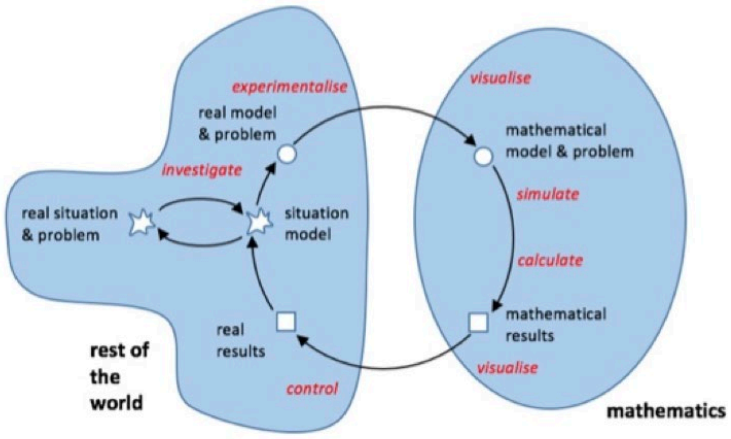

Figure 1. Mathematical modeling with digital technology (see Blum & Leiss, 2007, p. 207; Greefrath, 2011, p. 303)

Especially with regard to the design of modeling tasks, the broad spectrum of possibilities delivered by digital tools is important and involves new perspectives in presenting situations more realistically. Geiger (2011) claims there is a supportive function of digital tools for reality-based learning processes but also encourages to consider and investigate the affordances and constraints of digital tools with regard to modeling. Because of the theoretically considered possibilities of using digital tools in a meaningful way during modeling processes, investigations on this topic recently gained importance. Therefore, the considerations concerning the combination of the different steps in the modeling cycle and the variations of using technology are crucial for further investigations (Greefrath et al., 2018).

Next, we briefly present some of the rich research result on modeling processes. It was found that mathematical modeling tasks can come along with difficulties for students (e.g., Galbraith & Stillman, 2006). Until now, however, elaborate procedures based on video or interview studies have always been used to analyze modeling processes (e.g., Greefrath & Siller, 2017). This resulted both in small samples being considered as well as a small number of studies conducted in the field of modeling with digital tools. In the present paper, this view shall be extended by basing the sampling and description of processes on variables generated from computer-based log data.

RESEARCH QUESTION AND METHODS

On the one hand, automated assessment of modeling processes have not yet been described in empirical studies. But it can be assumed that this would allow for in-depth analyses of such processes. On the other hand, the sub-processes observed by Greefrath and Siller (2017) can be further investigated and enriched with respect to the concrete tool usage. In order to gain initial access to such processes, successful modelers are focused on first. Furthermore, the combination of self-regulated learning, CBLEs and mathematical modeling with technology is an open research field. To gain more information about self-regulated modeling processes within a CBLE, the following research question can be posed:

How can successful modeling processes within a CBLE be described using log data?

With regard to this research question, a study was conducted during distance learning in May 2019. Two secondary school classes (N=42, grade 9 from German Gymnasium, average age 14.56) took part and should work on five different modeling tasks offered in a CBLE. Since the sample of this survey is quite small and the coding of the modeling products is necessary to identify the successful modelers, we focus on a qualitative approach in this paper. The CBLE was pre-structured and included videos with information on the handling of GeoGebra to meet the properties of CBLEs described above. Furthermore, each modeling task should be solved with a pre-created GeoGebra-Applet. Except for the first one, all tasks were structured in the following way: on the first page, the situation of the task was presented by text, pictures or videos. The second page consisted of the GeoGebra-applet and on the third page, learners were asked to answer the task question, describe their approach, and validate how the unit, the chosen model, and the outcome made sense. In addition, between the three main pages, helpful strategies were described with the help of texts and pictures, which were intended to support especially in the case of hurdles and to stimulate self-regulated work. Furthermore, the first modeling task was more small-scale as it included smaller work requests. For example, the second page initially suggested simplifying the real situation. In addition, two different mathematical models were visualized, which were then to be implemented in GeoGebra. Afterwards, the solution should be validated, and a new, more precise model was to be found.

The items were developed with the authoring-tool *CBA-ItemBuilder* (Rölke, 2012) and delivered via a webpage. A wide range of log and process data (e.g., mouse movements, timestamps, tool-use in GeoGebra, answers in input-fields) were stored on a server at the University of Münster. On this basis, the latest state in the GeoGebra-applets was used to recover the constructions of the students with a custom program written in JavaScript. A visual coding of the final state was conducted. As done by Rellensmann et al. (2017), the modeling performance was assessed by estimating the accuracy of the solution on a 3-point scale. Hereby, the last snapshot from the GeoGebra-applet was considered to be representative of the final solution because all tasks should be solved in this way. A correct solution was coded with 2. A code of 1 was given for a solution that was incorrect due to estimation errors, not answering to the whole problem or, when the mathematical model was not fully adequate. The code 0 was awarded for an incorrect or missing solution. An average of 0.91 points

was scored among the 42 students. After the rating, the four best participants were selected to consider the most successful modeling processes with regard to the formulated research question. Their average modeling performance is 1.83 points.

To describe the modeling processes of the four selected students, the following variables were considered to be relevant and were extracted from log data: last mathematics grade, number of logins, total time spent within the CBLE, tool use, tutorial use.

RESULTS

The following Table 1 includes the extracted variables and both their specification for the four most successful participants as well as the average of each variable based on all 42 participants. The processes of the four participants, John, Laura, Sam and Max (fictional names), are described now. Hereby, special attention is paid to the task *playground*, where the students should find the best place for a new playground in the depicted park. In Figure 2, the four students' solutions are shown.

	No. Login	Grade	Total Time [Min]	No. Tools playground	No. Play GeoGebra tutorials
John	7	2	213	6	13
Laura	5	1	130	7	0
Sam	5	2	341	11	23
Max	2	2	79	3	6
Average (all participants)	3.12	2.37	191	4.10	9.32

Table 1. Exemplifying variables that describe processes properties within the CBLE

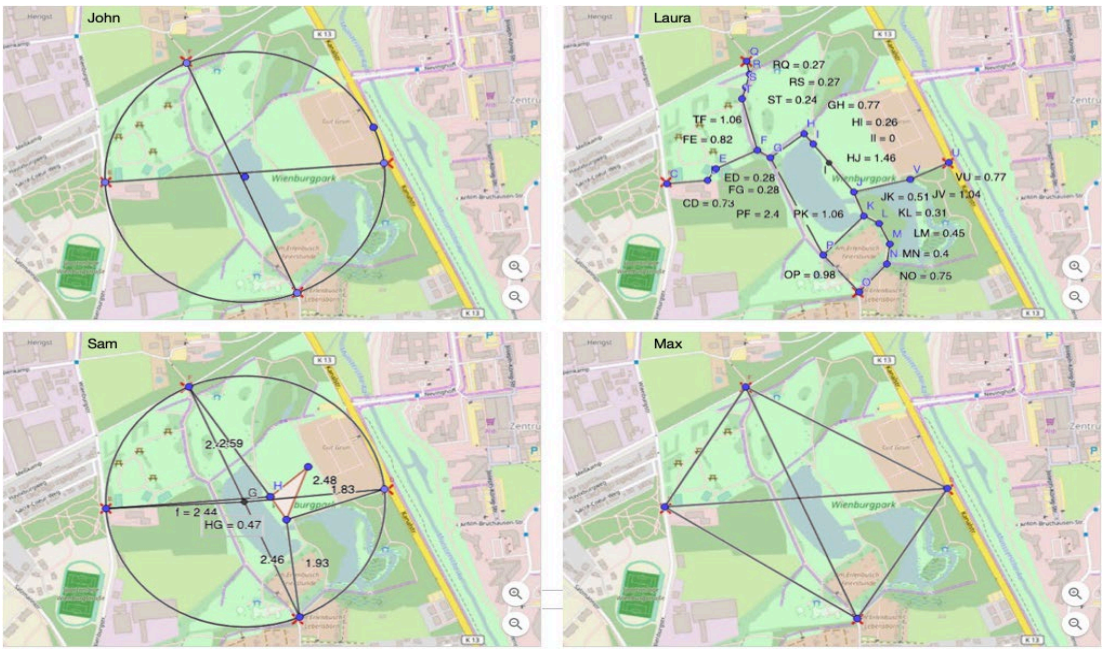

Figure 2. Four different final states in the GeoGebra-applet of the task *playground*

John

John logged in a total of 7 times, with two logins occurring within 10 minutes of each other. However, the rest were over the specified time period. In total, John was logged in on the tasks for 3.5 hours. He worked on one modeling task per logged-in day and additionally on the GeoGebra exercises on the first day. At the last login, no changes in answer fields or GeoGebra applets can be seen in the process data. Instead, only all tasks were clicked again. With regard to the task playground, John used six different tools: *segment*, *orthogonal*, *join*, *point*, *circle, two points* and *polygon*. It becomes obvious that he thinks about different solution possibilities, uses the "backwards"-button a few times and remains with the circle-solution. His final solution in the input field includes interpretation and validation: "The playground should be built mathematically in the center of the park. However, it is not realistically possible, because there is a pond. Therefore, I would build the playground somewhere next to the pond. This way, the way to it is longer from some entrances than from others, but the playground is still quite centered and easily accessible from all entrances."

Laura

Laura, who claimed to have the best mathematics grade, logged in five times. She worked on the GeoGebra exercise and the introductory task when she first logged in. Subsequently, as well as during the further logins, it can be observed that she sometimes switches between the tasks and also comes back to the GeoGebra exercises. However, she does not watch the tutorial videos. In total, she works in the CBLE for a little more than two hours on two different days. In the playground task, she changes tools a total of 19 times, using seven different tools: *move*, *point*, *join*, *segment*, *midpoint*, *distance* and *translate view*. The last sequence of tools consists of *point* and *distance*, with the view being moved again and again in the meantime. Accordingly, her solution approach is also focused on measuring the individual paths from the entrances. She describes her approach as follows: "I determined the lengths of the individual paths and picked out a point from which it seemed as if one was the same distance from all entrances. Since this point fit well, I refined it a bit." She also describes that she looked for possible obstacles on the paths. Accordingly, her approach is reality-based, and she does not move to the world of mathematics totally. She also does not consider whether other models might have produced a more accurate or better result, but she comes to a correct one.

Sam

Sam logged in on five different days and spent the most time (about 5.5 hours) with the CBLE. He watched the tutorial videos on GeoGebra a lot. Every time he logs in again, he first clicks through the task he solved last time and only then starts the following one. In some cases, he also changes his solution, so that a control can be assumed here. Sam's solution of the task playground is the best one (see Figure 2). It becomes obvious that he considers the lake and then measures where the best position for a new playground near to the lake could be. For example, he tries constructing the circle with different tools: at the beginning, he uses the Circle with one point tool and then drags the center point with the mouse. Afterwards, however, he uses the Circle with three points tool. He comments on his solution as follows: "To find the mathematically optimal point I drew a circle over all the entrances and found its center, since this point was in the lake I drew routes from the park entrances to a modeled surface for the realistic geographically optimal point and tried to place the surface so that the routes were similar in length." Thus, he used two different mathematical models, interpreted and validated them.

Max

Even though Max dealt with the CBLE the shortest of the four, he worked on all five tasks and also watched the GeoGebra tutorials. A total processing time of 1.5 hours is below the average of all

processing times. There is a total of 10 days between the two login times. Max works linearly and deals with the GeoGebra exercise as well as the first two tasks during the first login and the rest during the second access. His solution to the playground task is not complete and was therefore coded as 1. At this point, the insight is missing that modeling over a rectangle makes sense, which is only the case because the vertices are almost on a circle. Max uses only the three tools *Point*, *Segment*, *Delete* and then *Segment* again. This is even lower than the relatively low average value for the tools used by all participants. Furthermore, it follows that he considers only one mathematical model, which can also be seen in Figure 2. Nevertheless, he gives the correct answer and writes that the most ideal point would be in the lake, but this is not possible and therefore a place on the edge of the lake must be chosen, slightly disadvantaging an entrance side.

DISCUSSION

First of all, it can be concluded from the descriptions that the processes were different. John and Sam both worked in a very linear way, logging in regularly and solving one task after the next. Control mechanisms could also be observed, so that some characteristics of self-regulated work can be assumed (Crompton, 2013; Greene et al., 2011). Laura's return to the GeoGebra exercises may also be an indication that she sought for help and aimed at obtaining already mediated information about certain GeoGebra tools. Her frequent tool changes may indicate that her use of GeoGebra was rather uncertain. Max, on the other hand, worked through everything very quickly. This could be related to the fact that he also used few tools in the playground task and instead implemented the first considered mathematical model. He also does not go into further models in the description of his procedure. Nevertheless, he gives a correct answer to the general question.

In summary, sub-processes of modeling could be hypothesized based on log data (see Figure 1). It was also possible to determine specific additional functions that were made possible by the digital format. In line with previous empiricism (e.g., Galbraith & Stillman, 2006), indicators for difficulties such as frequent tool changes or access to explanations can also be defined and linked to possible successful strategies. It can also be observed that students used the opportunity to try out different mathematical models in the DGS by making use of the "back"-button. In addition, the intensely researched drag mode could be identified as a used tool during task processing (e.g., Arzarello et al., 2002). Interpreting the way of tutorial usage as an indicator for the familiarity with DGS, the following conclusions can be made. On the one hand, hurdles in the individual sub-processes can be supported by the means of CBLEs. On the other hand, the integration of various digital tools and media strengthened the opportunity of different, worthy solutions of the modeling problem. Thus, we can observe heterogeneity in tool use and modeling. Since the CBLE offers various opportunities, students can cope with their difficulties and find an individual but appropriate solution for the modeling problem, as can be seen in Figure 2.

CONCLUSION

This paper deals with different modeling processes within a CBLE and especially how the processes of successful modelers can be described by using log data. It can be concluded that successful modelers use the various embedded digital tools and media to cope with their individual difficulties during the modeling process and thus achieve different, valuable solutions.

For the first time, log data could be used for the analysis in the context of modeling problems, which offers a new focus on the handling of digital technologies and enables a detailed description. For example, it would be hardly feasible or very time-consuming to identify the different tools used in GeoGebra and to determine their switching frequency with the help of videos. Therefore, looking at

successful modeling processes in this way provides new clues as to how indicators of self-regulated learning can be established in the log data in future works. However, it should always be borne in mind that the interpretation of such indicators must be considered carefully. More qualitative approaches should also be used to validate the statements generated on the basis of variables from log data. It can be considered as a limitation of this study that students were not observed during processing or interviewed afterwards. Nevertheless, it seems very interesting to conduct similar studies with a larger sample. It would also be conceivable to perform a cluster analysis based on the extracted variables in order to identify processing types. Alternatively, a regression could be performed to identify predictive variables for successful or unsuccessful modeling. Overall, log and process data offer a promising means to gain new insights into technology use in mathematics education.

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SMART$_A$—ONLINE-DIAGNOSTIC TO REVEAL STUDENTS' ALGEBRAIC THINKING AND ENHANCE TEACHERS' DIAGNOSTIC COMPETENCIES

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Individual, diagnosis-guided support for learners is one of the most important factors in understanding mathematics and learning efficiently. Especially in the field of algebra, many students often still lack basic competencies to handle variables, algebraic expressions and equations in a proper way. Digitally supported diagnostic systems offer the possibility to perform a deep understanding-oriented diagnosis with little time. This is the aim of the Australian SMART-system (Specific Mathematics Assessments that Reveal Thinking (Stacey et al., 2018)), which is currently being adapted for use in German-speaking countries. SMART$_A$ is a twofold project to investigate the effects on students' understanding of algebra and on teachers' insight into formative assessment in the field of algebra.

Keywords: Algebraic thinking, formative assessment, online diagnostic, student thinking, teacher diagnostic.

INTRODUCTION

Individual, diagnosis-guided support for learners is one of the most important factors in understanding mathematics and learning efficiently (Wiliam, 2011; Black & Wiliam, 2009; Leuders & Prediger, 2017). Despite political, scientific and educational efforts, implementation of formative assessment in the classroom still appears to be challenging (Schütze et al., 2018). For sustainable individual support, teachers require diagnostic competencies and usually a not to be underestimated amount of time. To this demand, online diagnostic tools may seem to be a convenient solution. However, it must not be forgotten that diagnoses should not remain on a superficial level only focusing on solution rates such as "correct/incorrect" but should also target conceptual understanding. Nevertheless, at least in Germany, digital diagnostic systems often focus on solution rates and procedural fluency within narrowly defined tasks (Thurm, 2021). Thus, teachers only discover which tasks are mastered well and what solution rate their learner group has. Therefore, there is an urgent need to support teachers with in-depth understanding-oriented diagnostics that provide information on existing learner misconceptions and stages of understanding.

Deep understanding-oriented diagnosis can be performed with the help of digitally supported diagnostic systems that use a fast and intelligent evaluation (Stacey et al., 2018), which, for example, also analyses answer-patterns between individual diagnostic items (Steinle et al., 2009). This is the essence of SMART (Specific Mathematics Assessments that Reveal Thinking), which is a web-based diagnostic system that provides not only understanding-oriented diagnoses within a few minutes, but also further teaching recommendations and information on common misconceptions. Hence, on the one hand, SMART delivers quick, directly usable results and, on the other hand, it implicitly fosters teachers' pedagogical content knowledge and thus their diagnostic skills. As a part of this project, SMART tests are currently being adapted and translated for use in German-speaking countries in the frame of DZLM (German Centre for Mathematics Teacher Education) as a nationwide university

network for research and development of professional development (PD) programmes and teaching material. In addition, an accompanying PD programme for SMART tests is being designed to scale up the effects on teachers' diagnostic competencies and thereby students' competencies. The study sets out to investigate whether additional PD sessions are necessary for SMART to have a significant effect on teachers' as well as students' competencies.

For the first part of this project, called $SMART_A$, algebra was chosen as a topic because, especially in the field of algebra, there still seems to be a lack of basic competencies to handle variables, algebraic expressions and equations in a proper way (Arcavi et al., 2017). In Germany, this is reflected in a discrepancy between the expectations from universities for learners and actual school education (Neumann et al., 2017) and aspects of school mathematics are often seen as the main reason for this deviation (Biehler, 2018). In our contribution, we will present the first findings about the effects of SMART tests on teachers' design of algebra lessons and the development of students' algebraic competencies.

THEORETICAL BACKGROUND

The theoretical framework of $SMART_A$ comprises two main levels of the Three-Tetrahedron Model for content-related PD research (Prediger et al., 2019), here classroom level and teacher PD level concentrating on teachers' and students' competencies. On the teacher PD level, we focus on teachers' diagnostic competencies and expertise in the field of online formative assessment, while on the classroom level, we target students' learning of algebra.

In recent decades, research has shown that new technologies, such as digital formative tests or online diagnostic systems, can support students' mathematics performance in a variety of ways (Stacey & William, 2013). Although an integration of these digital tools into mathematics education is often recommended, the current use of technology in mathematics classrooms still remains low (Drijvers et al., 2016). This quantitative and qualitative gap between the potential of digital tools and the reality of teachers' use of them can be widely perceived (Bretscher, 2014), and research indicates that teachers are the most relevant factor for closing this gap, because they are ultimately responsible for which and how digital tools are used (Mumtaz, 2000; Thurm & Barzel, 2019). This fact highlights the importance of PD for teachers to support the implementation of digital tools (see also Thurm & Barzel, 2020).

This is not only the case for integrating tools like computer algebra or geometry software but particularly important with regards to digital formative assessment tools. Results have shown that formative assessment with technology is not much used in mathematics teaching and that, especially in times of lockdowns due to the COVID-19 pandemic, the use of digital formative assessment tools has even decreased (Drijvers et al., 2021). This is surprising since formative assessment – assessing the students' performance during the learning process and using this diagnostic information to improve their individual learning (Schütze et al., 2018) – can be very helpful in providing appropriate learning opportunities, especially in the challenging situation of distance learning.

Consequently, teachers should be supported in the integration of technology-based formative assessment into their teaching by high-quality teaching materials and online tools as well as corresponding PD. Studies have shown that teachers' PD is fundamental to improving teaching (Lipowsky & Rzejak, 2015). More specifically, Busch et al. (2015) found that PD programmes are able to improve teachers' diagnostic competencies from a corrective to a descriptive or analytical way of diagnosing that includes the application of pedagogical content knowledge (PCK). Helmke (2012) defines diagnostic competence as the ability to precisely assess the performance of a student, and it

can be operationalised as the accuracy of a teacher's assessment regarding student performance (Cullen & Shaw, 2000; Demaray & Elliott, 1998; Fuller, 2000).

The quality of PD can be ensured by following certain design principles, such as competence-orientation, participant-orientation, stimulating cooperation, various instruction formats, fostering (self-)reflection or case-relatedness (Barzel & Selter, 2015). For PD addressing technology innovations, these principles have to be complemented by focussing on self-efficacy to enable teachers to actually change their routines and integrate the innovation (Thurm & Barzel, 2019).

For the realisation of case-relatedness in PD programmes, a focus on one's own teaching and one's own students is much more effective than analysing other people's teaching (Seidel et al., 2011). This is the value of the SMART tests, which allow teachers to use short tests of 10-15 minutes to assess their students' competencies in a specific content focus. Learner response patterns across different diagnostic items are included in the SMART analysis. Following the test, the teacher receives an automated evaluation for each learner regarding existing misconceptions, individual level of understanding, gaps in prior knowledge, and frequent errors. In addition to this diagnostic information, the teacher gets access to targeted recommendations for appropriate learning support derived from this information (Steinle et al., 2009). SMART tests have been research-based developed and evaluated by analysis of more than 500,000 tests completed by students (Stacey et al., 2018).

The aim of a PD programme is to be effective on all of Lipowsky's (2014) four levels for successful PD: On the first level, teachers are satisfied with and accept the PD programme, but there is only a weak connection between satisfaction and changes in their knowledge and actions. The second stage refers to an actual change in teachers' competencies by enhancing teachers' knowledge and beliefs, while the third stage relates to changes in teachers' teaching practice and quality in the classroom. On the fourth level, the effectiveness manifests in the improvement of students' competencies (Lipowsky & Rzejak, 2015).

On the classroom level, SMART_A focuses on the learning of algebra. Although Küchemann and Malle described students' understanding of algebra and common misconceptions already in 1981 respectively 1993, learning algebra still appears to be challenging (Arcavi et al., 2017). For this study, we concentrate on the interpretation of variables involving common errors and typical misconceptions as, for example, *letter as object*.

Lucy bought 6 doughnuts for 12 dollars.

She wanted to work out how much each doughnut costs.

She wrote the equation $6d = 12$

In Lucy's equation, d stands for:

- ❍ one doughnut
- ❍ dollars
- ❍ the number of doughnuts
- ❍ doughnuts
- ❍ the cost of one doughnut

Figure 1. One of the SMART items testing for the *letter as object* misconception

RESEARCH QUESTIONS AND DESIGN OF THE STUDY

The research interest of $SMART_A$ is twofold as we focus on both PD and classroom levels. On the PD level, the aim is to find out to what extent the use of SMART tests improves teachers' general diagnostic competencies and especially whether SMART is effective as a means to enhance teachers' PCK. For this reason, a comparative research design was created to investigate whether SMART tests by themselves implicitly improve teachers' diagnostic competencies or if explicit PD is necessary, because studies have shown that especially PD programmes can support teachers to enhance their diagnostic competencies (see groups G1 and G2). On the classroom level, we examine how teachers make use of the diagnostic information in their teaching depending on the type of support they receive (see G1, G2, G3). In addition, we investigate the development of students' understanding of variables.

In total, the following three groups are compared:

G1: SMART diagnosis with teaching suggestions plus PD

G2: SMART diagnosis with teaching suggestions without PD

G3: SMART without diagnosis (only corrected student solutions) without PD

We investigate the following research questions:

1. How do teachers' diagnostic and support skills (in the area of variable comprehension) and teachers' self-efficacy beliefs about digital formative assessment develop through the use of the SMART tests?
2. What kind of support do teachers implement in their lessons depending on the type of support teachers receive?
3. How and for what purpose do teachers integrate SMART test results into their teaching?
4. How do students' competencies develop depending on the type of support teachers receive?

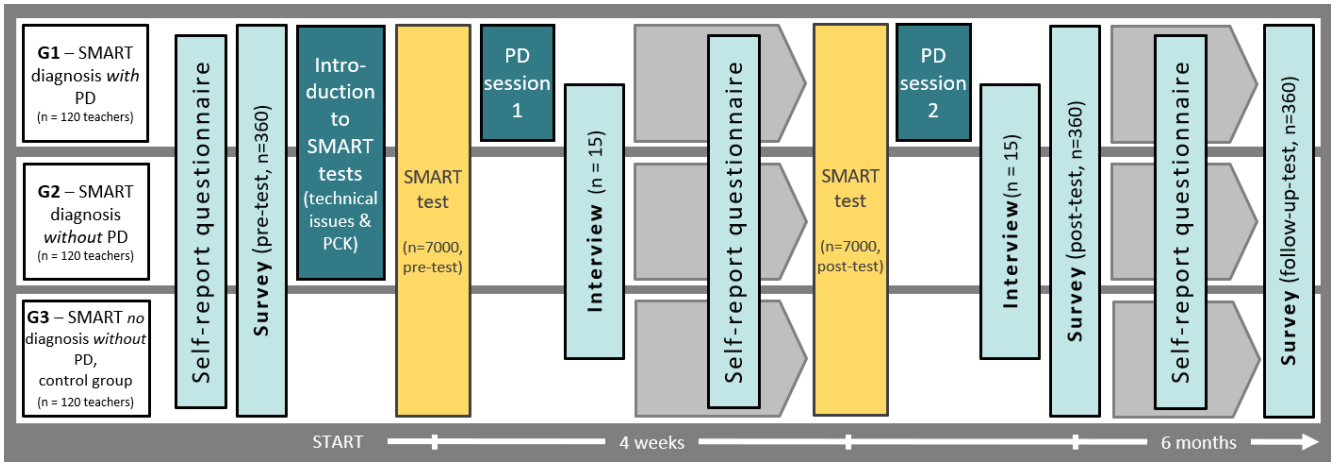

Figure 2. Project design

These questions are addressed by a randomised pre-test-treatment-post-test design, including quantitative as well as qualitative data collection in two different treatment groups and a control group. While both treatment groups will receive a video introduction to SMART tests, including the technical handling of SMART tests as well as relevant PCK, only teachers of group 1 (n = 120) will take part in two further PD sessions. The first PD session focuses on PCK and common misconceptions regarding algebra, and supports teachers in the development of targeted teaching

based on their students' SMART test results. In the second PD session, the implementation of those developed support concepts is evaluated, and the transfer to other topics that can be addressed by SMART tests is encouraged. Meanwhile, group 2 teachers (n = 120) will not receive further PD support in using SMART diagnoses; they only get access to a written report about diagnostic information and targeted recommendations. Teachers in the control group (n = 120) will not be supported by a video introduction or PD sessions. Another difference between the two treatment groups and the control group is that teachers in groups 1 and 2 will receive their students' SMART diagnoses as well as teaching suggestions to inform their further teaching, whereas teachers in the control group (G3) will only get access to the corrected solutions of their students and may use these at their own discretion. This design with three groups allows not only for comparison between teachers who did or did not receive additional PD, but also for scrutinising the effect of SMART tests compared to diagnostic tools that do not provide detailed diagnoses but only report on (in)correct solutions.

To answer the first question, diagnostic competencies of teachers in all groups are to be measured in pre-, post- and follow up-tests. For this purpose, a specific test instrument will be used that was developed by Busch et al. (2015) and will be adapted for the topic of algebra. In addition, self-efficacy beliefs regarding digital formative assessment will be surveyed.

The second research question will be answered with the help of self-report questionnaires that focus on the type of student support being implemented by teachers in their classrooms.

In order to scrutinise how teachers make sense of SMART test results, how they relate these to their teaching and their students, and with what intention they integrate the results into their teaching (question 3), interviews will be conducted with a sample of five teachers from each group. To answer question 3, quantitative results from the surveys will be combined with a qualitative analysis of these interviews in the sense of a mixed-methods design.

To investigate how online assessment tests for variables and algebraic expressions can support the development of students' algebraic competencies, SMART tests will be used not only to inform teachers about their students' achievements but also as a pre- and post-test to monitor and compare the students' progressions and change in misconceptions of all groups (question 4).

FIRST IMPRESSIONS

While adapting and piloting SMART tests in Germany, we have already obtained some initial findings. Besides the challenge to implement SMART into German language, and especially to transfer the material ranging from the single item to the corresponding teaching suggestions into the German culture of mathematics education, teachers' and students' competencies in the field of learning and teaching algebra could be observed.

As a first pilot, two volunteering teachers used the SMART tests "Values for letters" and "Letters for numbers or objects" with their 7th-grade students. Due to COVID restrictions, the teachers had to ask their students to fill in the test at home during an online lesson (pre-test) and as homework respectively in a lesson in which only half of the class was allowed to participate (post-test). As students did not necessarily attend both lessons or followed the instructions, only 18 ("Values for letters") respectively 15 students ("Letters for numbers or objects") completed both pre- and post-test. Another limitation of this pilot is that some of the students reportedly received help from adults while completing the test at home (although it had been made clear to the students that the test would not be used for grading). Therefore, student results need to be handled with care and further investigated, for example, by additional student interviews. In the following, we focus on the teachers

who were interviewed after administering the pre-test and asked to think aloud while receiving and reviewing their students' SMART results.

Our assumption that it would be worthwhile to investigate possibilities to support the development of diagnostic competencies in the field of algebra is corroborated by the interviews: Teachers seem to be unaware of typical mistakes, for example, the belief that the values that letters can take are somehow related to the variable's place in the alphabet:

> Teacher: I have never thought about there obviously being students who relate algebraic letters to their place in the alphabet. This is for me – well, I have NEVER thought about this[1].

Therefore, it seems to be indispensable to support teachers in enhancing their PCK regarding algebra. In the interviews, first indications can be found that the type of support provided to teachers is crucial. Although the teacher genuinely acknowledged the *letter as object* misconception as new and interesting information, he did not recognise any connection to his own teaching, for which he reportedly uses complete words like "fries" as variables and equation riddles with pictures of objects, which are popular on social media, as an introduction to algebra. This suggests that SMART diagnoses with teaching suggestions might not be sufficient to evoke a profound reflection of one's teaching and to change beliefs, but that additional PD sessions are necessary. In these PD sessions, it might be advisable to focus not only on PCK but also on diagnostic competencies because it does not appear to be a routine activity to receive diagnostic recommendations and implement those into the classroom as the interviewed teacher tended to retrace the correction of single student answers rather than planning targeted teaching activities based on the automatically diagnosed levels of understanding.

For now, we are looking forward to further results in the field of algebra. In the long term, our perspective is on more studies in other areas to gain more insight into students' and teachers' thinking and the challenges for PD to enhance mathematics teaching and learning on classroom and PD level.

NOTES

1. Translated by the authors; here the original German quotation from the teacher: "Da habe ich noch nie drüber nachgedacht, dass es auch offensichtlich Schüler gibt, die mit den Buchstaben halt Positionen im Alphabet verbinden. Das ist mir – also, da habe ich noch NIE drüber nachgedacht."

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COMPARING PROGRAMMING AND COMPUTATIONAL THINKING WITH MATHEMATICAL DIGITAL COMPETENCIES FROM AN IMPLEMENTATION PERSPECTIVE

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In this paper, we investigate the implementation strategies related to bringing programming and an increased focus on the use of digital tools into the mathematics curriculum in Sweden. Drawing on implementation theory, we take a starting point in two teacher training in-service modules that constitute a central aspect in the Swedish effort of implementing both programming and an increased use of digital tools in the Swedish curriculum. The paper thus aims to contribute in reaching an understanding of the specific challenges related to bringing programming into the mathematics curriculum from an implementation perspective.

Keywords: Comparative research, computational thinking, implementation research, mathematical digital competencies, programming.

INTRODUCTION AND STATE OF THE ART

In recent years, educational systems in many countries have increased their focus on programming and digital competencies. One key aspect is a change in direction, now focusing on support of students to become producers of technology rather than merely users. As part of this wave, Denmark, Norway, Sweden and several other countries are pursuing ambitious goals for implementing programming in compulsory education. The reasons behind such changes are many. One has to do with a wish from industry for more people pursuing a STEM career (Danmarks Vækstråd, 2016). Yet a reason to focus on programming may very well be to prepare students to meet a reality where computational and data-driven methods outperform classical mathematics paper and pencil calculations in tasks of mathematical modelling as well as problem solving. Hence, the increased focus on programming as an educational goal has potential consequences for mathematics teaching and learning. For example, teachers, schools, school owners and national educational systems face a range of decisions about the relation between mathematics teaching and technology teaching. These decisions concern content as well as which group of teachers are to be responsible for teaching students programming and not least in what school topics it should be taught. Sometimes programming is treated as a part of mathematics, but it can also be viewed as part of an integrated science topic, as a transdisciplinary element in all topics, or as a topic in its own right. Different countries are currently investing massively in pursuing different paths relating programming and digital mathematics (Bocconi et al., 2016; Vahrenhold et al., 2017). Still, there is currently no solid knowledge foundation on which to rely decisions about these matters. Although there are obvious synergies between digital mathematical competencies and programming, no systematic efforts in studying, describing and conceptualizing these have been made so far, nor in experimenting with their potentials in practice. Another challenge in exploiting these synergies is that mathematics teachers find programming to be outside their area of expertise (Misfeldt et al., 2019).

In spite of this, we currently see new initiatives seeking to implement programming as a part of the mathematics curriculum, thus exploiting synergies between them and simultaneously addressing a

societal need for programming skills. Although such initiatives can be considered similar to any other implementation of curriculum change, the novelty of programming as a school topic and the lack of knowledge about how to connect mathematics and programming makes it likely that this is a different challenge compared to other curriculum implementation processes. In light of these challenges, the aim of this paper is to investigate the specific challenges of bringing programming into the mathematics curriculum considered from an implementation perspective. We will pursue this aim by applying Century's and Cassata's (2016) five factors of implementation to compare the implementation strategies in two in-service teacher training modules focusing on 1) programming and 2) the use of digital tools in mathematics teaching, both in the Swedish educational system. Comparing these modules will allow us a deeper understanding of how the challenge of implementing programming relates to a more conventional yet seemingly similar implementation of curriculum change. Before we get to the case, we will first outline the relation between mathematics teaching and learning and programming.

The ambition of using computer-based constructions as a means to reform education has been around for the last 40 years and has led to educational ideas and innovations (e.g. programming languages for kids) that are currently applied when implementing programming and computing in compulsory schools (Bocconi et al., 2016; Brennan & Resnick, 2012). It was, however, not until Jeannette Wing's much-cited paper from 2006 was published that the effort of making computational thinking (CT) into an integrated part of compulsory education became mainstream (Bocconi et al., 2016). Wing (2006) described CT as decomposition, data representation and pattern recognition, abstractions and algorithms. Although Wing's (2006) work mostly addressed CT, educational research and policy also include elements of programming. In recent years, educational research has attempted to clarify and activate programming and CT as teachable competencies. We will refer to this trend as *programming* and *computational thinking* (PCT).

It is often highlighted how PCT relates to mathematical competencies such as abstraction, problem solving, modelling and algorithm building (Kafai & Burke, 2013). Mathematical competencies are well-described in the mathematics education literature, most frequently with reference to the Danish mathematics competencies – the so-called KOM – framework (Niss & Højgaard, 2011), in which a mathematical competency is defined as "(an individual's) well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49). The Danish mathematics programs, and in particular that of compulsory school, is heavily based on the KOM-framework, and also the Swedish curricula for compulsory school have been influenced by the KOM's description of mathematical competencies.

From an implementation point of view, it makes sense to align an inclusion of PCT with the Danish competencies approach of KOM. Yet, the KOM-framework itself does not explicitly address the notion of digital competency in relation to mathematics. This, however, is done by Geraniou and Jankvist (2019). They argue that much of students' mathematical work in the 21st century calls for a simultaneous activation of mathematical competencies and digital competencies in such an intricate manner that it makes sense to coin the two into Mathematical Digital Competencies (MDC). MDC involves "being aware of which digital tools to apply within different mathematical situations and contexts, and being aware of the different tools' capabilities and limitations" as well as "being able to use digital technology reflectively in problem solving and when learning mathematics" (p. 43). Furthermore, it entails "being able to engage in a techno-mathematical discourse" (p. 42). Both of these aspects of MDC can feed into PCT. But simultaneously, it is clearly possible to work with PCT without building explicitly on mathematics or MDC, as the case in the majority of the curriculum in England [1].

In the following section, we introduce the case on which we conduct our analysis in this paper, and next continue to describe the theoretical framework we apply and our approach to data collection and analysis.

IMPLEMENTING PROGRAMMING AND THE USE OF DIGITAL TOOLS IN SWEDEN

In 2017, the Swedish government decided that programming should be included in the mathematics curriculum from grade 1 through 12. This integration was connected to an attempt to raise the students' level of proficiency in algebra (Kilhamn & Bråting, 2019), and thus concerned all teachers and students of mathematics associated with these grade levels. Thus the Swedish educational system faced a major implementation challenge. To address it, a number of in-service training activities were initiated. Some of these are available under the programmes for teacher in-service training located at a digital portal. This portal was developed during an earlier national in-service training project called Boost for Mathematics. Although this project is completed, the portal remains as a platform for teacher training resources. It is on this portal that the resources for implementing programming are located. Despite such initiatives of support, Swedish mathematics teachers state rather clearly that they do not feel ready to conduct teaching in programming (Misfeldt et al., 2019).

The first author of this paper has been involved in developing some of the materials (Allsopp & Misfeldt, 2019). These programming materials are organized as four modules, each consisting of four key activities. The first two modules are entitled *About Programming*, and *Teaching with Programming.* The key ideas in these modules are about programming and how programming is taught. The third and fourth modules are entitled *Programming with Mathematics* and *Programming in Mathematics.* These explore the specific interfaces between mathematics and programming. In module 3, for instance, mathematics is used as a tool to develop programs (for example, computer games), whereas in module 4 programming is used to solve mathematical problems. The material is placed in the digital portal for in-service training in Sweden. In this paper, we focus on comparing module 1, focusing on teaching mathematics with digital tools, and module 2, focusing on programming as part of mathematics, from an implementation perspective. To do so, we draw on Century's and Cassata's (2016) five key aspects of implementation in education and mathematics education research, which we describe below.

THEORETICAL FRAMEWORK: FIVE KEY ASPECTS OF IMPLEMENTATION

In a central review of the literature around educational implementation research, Century and Cassata (2016) define implementation research as "the systematic inquiry regarding innovations enacted in controlled settings or in ordinary practice, the factors that influence innovation enactment, and the relationships between innovations, influential factors, and outcomes" (p. 170). Moreover, they suggest that implementation in education cannot be satisfactorily described as adoption of innovations without taking seriously five key aspects (also discussed more closely to mathematics education in Jankvist et al., 2019):

Characteristics of the individual users: The change that an educational innovation is aimed at generating is mediated by the people involved in the implementation process. Hence, it is important to know their individual characteristics. We distinguish between (a) *characteristics of the individual in relation to the innovation* (mathematical background, experience using the materials or resources involved in the innovation, etc.) and (b) *characteristics of the individual that exist independently of the innovation* (willingness to try new teaching methods, attitudes towards new artefacts in the classroom, etc.).

Organizational and environmental factors: In the case of an innovation implemented in a mathematics classroom, organizational factors refer, on the one hand, to the characteristics of the setting itself (number of students, characteristics of the physical space, access to material resources, etc.), and on the other hand to the collective beliefs and behaviours of the members of the class (identity, sociomathematical norms, didactical contract, etc.). Environmental factors refer to those outside the organization that have an influence on how an innovation is adopted and implemented (economic conditions, educational policies, priorities of government agencies, etc.).

Attributes of the innovation: The attributes of the innovation can influence its implementation. However, it is important to distinguish between the *actual* attributes of the innovation (objective characteristics) and the *perceived* attributes of the innovation (subjective characteristics perceived by the user). Of course, the perceived attributes may vary from user to user.

Implementation support strategies: It is important for an innovation initiative to be accompanied by an intentional and planned support for the end-users and their institutions. Such support strategies can consist of professional development, development or access to specific resources, etc.

Implementation over time: Another factor that influences the implementation of an innovation is time. Thus, it becomes relevant to study innovation endurance over time: how can we promote that an innovation, besides being adopted, is preserved over time until it is routinized? It is in this branch of the implementation research, where longitudinal studies will become essential to answer questions like the one previously stated.

Informed by the framework above, we seek to answer the following research question: What are the essential differences of the modules targeted at (1) programming and (2) the use of digital tools in mathematics teaching in Sweden considered from the perspective of Century's and Cassata's (2016) five factors of implementation?

METHOD AND DATA

In order to compare the two modules, we build on data from the above-mentioned digital portal for in-service training in Sweden (https://larportalen.skolverket.se). The in-service resources found at this digital portal function as an important part of the implementation of both programming and use of digital tools in mathematics in Sweden and is thus a natural entry point for investigating differences in the implementation strategies. In addition to these resources, we draw on results from a survey study sent to Swedish mathematics teachers (Misfeldt et al., 2019). Below, we describe these data sources and our approach to analyzing them.

Teaching Mathematics with Digital Tools 1 (module 1) focus on digital tools in mathematics teaching and contains eight elements: 1) The web as a resource; 2) Orchestration of mathematics education with the help of digital tools; 3) Dynamic representation with digital tools; 4) Formative classroom practice with response system; 5) Analysis of digital software; 6) Investigate and discover maths with digital tools; 7) Mathematics teaching based on the students' digital world; and 8) Mathematics teaching and development with digital tools. The portal includes descriptions of the necessary classroom equipment to carry out the activities in the module and a document that outline the theoretical foundations of the material and common pitfalls when using digital tools in the mathematics teaching[2]. Module 1 was developed in concordance with the in-service initiative "Boost for Mathematics".

Teaching Mathematics with Digital Tools 2 (module 2) focus on programming as a part of mathematics teaching and contains four elements: 1) About programming; 2) Teaching with programming; 3) Programming with Mathematics; and 4) Programming in Mathematics. Module 2

involve the similar support documents/resources as the ones described above for module 1. Contrary to module 1, module 2 was developed later as a direct response to changes in the mathematics curriculum standards in Sweden, starting in 2018.

Besides the content and resources of the two modules described above, the digital platform provides information that Swedish mathematics teachers are expected to work individually and in collegiate groups using the resources in the modules. The aim is for them to gain experience with teaching the content and inspiration of how the topics can be addressed in teaching. The platform also describes that all students ideally should have access to a device, but that they alternatively may work in pairs and share a device.

As the resources from the portal do not provide us information about the end-users, we supplement our empirical foundation with the results of a survey sent to Swedish mathematics teachers' (N=133) concerning their experience of teaching mathematics and programming, their conception of the relation between mathematics and programming, their conception of how programming should be implemented in mathematics and of how programming could help develop students' understanding of mathematical concepts, procedures, and problem-solving competency (Misfeldt et al., 2019). This survey provides us with empirical insights into the characteristics of the end-users in relation to the innovation, which is needed to fully understand the implementation strategies of the modules through the lenses of Century and Cassata (2016). Rather than summarizing all the results from the survey in our analysis, we include only the results needed in our analysis to answer the research question.

In our approach to analyze the data described above, we consider each of Century's and Cassata's (2016) factors of implementation for the intentions of the two modules in order to compare their strategies from an implementation point of view.

ANALYSIS: COMPARISON OF MODULES 1 AND 2

As stated, we now analyze and compare the implementation strategies in modules 1 and 2 according to Century's and Cassata's (2016) five factors of implementation. We will analyze modules 1 and 2 under the same heading and focus on describing how their elements and strategies compare in respect to each of the five factors.

Characteristics of the users: In the case of both modules, the users of the mathematics portal are Swedish teachers. As described by (Misfeldt et al., 2019), very few of the Swedish teachers included in the above-mentioned survey felt well prepared to teach programming, which is addressed in module 2. This is likely to be different from the case of module 1, where it is reasonable to anticipate a relatively higher level of proficiency concerning the use of digital tools or media in mathematics, since digital tools have been an integrated component of mathematics teaching for decades. Despite the similarity of the end-users, independently from the innovation, the situation is likely to differ completely regarding their relation to the innovation in modules 1 and 2.

Organizational and environmental factors: Concerning this factor, each of the modules has several unique characteristics. Module 1 was partly organized in concordance with the Boost for Mathematics project and was hence a part of a larger capacity-building program. Moreover, it represents an extension of an already existing and consolidated subject. Due to this characteristic of the module, at least some of the Swedish mathematics teachers are likely to have an expertise about the content of module 1. Contrary to module 1, module 2 is organized in relative isolation from existing initiatives. Moreover, the content of module 2, focusing on programming, represents a new topic of which Swedish mathematics teachers are not immediately expected to be proficient. From an organizational

perspective, this translates into differences between the two modules in terms of how their anchoring relate to existing initiatives and the availability of experts/resource persons in the current staff.

Attributes of the innovation: One way to describe the innovation is as a specific website/educational material. From that perspective, the two modules both include an overview of topic related concerns likely to emerge. In this respect, the modules are similar in format and only differ in the specificity of the aforementioned topic related concerns. Another way to describe the innovation is as a political decision. From that perspective, the two materials represent quite different innovations. The introduction of programming in the mathematics curriculum in module 2 represents a significant change in learning objectives, whereas the material about digital technologies and mathematics teaching developed for Boost for mathematics is much more concerned with enhancing and supporting the existing practice of mathematics teachers.

Implementation support strategies: Both collections of materials comes with a larger plan and implementation strategy as well as a package of further support initiatives. The material about programming in module 2 is situated in a recent political decision and comes as part of a number of other courses and initiatives allowing teachers of mathematics to learn about programming. The material about the use of digital tools in mathematics teaching was part of the Boost for Mathematics project, and thus inherited the support strategies from that initiative, such as meeting agendas and how organizational anchoring.

Implementation over time: Time plays a very different role in relation to the two materials. In relation to the material about programming, there is a clear starting point for implementing this change because it is connected to a change in the mathematics curriculum planned to begin in the fall of 2018. This is not the case in the modules about digital tools and media (module 1), which have been running for a long time and thus do not involve a similar date of launch.

DISCUSSION AND CONCLUSION

In this paper, we have investigated how the implementation strategies in modules 1 and 2 compare in relation to Century's and Cassata's (2016) five factors of implementation. The two modules share a number of obvious similarities in that they are both organized as in-service training initiatives located at a digital portal and supplied with similar support documents for teachers. The analysis has, however, revealed substantial differences between the two modules that are of great significance for their implementation. These differences include both innovation-specific and innovation-independent matters. Regarding the innovation specific differences, the fact that Swedish mathematics seems to consider programming to be outside their area of expertise triggers differences between the modules with regard to several of Century's and Cassata's (2016) factors. This difference translates into a substantial difference in the amount of available local support for the teachers, and it leads to two very different scenarios regarding the relation between end-users and the innovation in the two modules. These findings suggest that the grounds for implementing programming into mathematics represents a particular type of challenge that has to do with the nature of programming as a school topic. The analysis, however, also points to innovation independent differences that is likely to be of importance for the success of the implementation. Namely that module 1, contrary to module 2, is developed in relative isolation from existing initiatives. A future consideration could be how the implementation of initiatives related to programming can bridge ongoing activities with an existing infrastructure and format for resources and support.

We have generated the results summarized above by analyzing in-service teacher training resources found at the Swedish digital portal for teachers through the theoretical lenses offered by Century and Cassata (2016). These results represent a valuable contribution in terms of understanding the systemic

challenges involved in implementing programming into the mathematics curriculum, but needs to be supplemented with additional research in the near future. Although the resources at the digital portal constitute a cornerstone in the Swedish implementation strategy, there is also a need to investigate students' and teachers' practices and perceived challenges in programming lessons in mathematics. Understanding the practices and experiences of these actors is an essential key to optimally support the implementation of programming into the mathematics curriculum.

As any other framework, Century's and Cassata's (2016) concepts focus on some aspects of the implementation process at the cost of others. For example, the framework does not capture to what extent the decision to implement programming in mathematics was made in a top-down or bottom-up manner and any derived implications from this. Neither does it consider whether the implementation of programming has led to collateral changes in the education policy, such as changes in the national assessment. The insights generated in this paper constitute a first attempt at understanding some essential aspects of the implementation of programming in the mathematics curriculum. Any further studies should include the policy level in order to address issues identified in this paper.

NOTES

1. https://www.gov.uk/government/publications/national-curriculum-in-england-computing-programmes-of-study/national-curriculum-in-england-computing-programmes-of-study

2. https://larportalen.skolverket.se/LarportalenAPI/api-v2/document/path/larportalen/material/inriktningar/1-matematik/Gymnasieskola/443_REV_matematikundervisningmeddigitalaverktygGY/se-aven/pres/IKTGy_01_verktyg.docx

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MCM@HOME: AN APPROACH FOR SYNCHRONOUS DISTANCE LEARNING WITH MOBILE DEVICES

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Keywords: Digital learning paths, distance education, MCM@home, mobile learning, synchronous online teaching.

INTRODUCTION

Due to the Covid-19 crisis, instruction shifted from the classroom to the children's rooms in Spring 2020. As a result, lessons in Germany were no longer conducted synchronously but mostly asynchronously, leading to a loss of personal contact and a loss of familiar school structures (Wößmann et al., 2020). To address these issues, the MCM@home concept was developed in Spring 2020. Hereby, the smartphone app MathCityMap—originally aiming at mathematical outdoor education—is used for the purpose of distance education. In the following, three perspectives on MCM@home are presented, namely the view of learners, teachers and task authors.

THE MCM@HOME CONCEPT

Perspective of Learners

Following a low-tech approach, students only need a smartphone with an installed MathCityMap app (cost-, add-free and in line with GPDR) and an active internet connection to participate in lessons conducted with MCM@home. The app guides students through a digital learning path and shows the tasks. On demand, learners can involve up to three hints and a sample solution. Further, the app gives immediate feedback on the entered solution (Figure 1).

By using the feature Digital Classroom, all students work synchronously at a predefined time. Here, a chat is implemented to enable direct student-teacher interaction despite the spatial separation.

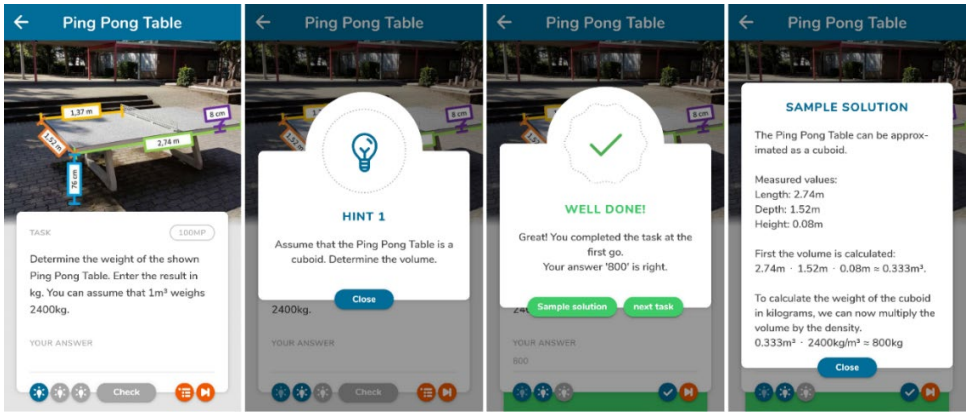

Figure 1. Students view of the MCM@home task "Ping Pong Table"

Perspective of Teachers

The working space for teachers is the MathCityMap web portal (https://mathcitymap.eu/). Within the Digital Classroom, teachers can easily monitor students' working progress in real-time by two

functionalities. Firstly, a class overview is provided, showing how the students solved the tasks of the digital learning path. It displays the number of invoked tasks and the quality of the entered solution per task. As students receive up to 100 points per task, their archived score is also shown. To analyse the working progress on an individual level, the e-portfolio allows teachers to retrace all interactions of a student with the app, such as the use of hints or entered solutions.

By using the chat function for sending text or voice messages as well as images, teachers can directly support the task solving process of their students. For a more detailed description of the Digital Classroom and its use in distance-education settings, see Larmann et al. (2022).

Perspective of Authors

Teachers can decide whether to use an already existing digital learning path or to create own tasks and learning paths. To create tasks, teachers simply need to fill in a predefined form in the web portal. It contains the task formulation, hints and a sample solution as well as a task image. Due to the wide range of different answer formats (e.g., exact value, interval, vector), the system offers the opportunity to create not only 'classic' computational tasks but also tasks for modelling, reasoning and problem solving. In addition, the task formats quiz or cloze text are available.

OUTLOOK AND WORKSHOP ACTIVITY

The MCM@home concept was developed in Spring 2020 based on the experiences of distance education in Germany. Until April 2021, MathCityMap experts created 56 digital learning paths in seven different languages, which were downloaded nearly 3.700 times to the app. This indicates a successful first dissemination progress of MCM@home in school praxis (Barlovits & Ludwig, 2021).

To fully meet the requirements of distance education (cf. Larmann et al., 2022), MCM@home will be further developed into a stand-alone system. This system, called ASYMPTOTE, will also consist of a smartphone app for learners and a web portal for teachers and task authors. It will extend MCM@home with a systemic adaptivity and a long-term analysis option.

In the workshop, the MCM@home concept will be presented as a technically low-barrier system for distance education. Subsequently, the workshop participants will get to know MCM@home from the three perspectives mentioned above: Firstly, they take on the role of students and work on a digital learning path within MCM@home. Secondly, from the teacher's perspective, the function of the digital classroom as a monitoring tool is discussed. Thirdly, the participants create as task authors their own tasks within the MCM@home system.

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AN APPROACH TO TEACHING DATA SCIENCE IN MIDDLE SCHOOL

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We present an innovative series of lessons using digital technologies for a data science project in middle school. In a data science project, on the one hand, the selection of data is of elementary importance, on the other hand, a technology has to be used that allows interesting data explorations and at the same time is easy to learn for students. As data, we use self-collected survey data from 492 young people regarding their leisure and media behaviour. For the data exploration, we chose the free and web-based data science platform CODAP, which allows an easy entry into data science. Student results show that CODAP is a suitable technology for doing data science in middle school and is positively received by the students.

Keywords: CODAP, data science, decision trees, statistical projects.

INTRODUCTION

Data science is an emergent field with fast-growing importance (Ridgway, 2016). In everyday life, one encounters data and conclusions inferred from data everywhere, for example in business, politics and society (Engel, 2017). Due to the general presence of data science, its statistical fundamentals should also be addressed in mathematics lessons. Today, data is analysed everywhere with the help of computers and, in the field of data science, often with artificial intelligence methods. This results in a connection to computer science lessons if one wants to address data science in school teaching. Because real data science problems are always situated in a certain context, there is a third component of a certain subject domain, such as politics or society. This results in the typical picture of data science at the intersection of statistics, computer science and domain knowledge as in Figure 1.

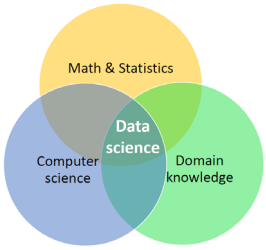

Figure 1. Data science as an interdisciplinary field

The basis with data provides a natural starting point for data science in maths and statistics lessons. The corresponding reasoning about data (Biehler et al., 2018) should be promoted as early as possible in schools (Ben-Zvi, 2018).

In this paper, we present an innovative approach to how data science in middle school can be carried out in a computer-based way. The approach for a teaching series is to lay the foundations in statistics and data exploration with technology in the first part of the teaching series. In the second part, a prediction model is developed by systematically creating a decision tree based on the data and the findings from the first part of the series. Preliminary results from an exploratory teaching experiment in a tenth-grade class, in which real, multivariate data were explored with the Common Online Data Analysis Platform (CODAP), are presented from this teaching series. The second part of the teaching series is currently being tested.

BACKGROUND

Data science is conducted here with a focus on statistical projects and the use of digital tools. In statistics education, so-called bottom-up software (Konold, 2007) are used for learning statistics. From a didactic point of view, the technological tools Fathom, TinkerPlots and CODAP are bottom-up tools, which are all based on a similar didactic principle. They represent so-called landscape software (Bakker, 2002), in which learners can move freely and pursue their own questions. According to this didactic principle, no graphical visualisations, such as pie charts or histograms, are given, but have to be created by certain actions. This process should create a deeper understanding of the representation in the learner.

In the series of lessons presented here, we use the CODAP environment (codap.concord.org), which is free and browser-based and provides the technological basis for the series of lessons.

Statistical projects are widely seen as an effective teaching strategy for learning data science (Gómez-Blancarte & Ortega, 2018) and can address fundamental ideas in statistics like data, representations, and variability (Burrill & Biehler, 2011). Research suggests a benefit in students' motivation by doing projects (Bilgin et al., 2015). Another benefit of an innovative data science project work is that

> [w]orking with SP [statistical projects] thus represents a strategy that can enrich curricula because each phase involved in developing a project entails the use of various statistical concepts and processes that go beyond the topics normally included in curricula. (Gómez-Blancarte & Ortega, 2018, p. 5)

Another important part of data science is predictive modelling (Ridgway et al., 2018) and especially machine learning. Machine learning encompasses various sub-areas (supervised learning, unsupervised learning, reinforcement learning, etc.) and a variety of different methods. One of the methods of supervised learning is decision trees, which are algorithmically created from data (Breiman et al., 1998). This method is one of many that can automatically solve classification or regression problems based on data. However, decision trees have unique features that make this method particularly suitable for use in schools. Engel et al. (2018) cite the following advantages, among others: Due to the hierarchical rule structures, decision trees are very easy to interpret. This enables the tracing of individual decision processes, but also the analysis of patterns in underlying multivariate data. Thus, decision trees are not only suitable as predictive tools, but also to search for explanations (conditional factors) in the data. Another particularly important advantage of the method over others is that, in a basic form, no higher mathematics is necessary to understand the algorithms.

In order to get to the decision tree method in class, the data basis must first be understood. Therefore, in our series of lessons, we work on data exploration in the first part and build on this in the second part. The paper describes ideas for the first part and presents some results from students' project work.

A SERIES OF LESSONS FOR DATA SCIENCE IN MIDDLE SCHOOL

Based on the approach of a statistical project and the technological tool CODAP, with which both data exploration and decision trees can be carried out, we have developed the teaching series "Data detectives at work".

The lessons address students with no prior knowledge of statistics or data science. In the first part, students' statistical thinking is promoted. For this purpose, we use the PPDAC framework by Wild and Pfannkuch (1999) with the phases problem, plan, data, analysis, and conclusion (Figure 2). In order to use data that is interesting for students, we started an online survey in cooperation with a German media association (https://www.mpfs.de). We call the resulting data JIM-PB 2021. These

data contain 492 cases of students between 10 and 20 years of age who answered 161 questions. The young people have given information on their leisure and media behaviour, for example about the frequency of reading books, magazines online or offline, playing computer games, using social media platforms, using YouTube, etc. An example question is: "How often do you watch LetsPlay videos on YouTube?" with possible answers 'daily', 'several times a week', 'once a week', 'twice a month', 'once a month', 'less often', 'never'. The resulting micro-data is analysed by the students, with media literacy as a background subject domain. Of course, data with 492 cases do not represent "big data", but with 161 variables it is truly multivariate and provides an opportunity for diverse explorations. For students, this is usually the first encounter with multivariate data in their school years. Likewise, frequency distributions and analysing statistical relationships between two variables are not typically encountered in the mathematics curriculum. The aim of this part is for students to investigate their own meaningful questions using real and interesting data and thus gain a first experience of a data science project.

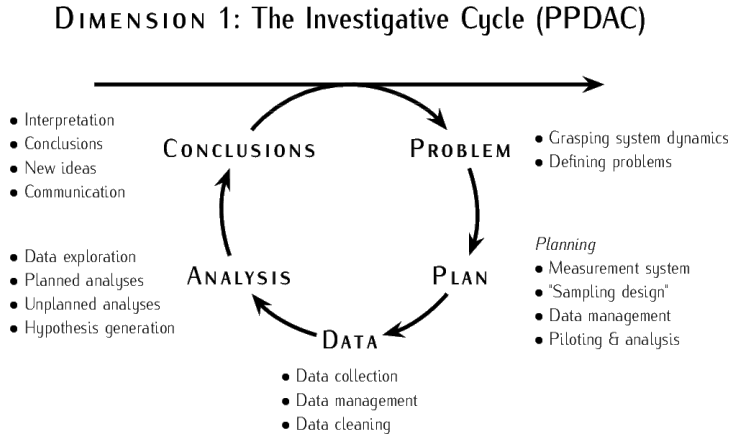

Figure 2. Data analysis cycle in "data detectives at work" according to Wild and Pfannkuch (1999)

We use CODAP as a digital tool for data analysis (for a detailed description, see Haldar et al., 2018). To be able to do data science with CODAP, one does not have to know how to write code, which is an important criterion for the use of the platform in middle school. CODAP is easy to learn and offers even inexperienced learners a quick start in data exploration. Thus, students can pursue their own questions with this digital tool.

Four learning objectives are the main guiding principles for the design of the first part of the teaching series:

- Students explore and analyse multivariate data.
- Students use basic terms of statistics and statistical concepts.
- Students use a digital tool such as CODAP for their data exploration.
- Students document and present their findings in an appropriate way.

Part one consists of eight lessons of 45 minutes each. In a data science project, the focus should be on the statistical and contextual content rather than on the procedures required to use the digital tool. Difficulties are known to arise when asking statistical questions (Arnold, 2013), exploring relationships between categorical variables (Watson & Callingham, 2014) and summarising findings in a presentation. These issues are specifically addressed during the first four sessions. There is a particular focus on percentage analysis when comparing distributions in lessons 3 and 4. The main project work takes place in lessons 5–7, when students explore the JIM-PB data along their own questions. In the project, students work in small groups on one topic each, for which they have to

pose questions and analyse the data themselves. The main context is targeted advertising, referring to four specific areas. These four areas represent the four topics for the group work: (1) Online newspapers, (2) TikTok, (3) Letsplay videos on Youtube, (4) Game consoles. The concluding lesson 8 is a reflection on the findings, the tool, and the data analysis process.

Part two consists of eight lessons too. Students use their findings from part one to create and understand the method of decision trees to predict a respective target group for the four areas.

IMPLEMENTATION OF THE FIRST PART OF "DATA DETECTIVES AT WORK"

All 13 students aged 15–17 from one tenth-grade class of a German middle school (German Realschule) participated in the regular lesson series in April 2021. The students had little prior knowledge of statistics. Due to the pandemic, a large part of the lessons took place online, which was easy to implement with CODAP because it is web-based. The presentations of the projects also took place digitally at the end. The students formed four groups for the project work.

In the first four lessons, students were taught how to represent the distribution of a categorical or numerical variable in CODAP, how to investigate relationships between two variables with CODAP using different types of percentages (row, column, cell) and how to represent absolute and relative frequencies with CODAP. It is well known from studies such as Watson and Callingham (2014) that learners have considerable difficulty in exploring relationships between two categorical variables. For this reason, special instructions were developed and discussed with the students.

We show some examples of the results from one typical student presentation. This shows how well students have mastered data analysis with the CODAP tool and what insights they have gained.

Results from the Presentation of Student Group 1 with the Topic of Online Newspapers

Student group 1 had three members. Their presentation consisted of seven slides with eleven diagrams, eight of them as a 7×7 table. In these tables, only row percentages were used. This is a default setting in CODAP when using percentages. Row percentages were used correctly in many places by all groups, but in some places, column or cell percentages would have been more appropriate with regard to the interpretations made by students. A typical example of student group 1, following next represents work done in a similar way by the other student groups.

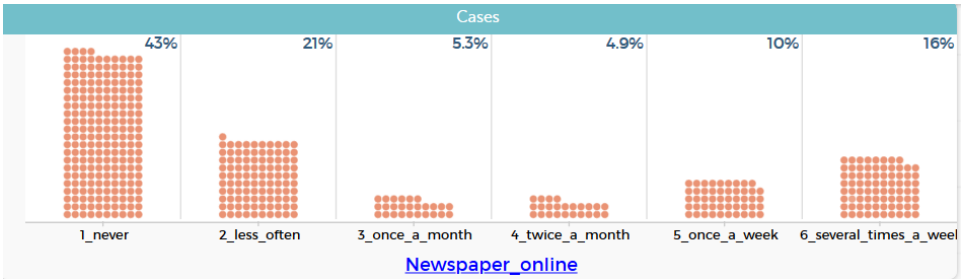

Figure 3. Students' diagram for "How often do you read newspapers online?"

At first, this group analysed the frequency of reading newspapers online with CODAP (Figure 3). As a result, they stated that 57% of the respondents read newspapers online, while 43% do not do so at all. Next, they wanted to investigate which social media are used by newspaper readers. To do this, they created the 7×6 table in Figure 4.

Students wrote for Figure 4: "If we look at the situation with Twitter, we see that active readers of online newspapers also use Twitter frequently." It can be assumed that they are referring to the

percentage figures shown in the upper right corner. Maybe the use of column percentages would have been more appropriate for an interpretation of frequent online newspaper reading.

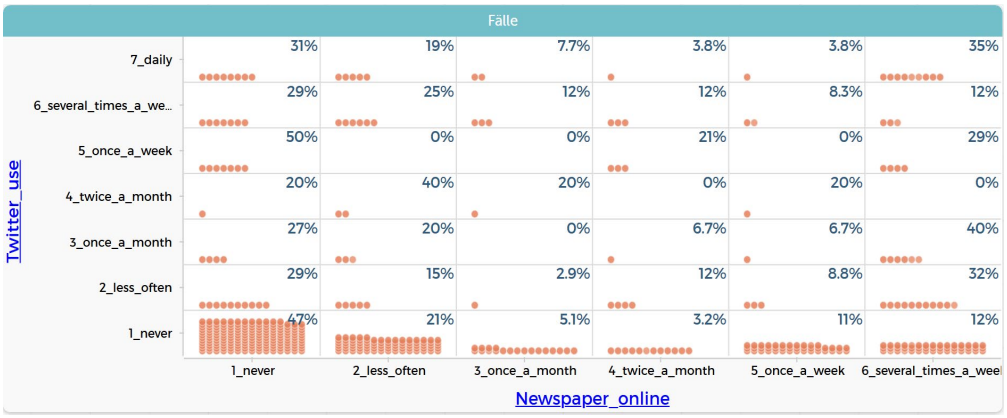

Figure 4. Students' diagram for "How often do newspaper readers use Twitter? (With row percentages)"

However, a large proportion never uses Twitter, which is why the results here are rather poorly usable. The situation is different for those who use Instagram (Figure 5).

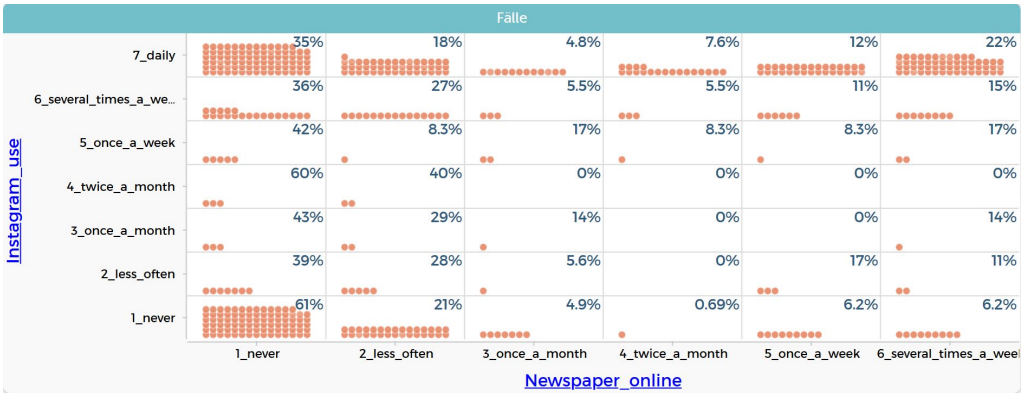

Figure 5. Students' diagram for "How often do newspaper readers use Instagram?" (With row percentages)

Students wrote here: "Instagram has a large active audience that uses the platform and reads newspapers online several times a week." It seems as if the students here have made an "and" association of the two variables in their interpretation.

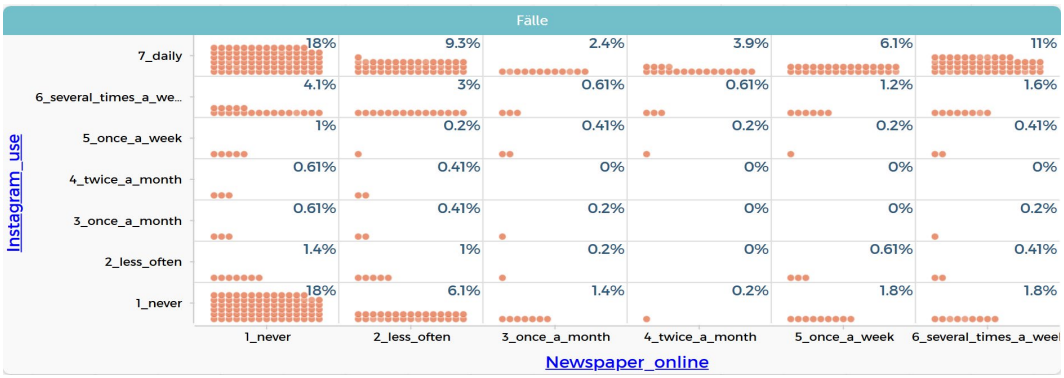

Figure 6. How often do newspaper readers use Instagram? (With cell percentages)

They are looking at people who read newspapers online AND use Instagram frequently. However, row percentages like in Figure 5 are not appropriate for such an interpretation, instead, cell percentages like in Figure 6 should have been used for this. Figure 6 shows a representation from which we can see that 11% of all respondents use Instagram daily and read newspapers online several times a week.

Furthermore, the students investigated which German school type newspapers are most frequently read online.

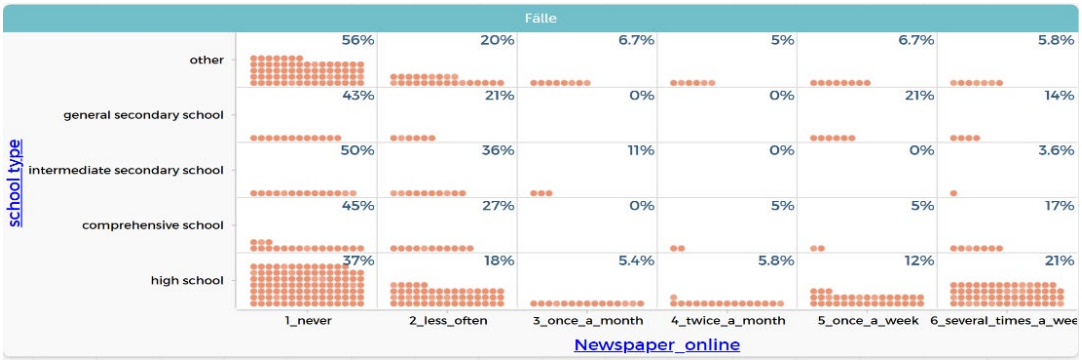

Figure 7. Students' diagram for "How often do students from different types of schools read newspapers online?" (With row percentages)

Students interpreted for Figure 7: "No matter how stereotypical it may sound, it is high school students who statistically read newspapers online more often", which is a correct interpretation.

The other graphs were correctly created and correctly interpreted by the group as well.

Results from the Presentation of Other Student Groups

Student group 2's presentation included six diagrams, four of them as a 7×7 table. In these tables, once row percentages, once column percentages, and twice cell percentages were used, all of them correctly. Student group 3's presentation included five diagrams, none of them as a 7×7 table. Student group 4's presentation included ten diagrams, and three times row percentages were used. Many interpretations of student group 4 would have required column or cell percentages, but were incorrectly assigned to row percentages. Interpreting 'small' diagrams like, e.g., comparing gender (with only two values: female/male) and another variable was done by all groups correctly.

Summary

Looking at all presentations of students, it can be seen that the interpretations always refer to only a few percentage values shown. On the one hand, it can be interpreted that the information content of the 7×7 visualisations was too high to be interpreted completely, but on the other hand that the students were able to extract the information that was important for them. One can see that all groups have dealt intensively with their topic, which can be shown by the number of slides and diagrams used in their final presentations. The percentages were not always applied correctly, but overall a lot of information was worked out using the data.

Students' Self-Assessment of Part One

After the lesson series of part one, the students completed an online survey on their attitudes and perceptions of the tool and the content. Students rated several items on a four-point Likert scale. Here we show selected results of four items.

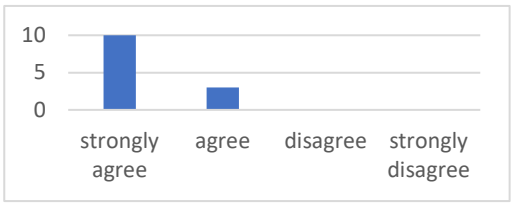

Figure 9. Item 1: Dealing with CODAP has been easy for me

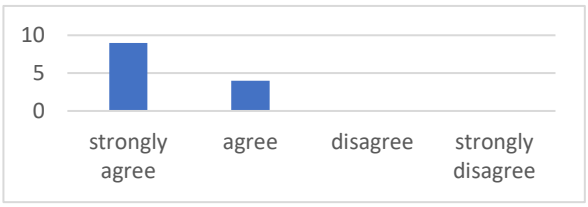

Figure 10. Item 2: I liked the interaction with CODAP

Results from item 1 and item 2 show that students felt competent in handling the tool CODAP and liked working with it. For the students' self-assessment concerning their statistical competencies, items 3 and 4 (in Figures 11 and 12) give an impression.

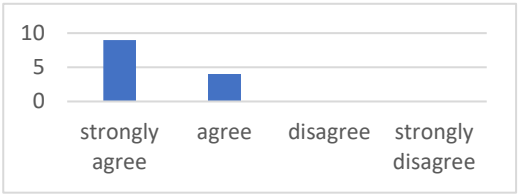

Figure 11. Item 3: I felt competent to interpret the charts

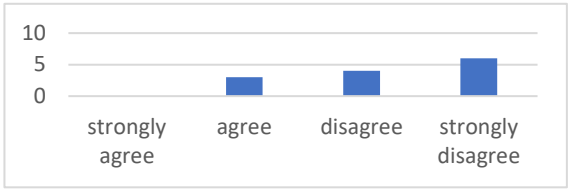

Figure 12. Item 4: I had difficulties in doing the data analysis on my own

The results show that all students felt competent to interpret the diagrams (although not all interpretations were correct), and only three students reported that they had difficulties when doing the data analysis with CODAP.

DISCUSSION

The innovative series of lessons on data science in middle school has generated enthusiasm among the students and led to interesting findings through data exploration. The students have made good use of statistical concepts to do data science. The resulting presentations have focused on many aspects of the different topics and answered many questions. Overall, few difficulties were encountered, including the relationship between two variables as reported by Watson and Callingham (2014). As expected, the CODAP tool could be used by the students for targeted data exploration after a minimal learning time. Thus, the approach presented here has proven successful in introducing an innovative data science project with real data in middle school.

The second part of the lesson series is currently being tested. The students continue to work with CODAP to create decision trees. For the first part of the innovative lesson series, CODAP has proven to be an excellent way to do data science in mathematics lessons, and now we are interested to see how it supports understanding decision trees.

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Appendix: Conference Programme

Schedule: summary

	Monday 13 September	Tuesday 14 September	Wednesday 15 September	Thursday 16 September
9:00 - 9:30		Plenary: Anna Baccaglini-Frank *(9:00 – 10:00)*	Plenary: Shai Olsher *(9:00 – 10:00)*	Paper session 5 *(9:00 – 10:30)*
9:30 - 10:00				
10:00 - 10:30		Coffee break	Coffee break	
10:30 - 11:00		Paper session 2 *(10:30 – 12:00)*	Paper session 4 *(10:30 – 12:00)*	Coffee break
11:00 - 11:30				Paper session 6 *(11:00 – 12:30)*
11:30 - 12:00				
12:00 - 12:30		Lunch *(12:00 – 13:00)*	Lunch *(12:00 – 13:30)*	Lunch *(12:30 – 13:30)*
12:30 - 13:00	Registration and lunch *(12:00 – 13:30)*			
13:00 - 13:30		Walk and talk *(13:00 – 14:30)*		
13:30 - 14:00	Openning ceremony		Workshops 2 *(13:30 – 15:00)*	Plenary: Chronis Kynigos *(13:30 – 14:30)*
14:15 - 14:30				
14:30 - 15:00	Paper session 1 *(14:15 – 15:45)*	Workshops 1 *(14:30 – 16:00)*		Closing ceremony
15:00 - 15:45			Excursion: Canal trip *(15:00 – 18:00)*	
15:45 - 16:00	Coffee break			
16:00 - 16:30	Plenary: Dan Meyer *(16:00 – 17:00)*	Coffee break		
16:30 - 17:00		Paper session 3 *(16:30 – 18:00)*		
17:00 - 17:30	Poster session and wine reception *(17:00 – 18:00)*			
17:30 - 18:00				
18:00 - 18:30				
18:30 - 19:00			Conference dinner *(Starting 18:45)*	
19:00 - later				

Day 1: Monday 13 September

12:00-13:30 – Registration and lunch
Aulaen (A141)
Time for checking in; receive credentials and materials, and having lunch

13:30-14:00 – Opening ceremony
Festsalen (A220)
Uffe Thomas Jankvist, IPC chair – Danish School of Education, Aarhus University, Denmark

14:15-15:45 – Paper session 1

Festsalen (A220)

- Rogier Bos and Winand Renkema: *Metaphor-based algebra animation*
- Philipp Larmann, Simon Barlovits and Matthias Ludwig: *MCM@home: Analysing a learning platform for synchronous distance education*
- Lena Frenken and Gilbert Greefrath: *Successful modelling processes in a computer-based learning environment*

Chair: Morten Blomhøj

Auditorium (D174)

- Francesca Ferrara, Giulia Ferrari and Ketty Savioli: *Children in movement towards STEAM: Coding and shapes at kindergarten*
- Melanie Platz: *"... Then it looks beautiful" - Preformal proving in primary school*
- Susanne Podworny and Yannik Fleischer: *An approach to teaching data science in middle school*

Chair: Julie Vangsøe Færch

15:45-16:00 – Coffee break
Aulaen (A141)

16:00-17:00 – Plenary lecture
Festsalen (A220)
Dan Meyer – Desmos, Oakland, CA, USA
Theme 4: Innovating with technologies
Pixels are pedagogy
Chair: Alison Clark-Wilson

17:00-18:30 – Poster session and wine reception
Festsalen (A220)

- Domenico Brunetto: *StreetMath: Supporting young migrants empowerment (cancelled)*
- Ben Davies, Eirini Geraniou, Cosette Crisan and Manolis Mavrikis: *Undergraduates' experiences with automated assessment in STACK* (cancelled)
- Morten Elkjær: *Equation Lab: Teaching equation solving in Virtual Reality using a modified dynamic balance model* (cancelled)
- Francesca Ferrara, Giulia Ferrari, Elvira Fernández-Ahumada and Natividad Adamuz-Povedano: *Gamifying CLIL within the mathematical context of fraction learning*
- Julie Vangsøe Færch: *Development and evaluation of primary school students' mathematical competencies via dynamic online learning environments*
- Inge Olav Hauge and Johan Lie: *Contributions of computational thinking and computer programming for development of critical democratic competence: Empowerment and agency*
- Ayse Kilic, Zeger-Jan Kock and Birgit Pepin: *Connectivity related issues in a modularised course involving mathematics*
- Evelyn Schirmer, Alexander K. Schüler-Meyer and Birgit E.U. Pepin: *Challenges of procedure-oriented cognitive conflict strategies for undergraduate students*
- Kinga Szücs: *Finding theorems and their proofs by using a calculator with CAS in university-level mathematics*
- Laura Wirth and Gilbert Greefrath: *Examining heuristic worked example videos in a collaborative setting: The conception of the project MoVie*
- Filip Moons, Ellen Vandervieren and Jozef Copaert: *Atomic, reusable feedback: A technology-mediated solution for assessing handwritten math tasks?*

Day 2: Tuesday 14 September

09:00-10:00 – Plenary lecture
Festsalen (A220)
Anna Baccaglini-Frank – University of Pisa, Italy
Theme 2: Making sense of 'classroom' practice
Shifts from teaching mathematics with technology to teaching mathematics through technology: A focus on mathematical discussion
Chair: Hans-Georg Weigand

10:00-10:30 – Coffee break
Aulaen (A141)

10:30-12:00 – Paper session 2

Room A401

- Rikke Maagaard Gregersen: *How about that algebra view in GeoGebra? A review on how task design may support algebraic reasoning in lower secondary school*
- Cecilie Carlsen Bach and Angelika Bikner-Ahsbahs: *When a digital tool guides mathematical communication*
- Annalisa Cusi, Agnese Ilaria Telloni and Katia Visconti: *Design of digital resources to scaffold metacognitive activities: Focus on students' reflections* (cancelled)

Chair: Anna Baccaglini-Frank

Room A405

- Katrin Klingbeil, Fabian Rösken, Daniel Thurm, Bärbel Barzel, Florian Schacht, Ulrich Kortenkamp, Kaye Stacey and Vicki Steinle: *$SMART_A$ – Online diagnostic to reveal students algebraic thinking and enhance teachers diagnostic competencies*
- Reinhard Oldenburg: *Relational thinking supported by an algebraic modeling tool on the web*
- Simeon Schwob and Paul Gudladt: *Successful communication as a part of teaching and learning mathematics in remote settings* (cancelled)

Chair: Stine Gerster Johansen

12:00-13:00 – Lunch
Aulaen (A141)

13:00-14:30 – Walk and talk
Festsalen (A220)
Walk tour in groups with a visit to Grundtvig's Church.

14:30-16:00 – Workshop session 1

Room A100a
- Chaim Ballin, Anatoli Kouropatov and Ofir Shafirovitz: *Workshop: Interactive digital environment for teaching and learning deductive geometry (FullProof): Design principles, functionality, pedagogy and results of implementation (Part I)*
  Room A104
- Theo van den Bogaart and Rogier Bos: Heuristic trees hackathon: *Designing and implementing support for mathematical problem solving (Part I)*
  Room A201
- Kendal Bahadirgil and Knud Nissen: *Maple Mathematics Suite - essential tools for STEM education*
  Room A203
- Lena Frenken: *Discovering the Possibilities of a Computer-Based Learning Environment on Mathematical Modelling*
  Room A408
- Mats Brunström, Maria Fahlgren, Mirela Vinerean and Yosief Wondmagegne: *Workshop on the design of tasks and feedback utilizing a combination of a dynamic mathematics software and a computer-aided assessment system*
  Room A412
- Yannik Fleischer and Susanne Podworny: *Teaching machine learning with decision trees in middle school using CODAP*
  Room A416
- Allan Tarp: *Develop children's innate mastery of many by bridging outside existence to inside essence in full sentences*

16:00-16:30 – Coffee break
Aulaen (A141)

16:30-18:00 – Paper session 3
Auditorium (D174)
- Raimundo Elicer and Andreas Lindenskov Tamborg: *Nature of the relations between programming and computational thinking and mathematics in Danish teaching resources*
- Liv Nøhr, Morten Misfeldt and Andreas Lindenskov Tamborg: *Teacher development in computational thinking and student performance in mathematics: A proxy-based TIMSS study*
- Eleonora Faggiano and Federica Mennuni: *Grasping sense and building meanings in mathematical distance contexts: The role of the teacher*
  Chair: Mathilde Kjær Pedersen

Room D170 (cancelled)

- Scott Courtney: *Exploring teachers' attempts to differentiate instruction in remote learning environments: The case of Claudia* (cancelled)
- Eirini Geraniou and Cosette Crisan: *Adapting classroom based practices to online teaching: A mathematics teacher's reflections* (cancelled)
- Melih Turgut, Iveta Kohanová, Jørn Ove Asklund, Øistein Gjøvik, Marit Buset Langfeldt and Hermund André Torkildsen: *Fourth graders explore a computational thinking task using Robot Emil: A multimodal analysis of pupils' thinking* (cancelled)

Day 3: Wednesday 15 September

09:00-10:00 – Plenary lecture
Festsalen (A220)
Shai Olsher – University of Haifa, Israel
Theme 3: Fostering mathematical collaborations
Te(a)ching to collaborate: Automatic assessment based grouping recommendations and implications for teaching
Chair: Eleonora Faggiano

10:00-10:30 – Coffee break
Aulaen (A141)

10:30-12:00 – Paper session 4

Festsalen (A220)

- Frederik Dilling and Julian Sommer: *Virtual Reality in Mathematics Education – Design of an Application for Multiview Projections*
- Alice Barana: *Understanding linear functions in an interactive digital learning environment*
- Cintia Scafa Urbaez Vilchez and Alice Lemmo: *A videogame as a tool to orchestrate productive mathematical discussions*

Chair: Eleonora Faggiano

Auditorium (D174)

- Manuela Subtil, António Domingos and Maria Alessandra Mariotti: *Graphing calculator in the connection between geometry and functions with the contribution of semiotic mediation*
- David Reid, Angelika Bikner-Ahsbahs, Thomas Janßen and Estela Vallejo-Vargas: *Forms of epistemic feedback*
- Zelha Tunç-Pekkan, Eylem Sayar and Isıl Ozturk: *Affordances of university based online laboratory school: Types of feedback*

Chair: Marianne Thomsen

12:00-13:30 – Lunch
Aulaen (A141)

13:30-15:00 – Workshop session 2

Room A100a

- Chaim Ballin, Anatoli Kouropatov and Ofir Shafirovitz: *Workshop: Interactive digital environment for teaching and learning deductive geometry (FullProof): Design principles, functionality, pedagogy and results of implementation (Part II)*
  Room A104
- Theo van den Bogaart and Rogier Bos: Heuristic trees hackathon: *Designing and implementing support for mathematical problem solving (Part II)*
  Room A203
- Filip Moons and Ellen Vandervieren: *Workshop - Writing atomic, reusable feedback to semi-automatedly assess handwritten math tasks*
  Room A405
- Philipp Larmann, Simon Barlovits and Matthias Ludwig: *MCM@home: An approach for synchronous distance learning with mobile devices*
  Room A414
- Frederik Dilling and Julian Sommer: *Mixed reality in mathematics education*
  Room A416
- Francesca Ferrara, Giulia Ferrari and Keyy Savioli: *Spatial and computational thinking at kindergarten through the aid of an educational robot*
  Room D166
- Douglas Butler: *Comparing the user interfaces of dynamic software* (cancelled)

15:00-18:00 – Excursion

Parking lot between buildings A and C

Ferry tour to the Copenhagen canals.
A bus will be expecting us on the campus parking lot.
The last stop of the tour will be **Christianshavn**, at a walking distance to the conference dinner.

18:45 – Conference dinner

Restaurant *Spiseloppen*
Address: Bådmandsstræde 43, 1407 Copenhagen (Christiania)

Day 4: Thursday 16 September

09:00-10:30 – Paper session 5

Festsalen (A220)

- Dimitris Diamantidis and Chronis Kynigos: *Digital media as tools fostering teacher creativity on designing tasks around an area of mathematical concepts*
- Bjarnheiður Kristinsdóttir, Freyja Hreinsdóttir and Zsolt Lavicza: *Developing silent video tasks' instructional sequence in collaboration with teachers*
- Tim Lutz: *Machine learning model for automated text classification of mathematical tasks (cancelled)*

Chair: Rikke Maagaard Gregersen

Room A203

- Marianne Thomsen and Uffe Thomas Jankvist: *Mathematical thinking in the interplay between historical original sources and GeoGebra*
- Maria Fahlgren, Mats Brunström, Mirela Vinerean and Yosief Wondmagegne: *Designing tasks and feedback utilizing a combination of a dynamic mathematics software and a computer-aided assessment system*
- Mathilde Kjær Pedersen: *The use of digital technologies for mathematical thinking competency*

Chair: Andreas Lindenskov Tamborg

10:30-11:00 – Coffee break
Aulaen (A141)

11:00-12:30 – Paper session 6

Festsalen (A220)

- Zelha Tunç-Pekkan, Rukiye Didem Taylan, Bengi Birgilil and İbrahim Burak Olmez: *An examination of pre-service mathematics teachers' experiences at an online school*
- Andreas Lindenskov Tamborg, Uffe Thomas Jankvist and Morten Misfeldt: *Comparing programming and computational thinking with mathematical digital competencies from an implementation perspective*
- Simone Jablonski, Eugenia Taranto, Matthias Ludwig and Maria Flavia Mammana: *Go online to go outdoors – a MOOC on MathCityMap*

Chair: Cecilie Carlsen Bach

Room A203

- Maxim Brnic and Gilbert Greefrath: *Does the gender matter? The use of a digital textbook compared to printed materials*
- Alexander Schüler-Meyer: *The purpose of handwriting with tablet-computers and smartpens in mathematical group work over distance*
- Stine Gerster Johansen: *A Review on Allgemeinbildung and Digital Technologies in Mathematics Education*

Chair: Raimundo Elicer

12:30-13:30 – Lunch
Aulaen (A141)

13:30-14:30 – Plenary lecture
Festsalen (A220)
Chronis Kynigos – National and Kapodistrian University of Athens, Greece; and Linnaeus University, Sweden
Theme 1: Designing technology
Embedding mathematics in socio-scientific games: The case of the mathematical in grappling with Wicked Problems
Chair: Hans-Georg Weigand

14:30-15:00 – Closing ceremony
Festsalen (A220)
Uffe Thomas Jankvist, IPC chair – Danish School of Education, Aarhus University, Denmark