A Characterization of Oeljeklaus-Toma Manifolds in Locally Conformally Kähler Geometry

Shuho Kanda
Graduate School of Mathematical Sciences, The University of Tokyo, Japan

1. Overview

LCK manifolds generalize Kähler manifolds and are studied in non-Kähler geometry. Vaisman manifolds are a subclass of LCK manifolds that are closer to being Kähler. Many non-Vaisman LCK manifolds exist, but few are easy to compute. Oeljeklaus-Toma (OT) manifolds form a class of compact non-Kähler manifolds. Some OT manifolds are the only known examples of compact LCK manifolds that have a solvmanifold structure (a quotient of a solvable Lie group by a lattice) and a non-Vaisman LCK structure. Thus, **OT manifolds are important examples in LCK geometry**. OT manifolds are a generalization of Inoue surfaces and are constructed essentially using number theory.

Questions

- Why have no other examples been found?
- Why does number theory play a crucial role in the only known case?

We prove that **if a certain class of solvmanifolds admits a left-invariant non-Vaisman LCK metric, then they must arise from the construction of OT manifolds**.

2. OT manifolds

We omit the definition of OT manifolds here but explain some key points. The information required to construct OT manifolds is as follows:

- An irreducible polynomial $f \in \mathbb{Q}[x]$ which has s real roots and $2t$ complex roots such that $s, t \geq 1$.
- Choose t complex roots such that, together with their conjugates, they form all complex roots.
- Take a subgroup $U \subset \mathcal{O}_K^\times$ of rank s satisfying some conditions, where $K \coloneqq \mathbb{Q}[x]/(f(x))$ is a field.

We can define a discrete group action $(U \ltimes \mathcal{O}_K) \curvearrowright (\mathbb{H}^s \times \mathbb{C}^t)$. We call its quotient $X(K,U) \coloneqq (\mathbb{H}^s \times \mathbb{C}^t)/(U \ltimes \mathcal{O}_K)$ an *OT manifold of type* (s,t). The properties of OT manifolds are as follows:

- OT manifolds are compact complex manifolds of dim $s+t$.
- OT manifolds do not admit any Kähler metric.
- OT manifolds of type $(s,1)$ admit a **non-Vaisman LCK metric**.
- OT manifolds are **solvmanifolds**.

The solvmanifold structure of an OT manifold is described as follows:

Definition (OT-like Lie groups)

Let $C = (c_{ij})_{ij} \in \mathrm{Mat}_{t\times s}(\mathbb{C})$ be a complex matrix such that

$$\mathrm{Re}(c_{1j}) + \cdots + \mathrm{Re}(c_{tj}) = -1/2 \qquad (1)$$

for all $1 \leq j \leq s$. We define a unimodular solvable Lie group $G_C = \mathbb{R}^s \ltimes_{\phi_C} (\mathbb{R}^s \oplus \mathbb{C}^t)$ where the map $\phi_C : \mathbb{R}^s \to \mathrm{GL}(\mathbb{R}^s \oplus \mathbb{C}^t)$ is the following:

$$\phi_C(t_1,\ldots,t_s) = \exp\left(\mathrm{diag}(t_1,\ldots,t_s,(\sum_{j=1}^s c_{ij}t_j)_{i=1}^t)\right).$$

We call a Lie group *OT-like* of type (s,t) if it is isomorphic to G_C for some matrix $C \in \mathrm{Mat}_{t\times s}(\mathbb{C})$.
If $\mathrm{Re}(c_{ij}) = -1/2t$ for all i,j, we call G_C *LCK OT-like*.

Note that all OT-like Lie group of type $(s,1)$ is LCK OT-like.

- For all OT manifolds $X(K,U)$ of type (s,t), there exists a matrix $C \in \mathrm{Mat}_{t\times s}(\mathbb{C})$ satisfying (1) and a lattice $\Gamma \subset G_C$ such that $X(K,U) \simeq \Gamma\backslash G_C$.
- LCK OT-like Lie groups admit a left-invariant non-Vaisman LCK metric.

As a result, OT manifolds of type $(s,1)$ admit a non-Vaisman LCK metric.

3. Result I

Main Theorem I

Let G_C be an OT-like Lie group of type (s,t). If G_C admits a simple lattice, then G_C is constructed from a simple OT manifold of type (s,t).

We call an OT manifold $X(K,U)$ *simple* if $\mathbb{Q}(U) = K$ holds. $X(K,U)$ is simple if and only if a lattice Γ such that $X(K,U) \simeq \Gamma\backslash G_C$ is simple. We prove that **any lattice of an LCK OT-like Lie group is simple**. An OT manifold admits an LCK metric if and only if its type is $(s,1)$. Thus we have

Corollary

Let G_C be an LCK OT-like Lie group of type (s,t). If G_C admits a lattice, then $t=1$ and G_C is constructed from an OT manifold of type $(s,1)$.

4. Result II

So, **what kind of solvable Lie group is an LCK OT-like Lie group?** It can be characterized within the class of meta-abelian Lie groups. The semi-direct product of two abelian Lie groups is called *meta-abelian*.

Main Theorem II

Let $G = \mathbb{R}^m \ltimes_\phi \mathbb{R}^n$ be a meta-abelian Lie group with $m = 1$ or 2. If G admits a left-invariant non-Vaisman LCK structure, then it is isomorphic to an LCK OT-like Lie group of type (s,t), where $m = s$ and $n = s+2t$.

What about the case when $m > 2$? Let $\mathfrak{g} = \mathbb{R}^m \ltimes_{d\phi} \mathbb{R}^n$ be the Lie algebra of G. If a left-invariant non-Vaisman LCK structure (G,J,g) satisfies $\mathfrak{g} = \mathbb{R}^n + J\mathbb{R}^n$, then G is isomorphic to an LCK OT-like Lie group. Whether this condition can be removed remains an open question.

5. Strategy of the proof

- To prove Main Theorem I, we use the following lemma proved by using Mostow's theorem:

If an OT-like Lie group $G_C = \mathbb{R}^s \ltimes_{\phi_C} (\mathbb{R}^s \oplus \mathbb{C}^t)$ admits a simple lattice, there exists $P \in \mathrm{GL}(n,\mathbb{R})$ and a lattice $\Gamma_1 \in \mathbb{R}^s$ such that

$$P(\phi_C(x))P^{-1} \in \mathrm{SL}(n,\mathbb{Z}), \quad \text{for all } x \in \Gamma_1,$$

where $n = s+2t$. As a result, we obtain a new lattice $\Gamma = \Gamma_1 \ltimes_{\phi_C} (P^{-1}\mathbb{Z}^n)$ such that the action $\Gamma_1 \curvearrowright (P^{-1}\mathbb{Z}^n)$ is simple.

We set $U \coloneqq \left\{P(\phi_C(x))P^{-1} \in \mathrm{SL}(n,\mathbb{Z}) \mid x \in \Gamma_1\right\}$. The group $U \subset \mathrm{SL}(n,\mathbb{Z})$ is a free abelian subgroup of rank s. Then we can define a $\mathbb{Q}$-algebra $K \coloneqq \mathbb{Q}[U] \subset \mathrm{Mat}(n,\mathbb{Q})$. Since the elements of K are diagonalized by P, K is reduced ring. As K is Artinian and reduced, it is isomorphic to a finite product of fields $K_1 \times \cdots \times K_r$. We can show that $r=1$ and $[K:\mathbb{Q}] = n$ by simplicity of the lattice. Since the characteristic polynomials of $A, A^{-1} \in U \subset \mathrm{GL}(n,\mathbb{Z})$ are monic and integer coefficients, it follows that $U \subset \mathcal{O}_K^\times$. By using the general theory of fields, it is ultimately shown that $X(K,U)$ is an OT manifold, and there exists a lattice Γ such that $X(K,U) \simeq \Gamma\backslash G_C$.

- To prove Main Theorem II, one only needs to carefully classify the structure using linear algebra. Yet, interestingly, the proof features the **Cauchy-Schwarz inequality** and the **Killing form**. The most non-trivial step is diagonalizing the action $d\phi : \mathbb{R}^s \curvearrowright (\mathbb{R}^s \oplus \mathbb{C}^t)$. First, this action decomposes into $\mathbb{R}^s \curvearrowright \mathbb{R}^s$ and $\mathbb{R}^s \curvearrowright \mathbb{C}^t$. The latter action is a sum of a skew-Hermitian action and a scalar multiplication, with respect to the original metric g. The former action is self-adjoint with respect to a new metric constructed by the Killing form. The Killing form is degenerate on a solvable Lie algebra in general, but it becomes non-degenerate when restricted to a certain subspace. Furthermore, its positive definiteness is shown using the Cauchy-Schwarz inequality.