Towards an Aubin-Yau theorem for transversally Kähler foliations

(Work in progress)

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Historical Motivation

Answered in 1978 by S.T. Yau and separately by T. Aubin in 1976, the Aubin-Yau conjecture has been sought after for decades and widely celebrated as the solution of a fully nonlinear PDE of Monge–Ampère type:

***Theorem* 1** ([6];[1]). *Let M be a compact complex manifold such that $c_1(M)$ admits a representative whose associated symmetric tensor is negative-definite. Then there exists a unique Kähler-Einstein metric with negative constant i.e. its Ricci form ρ and its Kähler form ω satisfy $\rho = -\omega$.* ■

Arising from applications, as well as a natural question for consideration, is the more general case of a **transversally Kähler foliation**.

Definition 1. Let $(M, \mathcal{F})$ be a foliated manifold. If:

- $\mathcal{F}$ is a Riemannian foliation and g is a bundle-like pseudometric on TM,
- $\exists \omega_0 \in \Omega^2_{\text{bas}}(M, \mathcal{F})$, nondenegerate on νF, and
- $\overline{J} := \overline{\omega_0}^{-1} \circ \overline{g} : \nu\mathcal{F} \to \nu\mathcal{F}$ is an integrable almost complex structure for any local leaf space which happens to be a smooth manifold

then $(M, \mathcal{F}, g, \omega_0)$ is a **transversally Kähler** foliation.

Research motivation: Study of LCK and Vaisman Manifolds

We're interested in studying and classifying LCK manifolds, which is a broad class of complex manifolds.

Definition 2. A complex Riemannian manifold M with real Hermitian form ω is **locally conformally Kähler (LCK)** if $\exists \theta \in \Omega^1(M)$ with $d\theta = 0$ and $d\omega = \omega \wedge \theta$.

Among LCK manifolds, Vaisman manifolds are a particularly well-behaved class; for instance they have:

- a canonical foliation
- an associated projective orbifold

Definition 3. An LCK manifold (M, ω, θ) is **Vaisman** if $\nabla\theta = 0$.

A rich source of examples for transversally Kähler foliations are Vaisman manifolds with their canonical foliations.

***Theorem* 2** ([4]). *Let (M, ω, θ) be a Vaisman manifold. Set $\Sigma := \langle \theta^\sharp, J\theta^\sharp \rangle$ (the canonical foliation). Then there exists $\omega_0 \in \Omega^2_{bas}(M, \Sigma)$ with respect to which (M, Σ) is a transversally Kähler foliation. More precisely, with*

$$\omega_0 := d^c\theta$$

it follows that

$$\omega = \omega_0 + \theta \wedge \theta^c$$

and ω_0 is a semi-positive, closed $(1,1)$-form, such that $\ker \omega_0 = \Sigma$. ■

Expected Main Result

***Theorem* 3** (Work in progress). *Let $(M, \mathcal{F})$ be a foliation of transversal Kähler type, with a fixed transversally holomorphic structure $\overline{J}$.*

Suppose $c_1(\nu\mathcal{F})$ can be represented by a basic form which is transversally negative (w.r.t. $\overline{J}$).

Then there exists a unique transversally Kähler, transversally Einstein metric with Einstein constant -1, i.e. for the transversally Kähler metric ω_0 we have $\omega_0 = -\rho$. ■

Here, for $(M, \mathcal{F}, \omega_0, \overline{g} \in \mathrm{Sym}^2(\nu\mathcal{F}))$ a transversally Kähler foliation, we consider ρ the basic 2-form which on $\nu\mathcal{F}$ is the skew-symmetric tensor obtained from the Ricci tensor of the Levi-Civita connection of $\overline{g}$ and $\overline{J}$.

Main Ideas

- Reformulation as a differential equation in basic functions.
- Passage from basic functions to functions on the global leaf space of a basic foliation.
- Adapting existence and uniqueness arguments.

Reformulation

Let $(M, \mathcal{F})$ be of transversal Kähler type, $\mathrm{codim}\mathcal{F} = q$. Find ω representing $-c_1(\nu\mathcal{F})$ and positive on $\nu\mathcal{F}$. Using:

Lemma 1 ([2, Proposition 3.5.1]). Let $(M, \mathcal{F})$ be a foliated transversally Kähler manifold with a fixed transversal holomorphic structure.

Let $\omega, \omega' \in \Omega^{1,1}_{\text{bas}}(M, \mathcal{F})$ with $[\omega] = [\omega']$ in $H^2_{\text{bas}}(M, \mathcal{F})$.

Then $\exists f \in \mathcal{C}^\infty_{\text{bas}}(M, \mathcal{F})$ with $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}f$. ■

can find a basic f with $\rho^\omega = -\omega + \sqrt{-1}\partial\bar{\partial}f$. Using:

Lemma 2 (Work in progress). Let $(M, \mathcal{F})$ be a foliated transversally Kähler manifold with fixed transversal holomorphic structure and suppose for some basic f we have $\partial\bar{\partial}f = 0$. Then $f = \text{const.}$ ■

arrive at an equation in basic functions $u \in \mathcal{C}^\infty_{\text{bas}}(M, \mathcal{F})$:

$$\log \frac{(\omega + \sqrt{-1}\partial\bar{\partial}u)^q}{\omega^q} - u = f \tag{1}$$

Passage to a simple foliation and adaptation

For a foliation $(M, \mathcal{F})$ of $\mathrm{codim}\mathcal{F} = q$, denote $\mathrm{Fr}(\nu F)$ the principal $GL(q, \mathbb{R})$ bundle of frames in $\nu\mathcal{F}$, and by $\mathcal{F}_T$ the lifted foliation on $\mathrm{Fr}(\nu\mathcal{F})$. The starting point is:

***Proposition* 1.** *Let*

- *$(M, \mathcal{F})$ be a foliated manifold with $codim(\mathcal{F}) = q$,*
- *G be a Lie subgroup of $GL(\mathbb{R}, q)$,*
- *E be a transverse G-structure on $Fr(\nu\mathcal{F})$*

Then $\mathcal{C}^\infty_{bas}(M, \mathcal{F}) \simeq \mathcal{C}^\infty(E, \mathcal{F}_T)^G$.

Definition 4. A smooth foliation is called **transversally parallelisable** if it admits a transversal $\{1\}$-structure.

***Theorem* 4.** *[3, Section 3.3] Let $(M, \mathcal{F})$ be a Riemannian foliation with transverse Levi-Civita connection ω on a transverse $O(q)$-structure E. Then $(E, \mathcal{F}_T|_E)$ is transversally parallelisable.*

***Theorem* 5.** *[3, Theorem 4.2] Let $(M, \mathcal{F})$ be a transversally parallelisable foliation. Then the basic associated foliation, $\mathcal{F}_b$, is a simple foliation.*

Combining 1, 4 and 5, we can see $\mathcal{C}^\infty_{\text{bas}}(M, \mathcal{F})$ as a subspace of $\mathcal{C}^\infty(E/\mathcal{F}_{T,b})^{O(q)}$, the latter consisting of smooth functions on a global compact manifold.

Thus, we can treat (1) as a PDE in $u \in \mathcal{C}^\infty(W)^{O(q)}$ where W is a compact smooth manifold.

Uniqueness follows via the negative-semidefiniteness of $\sqrt{-1}\partial\bar{\partial}u(\cdot, \overline{J}\cdot)$.

Work on existence follows the Schauder continutiy method: modifying (1) to a family of equations $(*)_{t\in[0,1]}$ with RHS tf instead of f and taking $I := \{t \in [0,1] : (*)_t \text{ has a solution }\}$, we show that I is open and closed (work in progress).

Application to Vaisman Manifolds

As a consequence of the Main Result and

***Theorem* 6.** *[5] Let (M, ω, θ) be an LCK manifold defined by the local system (L, ∇). Let ${}^C\nabla$ be the Chern connection of the holomorphic structure given by $\nabla^{0,1}$. Then the curvature of ${}^C\nabla$ is $R^{{}^C\nabla} = -2\sqrt{-1}d^c\theta$* ■

we also obtain:

***Theorem* 7.** *Let M be a Vaisman manifold with Lee form θ. Let L be the weight bundle of M.*

Suppose $K_M = L^{\otimes\alpha}$ for $\alpha \in \mathbb{R}^{<0}$.

Then for $\omega_0 := d^c\theta$, there exists a Vaisman metric ω such that $\rho^\omega = \omega_0$ on $\nu\Sigma$. ■

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