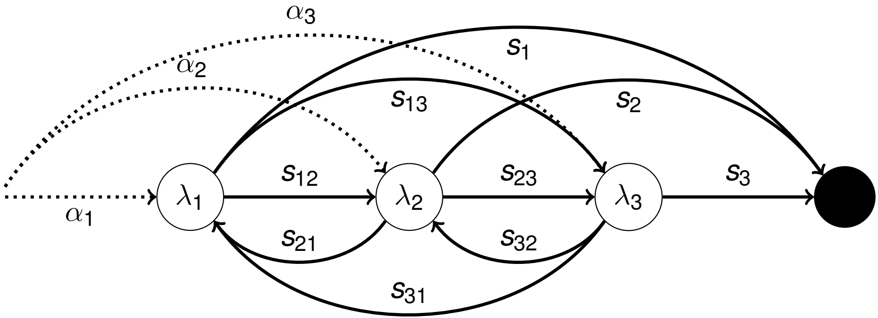
Clara Brimnes Gardner

Phase-type Representations for Exponential Distributions

Markov Jump Processes



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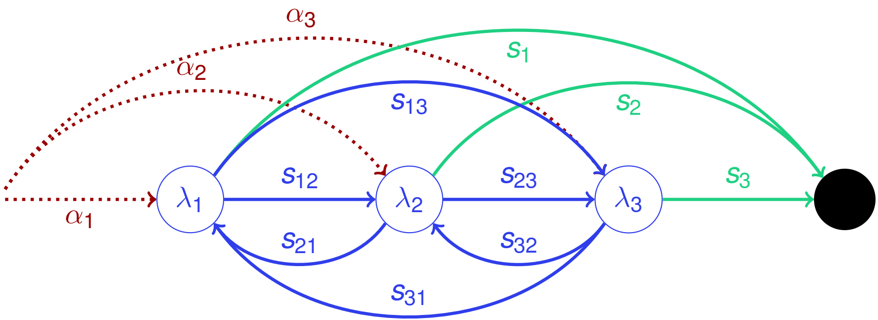
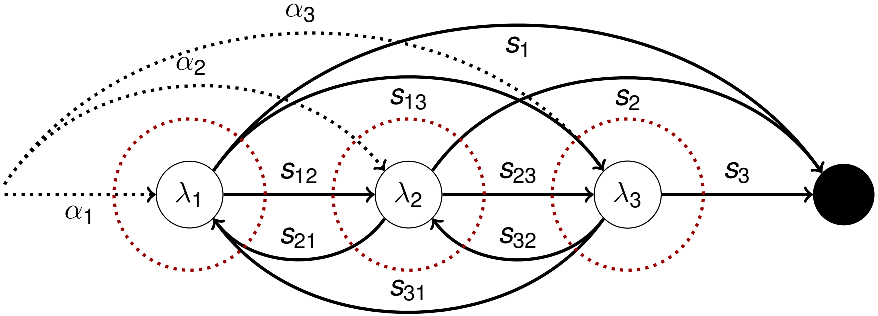


$$(\boldsymbol{\alpha}, \mathbf{S}) = \left(\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}, \begin{pmatrix} -\lambda_1 & s_{12} & s_{13} \\ s_{21} & -\lambda_2 & s_{23} \\ s_{31} & s_{32} & -\lambda_3 \end{pmatrix} \right) \text{ and } \mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = -\mathbf{S}\mathbf{e}.$$

Phase-type Distributions and the Exponential Distribution



$Z_{1,i} \sim \exp(\lambda_1)$ $Z_{2,i} \sim \exp(\lambda_2)$ $Z_{3,i} \sim \exp(\lambda_3)$

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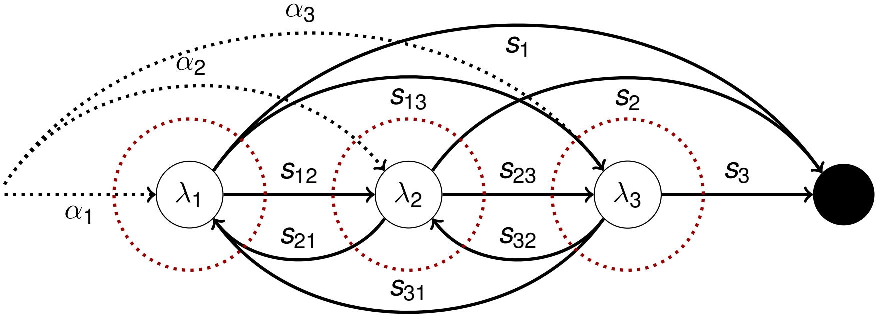
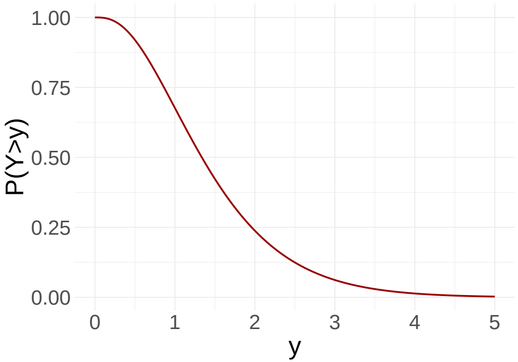


$Z_{1,i} \sim \exp(\lambda_1)$ $Z_{2,i} \sim \exp(\lambda_2)$ $Z_{3,i} \sim \exp(\lambda_3)$

$$Y = \sum_{i=1}^{p} \sum_{j=1}^{N_i} Z_{i,j},$$

Survival functions



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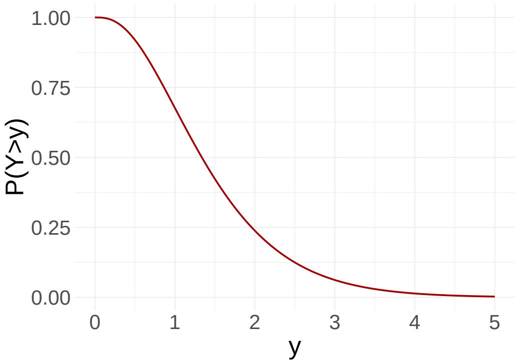

Exponential Distribution:

$$\mathbb{P}(Y > y) = \exp(-\lambda y),$$

Phase-type Distribution

$$\mathbb{P}(Y > y) = \boldsymbol{\alpha} \exp(\mathbf{S} y)\, \mathbf{e}.$$

When do we have

$$\exp(-\lambda y) = \boldsymbol{\alpha} \exp(\mathbf{S} y)\, \mathbf{e}?$$

The Jordan Normal Form

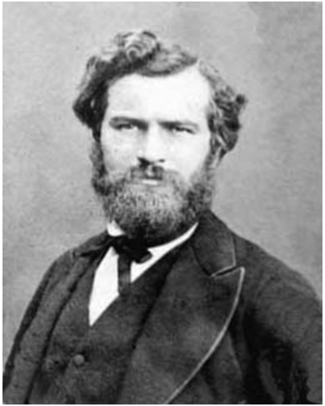

Figure: Picture from Wikipedia

The Jordan Normal Form: General Expression

$$\mathbf{S} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 & \cdots & \mathbf{T}_3 \end{pmatrix} \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_g \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_g \end{pmatrix}$$

with

$$\mathbf{T}_i = \begin{pmatrix} \mathbf{r}_{i,1} & \mathbf{r}_{i,2} & \cdots & \mathbf{r}_{i,b_i} \end{pmatrix}, \mathbf{J}_i = \begin{pmatrix} -\lambda_i & 1 & \cdots & 0 \\ 0 & -\lambda_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_i \end{pmatrix}, \mathbf{U}_i = \begin{pmatrix} \mathbf{l}_{i,b_i} \\ \mathbf{l}_{i,b_i-1} \\ \vdots \\ \mathbf{l}_{i,1} \end{pmatrix}$$

Example

$$\mathbf{S} = \begin{pmatrix} -4 & 1 & 2 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

The eigenvalues are $-\lambda_1 = -2$, $-\lambda_2 = -3$ and $-\lambda_3 = -4$.

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The eigenvalues are $-\lambda_1 = -2$, $-\lambda_2 = -3$ and $-\lambda_3 = -4$.
We can find three right eigenvectors: one for each eigenvalue

$$\mathbf{S}\mathbf{r}_1 = -\lambda_1 \mathbf{r}_1 \Rightarrow \mathbf{r}_1 = \begin{pmatrix} 1/2 & 1 & 0 & 0 \end{pmatrix}',$$
$$\mathbf{S}\mathbf{r}_2 = -\lambda_2 \mathbf{r}_2 \Rightarrow \mathbf{r}_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \end{pmatrix}',$$
$$\mathbf{S}\mathbf{r}_3 = -\lambda_3 \mathbf{r}_3 \Rightarrow \mathbf{r}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}'.$$

The Jordan Normal Form: Example

We find a generalized eigenvector of order $k = 2$

$$\mathbf{S} \cdot \mathbf{r}_{2,2} = -\lambda_2 \cdot \mathbf{r}_{2,2} + \mathbf{r}_2 \Rightarrow \mathbf{r}_{2,2} = \begin{pmatrix} -7/3 & -1/3 & 0 & 1/3 \end{pmatrix}$$

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Then we can write $\mathbf{S}$ as

$$\mathbf{S} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{r}_{1,1} & \mathbf{r}_{2,1} & \mathbf{r}_{2,2} & \mathbf{r}_{3,1} \end{pmatrix} \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 1 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & -\lambda_3 \end{pmatrix} \begin{pmatrix} \mathbf{l}_{1,1} \\ \mathbf{l}_{2,2} \\ \mathbf{l}_{2,1} \\ \mathbf{l}_{3,1} \end{pmatrix}$$

Jordan block of order 2

The Jordan Normal Form: Compact Notation

$$\mathbf{S} = \mathbf{T}\mathbf{J}\mathbf{T}^{-1} = \mathbf{T}\left(\sum_{i=1}^{g}\left((-\lambda_i)\mathbf{E}_i + \mathbf{N}_i\right)\right)\mathbf{T}^{-1},$$

From previous example

$$\mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{N}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

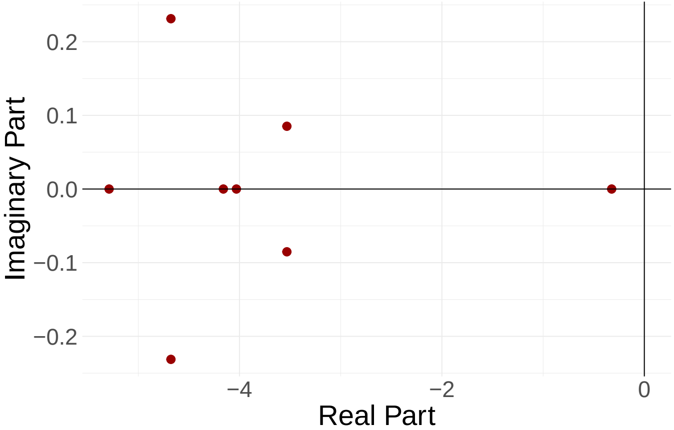
The Jordan Normal Form and the Matrix Exponential

$$\begin{aligned}\exp\left(\mathbf{S}y\right) &= \mathbf{T}\exp\left(\mathbf{J}y\right)\mathbf{T}^{-1}\\ &= \sum_{j=1}^{g}\sum_{k=1}^{B_j}\mathbf{M}_{j,k}\frac{y^{k-1}}{(k-1)!}\exp(-\lambda_i y),\end{aligned}$$

with

$$\mathbf{M}_{j,k} = \mathbf{T}\left(\sum_{i\in A_j}\mathbf{E}_i\mathbf{N}_i^{k-1}\mathbf{E}_i\right)\mathbf{T}^{-1}.$$

A look at the Eigenvalues



The Survival Function

$$\mathbb{P}(Y > y) = = \boldsymbol{\alpha} \exp(\mathbf{S}y)\,\mathbf{e} = \boldsymbol{\alpha} \sum_{i=1}^{g} \sum_{k=1}^{B_i} \mathbf{M}_{i,k} \frac{y^{k-1}}{(k-1)!} \exp(-\lambda_i y)\,\mathbf{e}$$

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$$= c_{1,B_1} \frac{y^{k-1}}{(k-1)!} \exp(-\lambda_1 y) + \sum_{k=1}^{B_1-1} c_{1,k} \frac{y^{k-1}}{(k-1)!} \exp(-\lambda_1 y) +$$

$$\sum_{i=2}^{g} \sum_{k=1}^{B_i} c_{i,k} \frac{y^{k-1}}{(k-1)!} \exp(-\lambda_i y)$$

with

$$c_{i,k} = \alpha \mathbf{M}_{i,k} \mathbf{e} = \alpha \mathbf{T} \left(\sum_{j \in A_i} \mathbf{E}_j \mathbf{N}_j^{k-1} \mathbf{E}_j \right) \mathbf{T}^{-1} \mathbf{e}.$$

Paper from Bean and Green

PERGAMON

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When Is a MAP Poisson?

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Abstract—The departure process of a queue is important in the analysis of networks of queues, as it may be the arrival process to another queue in the network. A simple description of the departure process could enable a tractable analysis of a network, saving costly simulation or avoiding the errors of approximation techniques.

In a recent paper, Olivier and Walrand [1] conjectured that the departure process of a MAP/PH/1 queue is not a MAP unless the queue is a stationary $M/M/1$ queue. This conjecture was prompted by their claim that the departure process of an MMPP/M/1 queue is not a MAP unless the queue is a stationary $M/M/1$ queue. We note that their proof has an algebraic error, see [2], which leaves the above question of whether the departure process of an MMPP/PH/1 queue is a MAP, still open.

There is also a more fundamental problem with Olivier and Walrand's proof. In order to identify stationary $M/M/1$ queues, it is essential to be able determine from its generator when a stationary MAP is a Poisson process. This is not discussed in [1], nor does it appear to have been discussed elsewhere in the literature. This deficiency is remedied using ideas from nonlinear filtering theory, to give a characterisation as to when a stationary MAP is a Poisson process.

Irreducible Generator Matrices

Definition (Irreducible Generators)

A generator matrix, $\mathbf{S}$ is said to be irreducible if all states communicate.

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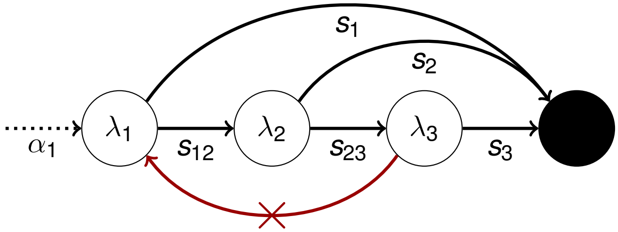

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A phase-type representation,$(\alpha, \mathbf{S})$, is said to be irreducible if there is a positive probability of visiting any of the transient states.

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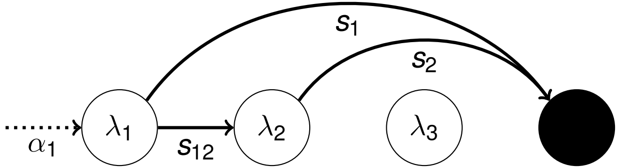

The Transition Probability Matrix

The transition probabilities

$$\mathbb{P}(X_{y+t} = j \mid X_t = i) = p_{ij}(y), \quad t, y > 0,$$

are collected in the transition probability matrix

$$\{p_{ij}(y)\} = P(y) = \exp(\mathbf{S}y).$$

Using the Jordan normal form we obtain:

$$\exp(\mathbf{S}y) = \sum_{i=1}^{g} \sum_{k=1}^{B_i} \mathbf{M}_{i,k} \frac{y^{k-1}}{(k-1)!} \exp\left(-\lambda_i y\right)$$

Probabilistic Argument: Non-negativity

$$\exp(\mathbf{S}y) = M_{1,B_1}\frac{y^{k-1}}{(k-1)!}\exp\left(-\lambda_1 y\right) + \sum_{k=1}^{B_1-1} M_{1,k}\frac{y^{k-1}}{(k-1)!}\exp\left(-\lambda_1 y\right) + \sum_{i=2}^{g}\sum_{k=1}^{B_i} M_{i,k}\frac{y^{k-1}}{(k-1)!}\exp(-\lambda_i y)$$

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Conclusion: Let $\{m^{l,k}_{1,B_1}\}$ be the elements of M_{1,B_1}. Then

$$m^{l,k}_{1,B_1} \geq 0.$$

Linear Algebra Argument: Positivity

The matrix M_{1,B_1} can be calculated as

$$M_{1,B_1} = \mathbf{T}\left(\sum_{j\in A_i} \mathbf{E}_j \mathbf{N}_j^{k-1} \mathbf{E}_j\right)\mathbf{T}^{-1}$$
$$= \mathbf{r}_{i,1}\mathbf{l}_{i,1}.$$

But $\mathbf{r}_{i,1} \neq \mathbf{0}$ and $\mathbf{l}_{i,1} \neq \mathbf{0}$.

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But $\mathbf{r}_{i,1} \neq \mathbf{0}$ and $\mathbf{l}_{i,1} \neq \mathbf{0}$.
Conclusion: M_{1,B_1} has at least one positive element.

What About the Survival Function?

$$\begin{aligned}\mathbb{P}(Y > t + y) &= \boldsymbol{\alpha} \exp\left(\mathbf{S}(t + y)\right) \mathbf{e} \\ &= \boldsymbol{\alpha} \exp(\mathbf{S}t) \exp\left(\mathbf{S}(y)\right) \mathbf{e} \\ &= \mathbf{q}(t) \exp(\mathbf{S}y)\mathbf{e},\end{aligned}$$

where $\mathbf{q}(t) = \boldsymbol{\alpha} \exp(\mathbf{S}t)$.

What About the Survival Function?

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where $\mathbf{q}(t) = \alpha \exp(\mathbf{S}t)$.

If $(\alpha, \mathbf{S})$ is an irreducible representation, then $\mathbf{q}(t) > 0$.

Survival Function II

$$
\begin{aligned}
\mathbb{P}(y > t + y) &= \mathbf{q}(t)\exp(\mathbf{S}y)\mathbf{e} \\
&= \mathbf{q}(t)M_{1,B_1}\mathbf{e}\frac{y^{k-1}}{(k-1)!}\exp\left(-\lambda_1 y\right) + \\
&\sum_{k=1}^{B_1-1}\mathbf{q}(t)M_{1,k}\mathbf{e}\frac{y^{k-1}}{(k-1)!}\exp\left(-\lambda_1 y\right) + \\
&\sum_{i=2}^{g}\sum_{k=1}^{B_i}\mathbf{q}(t)M_{i,k}\mathbf{e}\frac{y^{k-1}}{(k-1)!}\exp(-\lambda_i y)
\end{aligned}
$$

Theorem

Theorem (Exponential Phase-type Representations)

Let $Y \sim PH(\boldsymbol{\alpha}, \mathbf{S})$ *such that the representation is irreducible.* Y *is exponentially distributed if and only if* $B_1 = 1$ *and*

$$c_{j,k} = \boldsymbol{\alpha}\mathbf{T}\left(\sum_{i \in A_j} \mathbf{E}_i \mathbf{N}_i^{\nu-1} \mathbf{E}_i\right)\mathbf{T}^{-1}\mathbf{e} = 0.$$

for $\{j, k\} \in \{2, \dots, g\} \times \{1, \dots, B_j\}$.

Theorem: Distinct Eigenvalues

Theorem

Let $Y \sim PH(\alpha, \mathbf{S})$ such that the representation is irreducible, and each Jordan block has a distinct eigenvalue. Y is exponentially distributed if and only if $B_1 = 1$ and

$$\alpha \mathbf{r}_{j,\nu} = 0 \text{ or } \mathbf{l}_{j,B_j+1-\nu}\mathbf{e} = 0, \tag{1}$$

for all $\{j, \nu\} \in \{2, 3, \ldots, g\} \times \{1, 2, \ldots, B_j\}$.

Example

Example: three distinct eigenvalues $-\lambda_1$, $-\lambda_2$ and $-\lambda_3$ each with algebraic and geometric multiplicity :

$$\begin{aligned}\mathbb{P}(Y > y) &= \boldsymbol{\alpha} \exp(\mathbf{S}y)\,\mathbf{e} \\ &= \boldsymbol{\alpha}\mathbf{r}_1\mathbf{l}_1\mathbf{e} \exp(-\lambda_1 y) + \boldsymbol{\alpha}\mathbf{r}_2\mathbf{l}_2\mathbf{e} \exp(-\lambda_2 y) + \boldsymbol{\alpha}\mathbf{r}_3\mathbf{l}_3\mathbf{e} \exp(-\lambda_3 y)\end{aligned}$$

For Y to be exponentially distributed we require:

$$\begin{aligned}\boldsymbol{\alpha}\mathbf{r}_2 = 0 \text{ or } \mathbf{l}_2\mathbf{e} = 0, \\ \boldsymbol{\alpha}\mathbf{r}_3 = 0 \text{ or } \mathbf{l}_3\mathbf{e} = 0.\end{aligned}$$

PH-simplicity

Definition (O'Cinneide)

Let PH($\mathbf{S}$) be the set of all phase-type distributions with a representation of the form $(\alpha, \mathbf{S})$. The generator $\mathbf{S}$ is said to be PH-simple if each distribution in PH($\mathbf{S}$) has a unique representation of the form $(\alpha, \mathbf{S})$.

Theorem (O'Cinneide)

A PH-generator $\mathbf{S}$ is PH-simple if and only if it has no left eigenvector orthogonal to $\mathbf{e}$.

Comparison

Theorem

Let $Y \sim PH(\alpha, \mathbf{S})$ such that the representation is irreducible, and each Jordan block has a distinct eigenvalue. Y is exponentially distributed if and only if $B_1 = 1$ and

$$\alpha \mathbf{r}_{j,\nu} = 0 \textit{ or } \mathbf{l}_{j,B_j+1-\nu}\mathbf{e} = 0, \tag{2}$$

for all $\{j, \nu\} \in \{2, 3, \ldots, g\} \times \{1, 2, \ldots, B_j\}$.

If $\mathbf{S}$ is simple, then $\mathbf{l}_{i,1}\mathbf{e} \neq 0$. This implies that Y can only be exponentially distributed with $\alpha \mathbf{r}_i = 0$. This happens if α is a left eigenvector.