Quantitative Equidistribution on Hyperbolic Surfaces and Arithmetic Applications

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Equidistribution

Let M be a topological space and μ a (Borel) probability measure on M. Let μ_k be a sequence of (Borel) probability measures on M.

Definition

The sequence $\{\mu_k\}$ equidistributes on M w.r.t. μ if

$$\lim_{k \to \infty} \int_M f(x)\, d\mu_k(x) = \int_M f(x)\, d\mu(x)$$

for all $f \in C_b(M)$.

Example

Let $\{x_n\}$ be a sequence in M and let μ_k denote the probability measure

$$\mu_k(B) := \frac{1}{k} \sum_{n=1}^{k} \delta_{x_n}(B) = \frac{1}{k} \# \{1 \leq n \leq k : x_n \in B\},$$

where δ_{x_n} is the point mass at the point $x_n \in M$.

Example: Kronecker–Weyl Equidistribution

Example

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be irrational. Let $M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let

$$\mu_k(B) := \frac{1}{k} \sum_{n=1}^{k} \delta_{\{\alpha n\}}(B) = \frac{1}{k} \# \{1 \leq n \leq k : \alpha n \pmod 1 \in B\}.$$

Then $\{\mu_k\}$ equidistributes on M w.r.t. the Lebesgue measure.

Portmanteau Theorem

Theorem

Let (M, d) be a metric space. The following are equivalent:

- $\{\mu_k\}$ *equidistributes on* M *w.r.t.* μ*;*
- $\lim_{k\to\infty} \int_M F(x)\, d\mu_k(x) = \int_M F(x)\, d\mu(x)$ *for every bounded Lipschitz function* $F : M \to \mathbb{R}$ *(i.e.* $\sup_{\substack{x,y\in M\\ x\neq y}} \frac{|F(x)-F(y)|}{d(x,y)} < \infty$*);*
- $\lim_{k\to\infty} \mu_k(B) = \mu(B)$ *for every (Borel)* μ*-continuity set* B *(i.e.* $\mu(\partial B) = 0$*).*

Weyl Equidistribution Criterion

Theorem

Let $\mathcal{F}$ be a subset of $C_b(M)$ s.t. the set of finite linear combinations of elements of $\mathcal{F}$ is dense in $C_b(M)$. Then $\{\mu_k\}$ equidistributes on M w.r.t. μ iff

$$\lim_{k\to\infty} \int_M f(x)\, d\mu_k(x) = \int_M f(x)\, d\mu(x)$$

for every $f \in \mathcal{F}$.

Example

Let $M = \mathbb{R}/\mathbb{Z}$ and $\mathcal{F} = \{e^{2\pi i m x} : m \in \mathbb{Z}\}$. By the Stone–Weierstraß theorem, the set of finite linear combinations of elements of $\mathcal{F}$ is dense in $C_b(M)$.

Quantitative Equidistribution: Discrepancy

Equidistribution is a *qualitative* statement.
To make it *quantitative*, one must choose some method to quantify the rate of equidistribution.

Definition

Let $\mathcal{B}$ be a collection of sets. The $\mathcal{B}$-*discrepancy* between probability measures ν_1, ν_2 is

$$\mathcal{D}_{\mathcal{B}}(\nu_1, \nu_2) := \sup_{B \in \mathcal{B}} |\nu_1(B) - \nu_2(B)|.$$

If $\mathcal{B}$ generates the Borel sets on M, then $\lim_{k \to \infty} \mathcal{D}_{\mathcal{B}}(\mu_k, \mu) = 0$ implies that $\{\mu_k\}$ equidistributes on M w.r.t. μ.

Example

- If M is Euclidean, one can take $\mathcal{B}$ to be all boxes.
- If M is a vector space, one can take $\mathcal{B}$ to be all convex sets.
- If M is a metric space, one can take $\mathcal{B}$ to be all balls.

Quantitative Equidistribution: Discrepancy

Goal

If $\{\mu_k\}$ equidistributes on M w.r.t. μ, give upper bounds for $\mathcal{D}_{\mathcal{B}}(\mu_k, \mu)$ as $k \to \infty$ for a given collection of sets $\mathcal{B}$.

Theorem (Erdős–Turán Inequality)

Take $\mathcal{B}$ to be all intervals I in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, so that $\mathcal{D}_{\mathcal{B}} = \mathcal{D}_{\mathrm{box}}$. Then for any $T \geq 1$,

$$\mathcal{D}_{\mathrm{box}}(\mu_k, \mu) \ll \frac{1}{T} + \sum_{1 \leq |m| \leq T} \frac{1}{|m|} \left| \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i m x} \, d\mu_k(x) - \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i m x} \, d\mu(x) \right|.$$

Note $\int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i m x} \, d\mu(x) = 0$ for all $m \in \mathbb{Z} \setminus \{0\}$ if μ is Lebesgue.

In practise, the parameter T is chosen to minimise the RHS; enlarging T decreases the size of the first term but increases the length of the sum.

Quantitative Equidistribution: Shrinking Target Problems

Another quantification of equidistribution is to allow the sets B to shrink as $k \to \infty$.

Definition

Let $\mathcal{B}$ be a collection of (Borel μ-continuity) sets. Let $\{B_k\}$ be a sequence in $\mathcal{B}$ such that $\mu(B_k) > 0$ and $\mu(B_k) \to 0$ as $k \to \infty$. We say that $\{\mu_k\}$ equidistributes on the shrinking sets B_k w.r.t. μ if

$$\lim_{k \to \infty} \frac{\mu_k(B_k)}{\mu(B_k)} = 1.$$

Example

- If M is Euclidean, one can take $\mathcal{B}$ to be all cubes and $\{B_k\}$ to be a sequence of rescaled cubes whose side lengths converge to 0.
- If M is a metric space, one can take $\mathcal{B}$ to be all balls and $\{B_k\}$ to be a sequence of balls $B_{r_k}(x)$ with x fixed and $r_k \to 0$.

Quantitative Equidistribution: Wasserstein Distance

These quantifications of equidistribution depend on choices of sets.

Question

Are there more intrinsic quantifications of the rate of equidistribution?

Definition

Let (M, d) be a metric space and let ν_1, ν_2 be probability measures on M. The 1-*Wasserstein distance* between ν_1 and ν_2 is

$$\mathscr{W}_1(\nu_1, \nu_2) := \sup_{F \in \mathrm{Lip}_1(M)} \left| \int_M F(x)\, d\nu_1(x) - \int_M F(x)\, d\nu_2(x) \right|,$$

where $\mathrm{Lip}_1(M)$ denotes the space of Lipschitz functions $F : M \to \mathbb{R}$ for which $\sup_{\substack{x,y \in M \\ x \neq y}} \frac{|F(x)-F(y)|}{d(x,y)} \leq 1$.

Note that F need *not* be bounded (unless, say, M is compact). In general, $\mathscr{W}_1(\nu_1, \nu_2)$ need not be finite.

Quantitative Equidistribution: Wasserstein Distance

Definition

Let (M, d) be a metric space and let ν_1, ν_2 be (Borel) probability measures on M. For $p \in [1, \infty)$, the *p-Wasserstein distance* between ν_1 and ν_2 is

$$\mathscr{W}_p(\nu_1, \nu_2) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int_M d(x, y)^p \, d\pi(x, y),$$

where $\Pi(\nu_1, \nu_2) \ni \pi$ denotes the set of couplings of ν_1 and ν_2: joint probability measures on $M \times M$ whose marginals are ν_1 and ν_2, so that $\pi(B \times M) = \nu_1(B)$ and $\pi(M \times B) = \nu_2(B)$ for every Borel set $B \subseteq M$.

A necessary and sufficient condition for this to be finite is that $\int_M d(x, y)^p \, d\nu_1(x), \int_M d(x, y)^p \, d\nu_2(x) < \infty$ for some $y \in M$.

The p-Wasserstein distances arise in the theory of optimal transport. They are also commonly used in probability theory as a distance function on probability measures.

Quantitative Equidistribution: Wasserstein Distance

Theorem (Kantorovich–Rubinstein Duality)

These two definitions of the 1*-Wasserstein distance agree:*

$$\inf_{\pi \in \Pi(\nu_1,\nu_2)} \int_M d(x,y)\, d\pi(x,y) = \sup_{F \in \mathrm{Lip}_1(M)} \left| \int_M F(x)\, d\nu_1(x) - \int_M F(x)\, d\nu_2(x) \right|.$$

By the Portmanteau theorem, $\lim_{k\to\infty} \mathscr{W}_1(\mu_k, \mu) = 0$ implies that $\{\mu_k\}$ equidistributes on M w.r.t. μ.

The converse holds if $\int_M d(x,y)\, d\mu_k(x)$ and $\int_M d(x,y)\, d\mu(x)$ are finite for some $y \in M$.

Question

How can one bound $\mathscr{W}_1(\nu_1, \nu_2)$?

Inequalities for the Wasserstein Distance

There is an analogue of the Erdős–Turán inequality for the 1-Wasserstein distance on $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$.

Theorem (Bobkov–Ledoux, Borda)

Let ν_1, ν_2 be probability measures on $\mathbb{T}^n$. Then for any $T \geq 1$,

$$\mathscr{W}_1(\nu_1, \nu_2) \ll \frac{\sqrt{n}}{T}$$

$$+ \left(\sum_{1 \leq \|m\|_\infty \leq T} \frac{1}{\|m\|_2^2} \left| \int_{\mathbb{T}^n} e^{-2\pi i m \cdot x} \, d\nu_1(x) - \int_{\mathbb{T}^n} e^{-2\pi i m \cdot x} \, d\nu_2(x) \right|^2 \right)^{1/2}.$$

Recently used by Kowalski–Untrau for some problems in number theory involving exponential sums over finite fields.

Goal

Prove an analogous inequality on finite volume hyperbolic surfaces $\Gamma \backslash \mathbb{H}$ and apply this to arithmetic equidistribution problems.

Inequalities for the Wasserstein Distance

Theorem (H. (2025+))

Let Γ be a cocompact lattice in $\mathbb{H}$. Let ν_1, ν_2 be probability measures on $\Gamma\backslash\mathbb{H}$. Then for any $T \geq 1$,

$$\mathscr{W}_1(\nu_1,\nu_2) \ll \frac{1}{T} + \left(\sum_{f \in \mathcal{B}} \frac{e^{-\frac{t_f^2}{T^2}}}{\frac{1}{4}+t_f^2} \left| \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\nu_1(z) - \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\nu_2(z)\right|^2\right)^{1/2}.$$

Inequalities for the Wasserstein Distance

Observations:

- Distance function on $\Gamma\backslash\mathbb{H}$ is $\rho_\Gamma(z,w) := \min_{\gamma\in\Gamma}\rho(z,\gamma w)$ with $\rho(z,w) := 2\operatorname{arsinh}\frac{|z-w|}{2\sqrt{\Im(z)\Im(w)}}$;
- $e^{-2\pi i m\cdot x}$ replaced by $f \in \mathcal{B}$, basis of nonconstant Maaß forms (Laplacian eigenfunctions) on $\Gamma\backslash\mathbb{H}$, L^2-normalised w.r.t. $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure μ;
- $\|m\|_2^2$ replaced by Laplacian eigenvalue $\frac{1}{4}+t_f^2$;
- Sharp cutoff $1_{\|m\|_\infty\leq T}$ replaced by rapidly decaying weight $e^{-\frac{t_f^2}{T^2}}$.
- Similar inequality holds for cofinite but noncocompact lattices Γ:
  - Also a continuous contribution from Eisenstein series at each cusp;
  - Since $\Gamma\backslash\mathbb{H}$ is noncompact, need to assume a natural non-escape of mass condition on ν_1 and ν_2 for $\mathscr{W}_1(\nu_1,\nu_2)$ to be finite.

Sketch of proof 1/4.

Want to bound

$$\sup_{F \in \mathrm{Lip}_1(\Gamma\backslash\mathbb{H})} \left| \int_{\Gamma\backslash\mathbb{H}} F(z)\, d\nu_1(z) - \int_{\Gamma\backslash\mathbb{H}} F(z)\, d\nu_2(z) \right|.$$

By a smoothing argument, suffices to assume F is smooth.

Key trick: convolve with a point-pair invariant kernel

$$K(z,w) := \sum_{\gamma\in\Gamma} k(u(z,w)), \quad u(z,w) = \sinh^2 \frac{\rho(z,w)}{2} = \frac{|z-w|^2}{4\Im(z)\Im(w)}.$$

Choose $k : (0,\infty) \to [0,\infty)$ dependent on T such that

- $4\pi \int_0^\infty k(u)\, du = 1$;
- $\int_0^\infty k(u) \operatorname{arsinh} \sqrt{u}\, du \ll \frac{1}{T}$;
- $h(t) := 4\pi \int_0^\infty k(u) P_{-\frac{1}{2}+it}(1+2u)\, du = e^{-\frac{t^2+\frac{1}{4}}{2T^2}}$.

Sketch of proof 2/4.

Assumptions on k ensure that $\int_{\Gamma\backslash\mathbb{H}} K(z,w)\,d\mu(w) = 1$ independently of z. By the triangle inequality,

$$\left|\int_{\Gamma\backslash\mathbb{H}} F(z)\,d\nu_1(z) - \int_{\Gamma\backslash\mathbb{H}} F(z)\,d\nu_2(z)\right|$$

$$\leq \sum_{j=1}^{2} \left|\int_{\Gamma\backslash\mathbb{H}} \int_{\Gamma\backslash\mathbb{H}} (F(z) - F(w))K(z,w)\,d\mu(w)\,d\nu_j(z)\right|$$

$$+ \left|\int_{\Gamma\backslash\mathbb{H}} \int_{\Gamma\backslash\mathbb{H}} F(w)K(z,w)\,d\mu(w)\,d\nu_1(z)\right.$$

$$\left. - \int_{\Gamma\backslash\mathbb{H}} \int_{\Gamma\backslash\mathbb{H}} F(w)K(z,w)\,d\mu(w)\,d\nu_2(z)\right|.$$

By Lipschitz assumption together with unfolding, first two terms are at most $16\pi \int_0^\infty k(u) \operatorname{arsinh} \sqrt{u}\,du$ independently of F. By our construction of k, this is $\ll \frac{1}{T}$.

Sketch of proof 3/4.

By Parseval, Cauchy–Schwarz, assumptions on k, and properties of the Selberg–Harish-Chandra transform,

$$\left|\int_{\Gamma\backslash\mathbb{H}}\int_{\Gamma\backslash\mathbb{H}} F(w)K(z,w)\,d\mu(w)\,d\nu_1(z) - \int_{\Gamma\backslash\mathbb{H}}\int_{\Gamma\backslash\mathbb{H}} F(w)K(z,w)\,d\mu(w)\,d\nu_2(z)\right|^2 \leq \left(\sum_{f\in\mathcal{B}}\left(\frac{1}{4}+t_f^2\right)|\langle F,f\rangle|^2\right) \times \left(\sum_{f\in\mathcal{B}}\frac{e^{-\frac{t_f^2}{T^2}}}{\frac{1}{4}+t_f^2}\left|\int_{\Gamma\backslash\mathbb{H}} f(z)\,d\nu_1(z) - \int_{\Gamma\backslash\mathbb{H}} f(z)\,d\nu_2(z)\right|^2\right).$$

Sketch of proof 4/4.

By Parseval in reverse, Green's identity, and Lipschitz assumption,

$$\begin{aligned}\sum_{f \in \mathcal{B}} \left(\frac{1}{4} + t_f^2\right) |\langle F, f\rangle|^2 &= \langle \Delta F, F\rangle \\ &= 4 \int_{\Gamma\backslash\mathbb{H}} \Im(z)^2 \left|\frac{\partial F}{\partial z}\right|^2 \, d\mu(z) \\ &\leq 1\end{aligned}$$

independently of $F \in \mathrm{Lip}_1(\Gamma\backslash\mathbb{H})$. □

Binary Quadratic Forms

Definition

An integral binary quadratic form Q is a homogeneous polynomial

$$Q(x, y) = ax^2 + bxy + cy^2$$

for which $a, b, c \in \mathbb{Z}$.

For brevity, we write $Q = [a, b, c]$.

- The discriminant of Q is $b^2 - 4ac$.
- Q is primitive if $(a, b, c) = 1$.
- Q is positive definite if $D < 0$ and $a, c > 0$.

Let D be a fundamental discriminant.
We let $\mathcal{Q}_D$ denote the set of primitive integral binary quadratic forms of discriminant D that are positive definite if $D < 0$.

Binary Quadratic Forms and Narrow Ideal Classes

The group $\Gamma := \mathrm{SL}_2(Z) \ni \gamma$ acts on $\mathcal{Q}_D$ via

$$(\gamma \cdot Q)(x, y) := Q\left(\gamma \begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Proposition

The set $\Gamma \backslash \mathcal{Q}_D$ is isomorphic to the narrow class group Cl_D^+ of the quadratic field $\mathbb{Q}(\sqrt{D})$.

$$Q = [a, b, c] \mapsto \begin{cases} \dfrac{-b + \sqrt{D}}{2a}\mathbb{Z} + \mathbb{Z} & \text{if } a > 0, \\ \dfrac{b + \sqrt{D}}{-2a}\mathbb{Z} + \mathbb{Z} & \text{if } a < 0, \end{cases}$$

$$\mathfrak{a} = w\mathbb{Z} + \mathbb{Z} \mapsto \frac{\mathrm{N}(x - wy)}{\mathrm{N}(\mathfrak{a})}, \qquad w \in \mathbb{Q}(\sqrt{D}), \ w > \sigma(w).$$

Heegner Points

Let $D < 0$. For each $Q = [a, b, c] \in \mathcal{Q}_D$, define the point

$$z_Q := \frac{-b + i\sqrt{-D}}{2a} \in \mathbb{H}.$$

The orbit $\{\Gamma z_Q\}$ is a countable collection of points in $\mathbb{H}$ associated to the equivalence class $\Gamma Q \in \Gamma\backslash\mathcal{Q}_D$, or equivalently a single point on the modular surface $\Gamma\backslash\mathbb{H}$.

We call such a point a *Heegner point* or *CM point*. We let $z_A \in \Gamma\backslash\mathbb{H}$ denote such a point associated to an ideal class $A \in \mathrm{Cl}_D$, or equivalently an element ΓQ of $\Gamma\backslash\mathcal{Q}_D$.

For each $D < 0$, there are h_D such points, where $h_D := \#\mathrm{Cl}_D$ is the class number of $\mathbb{Q}(\sqrt{D})$. By the class number formula, the number of Heegner points is $\approx \sqrt{-D}$.

Heegner Points

Definition

For $D < 0$, we define a probability measure μ_D on $\Gamma\backslash\mathbb{H}$ by

$$\mu_D(B) := \frac{\#\{A \in \mathrm{Cl}_D : z_A \in B\}}{h_D} \quad \text{for } B \subset \Gamma\backslash\mathbb{H},$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) := \frac{1}{h_D} \sum_{A \in \mathrm{Cl}_D} f(z_A) \quad \text{for } f : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

By the class number formula,

$$h_D = \frac{\omega_D}{2\pi} \sqrt{-D} L(1, \chi_D).$$

Let $D > 0$. For each $Q = [a, b, c] \in \mathcal{Q}_D$, define the geodesic

$$\mathcal{C}_Q := \left\{z \in \mathbb{H} : a|z|^2 + b\Re(z) + c = 0\right\} \subset \mathbb{H}.$$

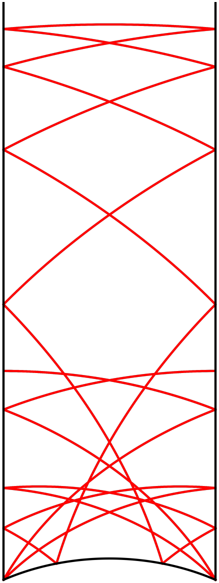
The orbit $\{\Gamma\mathcal{C}_Q\}$ is a countable collection of geodesics in $\mathbb{H}$ associated to the equivalence class $\Gamma Q \in \Gamma\backslash\mathcal{Q}_D$, or equivalently a single closed geodesic on the modular surface $\Gamma\backslash\mathbb{H}$.

We let $\mathcal{C}_A \in \Gamma\backslash\mathbb{H}$ denote such a closed geodesic associated to an ideal class $A \in \mathrm{Cl}_D^+$, or equivalently an element ΓQ of $\Gamma\backslash\mathcal{Q}_D$.

For each $D > 0$, there are h_D^+ such closed geodesics, where $h_D^+ := \#\mathrm{Cl}_D^+$ is the narrow class number of $\mathbb{Q}(\sqrt{D})$. Each has length $2\log\epsilon_D$, where ϵ_D is the least totally positive unit in $\mathbb{Q}(\sqrt{D})$. By the class number formula, the sum of lengths of closed geodesics is $\approx \sqrt{D}$.

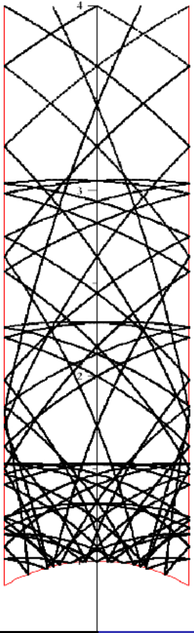
Example: $D = 19$

(Image: Constantin Kogler)



Example: $D = 377$

(Image: Einsiedler–Lindenstrauss–Michel–Venkatesh)



Definition

For $D > 0$, we define a probability measure μ_D on $\Gamma\backslash\mathbb{H}$ by

$$\mu_D(B) := \frac{\sum_{A \in \mathrm{Cl}_D^+} \ell(\mathcal{C}_A \cap B)}{2h_D^+ \log \epsilon_D} \quad \text{for } B \subset \Gamma\backslash\mathbb{H},$$

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) := \frac{1}{2h_D^+ \log \epsilon_D} \sum_{A \in \mathrm{Cl}_D^+} \int_{\mathcal{C}_A} f(z)\, ds \quad \text{for } f : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

Here $\ell(\mathcal{C}) := \int_{\mathcal{C}} ds$ with $ds^2 = y^{-2}\, dx^2 + y^{-2}\, dy^2$ the length element on $\mathbb{H}$.

By the class number formula,

$$\sum_{A \in \mathrm{Cl}_D^+} \ell(\mathcal{C}_A) = 2h_D^+ \log \epsilon_D = 2\sqrt{D} L(1, \chi_D).$$

Equidistribution of Heegner Points and Closed Geodesics

Theorem (Duke (1988))

(1) *As $D \to -\infty$ along negative fundamental discriminants, the probability measures μ_D equidistribute on $\Gamma\backslash\mathbb{H}$ with respect to the $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure $d\mu = \frac{3}{\pi}\frac{dx\,dy}{y^2}$.*

(2) *As $D \to \infty$ along positive fundamental discriminants, the probability measures μ_D equidistribute on $\Gamma\backslash\mathbb{H}$ with respect to the $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure $d\mu = \frac{3}{\pi}\frac{dx\,dy}{y^2}$.*

Theorem (H. (2025+))

(1) *As $D \to -\infty$ along negative fundamental discriminants, $\mathscr{W}_1(\mu_D, \mu) \ll_\varepsilon |D|^{-\frac{1}{12}+\varepsilon}$. Assuming GLH, we have the stronger bound $\mathscr{W}_1(\mu_D, \mu) \ll_\varepsilon |D|^{-\frac{1}{4}+\varepsilon}$.*

(2) *As $D \to \infty$ along positive fundamental discriminants, $\mathscr{W}_1(\mu_D, \mu) \ll_\varepsilon D^{-\frac{1}{12}+\varepsilon}$. Assuming GLH, we have the stronger bound $\mathscr{W}_1(\mu_D, \mu) \ll_\varepsilon D^{-\frac{1}{4}+\varepsilon}$.*

Sketch of proof 1/2.

We must bound

$$\frac{1}{T} + \left(\sum_{f \in \mathcal{B}} \frac{e^{-\frac{t_f^2}{T^2}}}{\frac{1}{4} + t_f^2} \left| \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) - \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu(z) \right|^2 \right)^{1/2}$$

and then choose $T \geq 1$ to minimise this.

Since f is a cusp form, $\int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu(z) = 0$.

By Waldspurger's theorem,

$$\left| \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu_D(z) \right|^2 \approx \frac{H_{\mathrm{sgn}(D)}(t_f)}{\sqrt{|D|} L(1, \chi_D)^2} \frac{L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_D\right)}{L(1, \mathrm{ad}\, f)},$$

where $H_+(t_f) \asymp \frac{1}{t_f}$ and $H_-(t_f) \asymp 1$.

Sketch of proof 2/2.

We roughly have that

$$\mathscr{W}_1(\mu_D, \mu) \ll_\varepsilon \frac{1}{T} + |D|^{-\frac{1}{4}+\varepsilon} \left(\sum_{t_f \ll T} \frac{H_{\mathrm{sgn}(D)}(t_f)}{t_f^2} \frac{L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_D\right)}{L(1, \mathrm{ad}\, f)} \right)^{1/2}.$$

To bound this moment of L-function unconditionally, we use Hölder's inequality and the cubic moment bound of Young (following Conrey–Iwaniec)

$$\sum_{t_f \sim T} \frac{L\left(\frac{1}{2}, f \otimes \chi_D\right)^3}{L(1, \mathrm{ad}\, f)} \ll_\varepsilon |D|^{1+\varepsilon} T^{2+\varepsilon}.$$

We get

$$\mathscr{W}_1(\mu_D, \mu) \ll_\varepsilon \frac{1}{T} + |D|^{-\frac{1}{12}+\varepsilon} T^\varepsilon; \qquad \text{take } T = |D|^{\frac{1}{12}}.$$

□

Quantitative Equidistribution for QUE

A similar method gives bounds for mass equidistribution of Maaß cusp forms assuming GLH.

Theorem (H. (2025+))

Let $g \in \mathcal{B}$ be an L^2-normalised Hecke–Maaß cusp form on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, and define the probability measure $d\mu_g(z) := |g(z)|^2 \, d\mu(z)$. Assuming GLH,

$$\mathscr{W}_1(\mu_g, \mu) \ll_\varepsilon t_g^{-\frac{1}{2}+\varepsilon}.$$

Remaining Questions

Question

Are these conditional bounds for the 1-Wasserstein distance essentially sharp?

(The answer is surely yes.)

Question

Can one prove unconditional bounds for the 1-Wasserstein distance concerning the mass equidistribution of holomorphic Hecke cusp forms in the weight or level aspects?

(Holowinsky's method is not directly applicable since it deals with incomplete Eisenstein series rather than real-analytic Eisenstein series.)

Thank you!