Galois Representations and Deformations – Exercise Sheet 1

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Let Γ be a profinite group, F a topological field, and $\rho : \Gamma \to \mathrm{GL}_n(F)$ a continuous representation.

1. Exercise (a) Assume that $F = \overline{\mathbb{F}_p}$ equipped with discrete topology. Show that $\rho(\Gamma)$ is finite and deduce that ρ factors through $\mathrm{GL}_n(\mathbb{F}_q)$ for some finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$.

(b) Assume that $F = \overline{\mathbb{Q}_p}$ with its usual topology. Use Baire category theorem (note that $\mathrm{GL}_n(F)$ is locally compact and Hausdorff) to show that ρ factors through $\mathrm{GL_n(E)}$ for some finite extension E of $\mathbb{Q}_p$.
Hint: The set of all subfields E of $\overline{\mathbb{Q}}_p$ that are finite over $\mathbb{Q}_p$ is countable. (Why?) Hence $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is the countable union of the corresponding $\mathrm{GL}_n(E)$.
Also, there are notes by Keith Conrad on *Compact subgroups ...*.

(c) Let E be as in part (b) and O_E its ring of integers. Show that there exists a rank n, O_E-lattice Λ in E^n that is stable under the action of Γ induced by ρ and deduce that in a suitable basis (after a conjugation) we can assume $\rho : \Gamma \to \mathrm{GL}_n(O_E)$.
Hint: Let $L \subset E^n$ be any O_E-lattice of rank n. Show that there exists a normal open subgroup $H \subset \Gamma$ that stabilizes L. Intersect suitable translates of L to find Λ.

(d) Assume that $F = \mathbb{C}$ equipped with usual complex topology. Show that $\rho(\Gamma)$ is finite and deduce that (up to conjugation) ρ factors through $\mathrm{GL}_n(L)$ for some number field L.
Hint: Let $\|\cdot\|$ be the spectral norm on matrices, i.e., $\|\alpha\|$ is the square root of the largest absolute value of an eigenvalue of $\alpha^t\alpha$, and consider the open set $U := \{\alpha \in \mathrm{GL}_n(\mathbb{C}) \mid \|U - 1\| < 1/2\}$. Show that U contains no non-trivial subgroup of $\mathrm{GL}_n(\mathbb{C})$ by analyzing $\{\alpha^n \mid n \in \mathbb{Z}\}$ for any $\alpha \in U$.

(e) Let L be as in part (d) and O_L its ring of integers. Show that there exists a rank n, O_L-lattice Λ in L^n that is stable under the action of Γ induced by ρ. Show further that there is a number field $L' \supset L$ such that after a suitable base change ρ takes values in $\mathrm{GL}_n(O_{L'})$.

For a matrix $\alpha \in M_n(A)$ over a ring A, we denote by $\mathrm{cp}_\alpha \in A[X]$ its (monic degree n) characteristic polynomial. For a representation ρ as above, we denote by cp_ρ the map $\Gamma \to F[X], g \mapsto \mathrm{cp}_{\rho(g)}$. For a finite length module M over a (not necessarily commutative) ring R its semisimplification is a finite length semisimple module M^{ss} that has the same Jordan-Hölder factors, counted with multiplicity as M.

2. Exercise Let $\rho : \Gamma \to \mathrm{GL}_n(F)$ be as above.

(a) Use the Jordan-Holder theorem for modules over rings to show that every finite dimensional representation of a group has a unique semi-simplification (up to isomorphism).

(b) Show that a finite dimensional representation and its semi-simplification have the same characteristic polynomial. In particular, $\rho^{\mathrm{ss}} : \Gamma \to \mathrm{GL}_n(F)$ is a semisimple representation such that $\mathrm{cp}_\rho = \mathrm{cp}_{\rho^{\mathrm{ss}}}$ – and the unique such by the Brauer-Nesbitt Theorem (see below).

(c) Suppose $F \supset \mathbb{Q}_p$ is finite with ring of integers $\mathcal{O}_F$, uniformizer π and residue field k_F. Let $\Lambda \subset F^n$ be a Γ-stable sublattice for the action via ρ, and denote by $\rho_\Lambda : \Gamma \to \mathrm{GL}_n(\mathcal{O}_F)$ the representation with respect to an $\mathcal{O}_F$-basis of Λ. Clearly the reduction $\mathcal{O}_F \to k_F$ induces a representation $\overline{\rho}_\Lambda : \Gamma \to \mathrm{GL}_n(k_F)$. Show that $\overline{\rho}_\Lambda^{\mathrm{ss}}$ is independent of Λ.
Hint: Show that $\mathrm{cp}_\rho = \mathrm{cp}_{\rho_\lambda}$ under $\mathcal{O}_F[X] \subset F[X]$, and that $\mathrm{cp}_{\overline{\rho}_\Lambda} = \mathrm{cp}_{\rho_\Lambda} \pmod{\pi}$.

(d) Let F be as in (b). Show that there exists a finite extension $F' \supset F$ with residue field $k_{F'}$, and a Γ-stable lattice $\Lambda' \subset (F')^n$ for the induced action by Γ such that (i) $\rho_{\Lambda'}$ takes values in $\mathrm{GL}_n(\mathcal{O}_{F'})$ and $\overline{\rho}_{\Lambda'} : \Gamma \to \mathrm{GL}_n(k_{F'})$ is semisimple.
Hint: Combine (a) and part (c) of Exercise 1 to show that $\rho_\Lambda \pmod{\pi}$ is block upper triangular with respect to a suitable $\mathcal{O}_F$-basis. One can now conjugate ρ_Λ by a suitable diagonal matrix over an extension of the form $F' = F(\sqrt[m]{\pi})$ to obtain the wanted Λ'.

(e) Show that a Λ' as in (d) need in general not exist over F.

Recall that the Brauer–Nesbitt theorem states that if $r, r' : G \to \mathrm{GL}_n(E)$ are two semi-simple (not necessarily continuous) representation over any field E, then ρ and ρ' are isomorphic (conjugate) if and only if they have the same characteristic polynomial. If $\mathrm{char}(E)$ is 0 or strictly greater than n, then it is enough for the two representation to have the same trace.

3. Exercise Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Let $r, r' : G \to \mathrm{GL}_n(E)$ and $r_\mathcal{O}, r_\mathcal{O} : G \to \mathrm{GL}_n(\mathcal{O})$ be representations of G, and $\overline{r_\mathcal{O}}$ be the reduction $r_\mathcal{O} \otimes_\mathcal{O} k$.

(a) Assume that r_E is irreducible and $\mathrm{Tr}(r_E) = \mathrm{Tr}(r'_E)$. Show that r'_E is irreducible and $r_E \simeq r'_E$.

(b) Assume that $\overline{r_\mathcal{O}} : G \to \mathrm{GL}_n(k)$ is irreducible and $\mathrm{Tr}(r_\mathcal{O}) = \mathrm{Tr}(r'_\mathcal{O})$. Show that $r_\mathcal{O} \simeq r'_\mathcal{O}$.
Hint: Show that any two $\mathcal{O}$-lattices Λ, Λ' in E^n of rank n that are stable under G are homothetic; first rescale Λ' such that $\Lambda \supset \Lambda' + \pi\Lambda \supsetneq \pi\Lambda$ for π a uniformizer of E.

(c) Give a counter-example to the Brauer-Nesbitt theorem without the semi-simplicity condition.

Let K be a number field and S a finite subset of the set Pl_K of places of K. For $v \in \mathrm{Pl}_K^{\mathrm{fin}} \backslash S$, let Frob_v be a Frobenius automorphism in $\mathrm{G}_{K,S}$ (unique up to conjugation in $G_{K,S}$). Let E be a p-adic field.

4. Exercise Let $\rho, \rho' : G_{K,S} \to \mathrm{GL}_n(E)$ be semisimple continuous for E a p-adic field. Show that the following are equivalent:

(i) $\rho \simeq \rho'$.

(ii) $\mathrm{cp}_{\rho(\mathrm{Frob}_v)} = \mathrm{cp}_{\rho'(\mathrm{Frob}_v)}$ for all $v \in \mathrm{Pl}_K^{\mathrm{fin}} \backslash S$

(iii) $\mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\rho'(g))$ for all $g \in G_{K,S}$.

(iv) $\mathrm{Tr}(\rho(\mathrm{Frob}_v)) = \mathrm{Tr}(\rho'(\mathrm{Frob}_v))$ for all $v \in \mathrm{Pl}_K^{\mathrm{fin}} \backslash S$.

Hints: (iv)$\Rightarrow$(iii) uses the Čebotarev Density Theorem.
(iii)$\Rightarrow$(i) may use Brauer-Nesibtt, or Bourbaki, Algébre , ch. 8, §12, n I, prop. 3.

5. Exercise Show that a 2-dimensional odd ℓ-adic representation of $\Gamma_\mathbb{Q}$ is irreducible if and only if it is absolutely irreducible.