Galois Representations and Deformations – Exercise Sheet 2

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Let Γ be a profinite group. For p a prime, by Γ_p we denote the maximal pro-p quotient of Γ.
Let also E be a p-adic field with ring of integers $\mathcal{O}$, uniformizer $\varpi_{\mathcal{O}}$ and residue field $\mathbb{F}$. Denote by $\mathrm{CNL}_{\mathcal{O}}$ the category of complete noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}$ and by $\mathrm{Ar}_{\mathcal{O}}$ the full subcategory on Artin rings.
Let $A \in \mathrm{CNL}_{\mathcal{O}}$ and $\mathfrak{m}_A$ its maximal ideal. Then the graded ring associated with A is defined as

$$\mathrm{gr}(A) = k \oplus (\mathfrak{m}_A/\mathfrak{m}_A^2) \oplus (\mathfrak{m}_A^2/\mathfrak{m}_A^3) \oplus \cdots$$

with the evident addition and multiplication. Let $\mathfrak{m}_A = (x_1, \cdots, x_n)$, then one easily sees that $[x_1], \cdots, [x_n] \in \mathfrak{m}_A/\mathfrak{m}_A^2$ generate $\mathrm{gr}(A)$ as a k-algebra. gr is clearly a functor, a morphism $\phi : A \to B$ in $\mathrm{CNL}_{\mathcal{O}}$ induces a morphism $\mathrm{gr}(\phi) : \mathrm{gr}(A) \to \mathrm{gr}(B)$. It is not hard to see that if $\mathrm{gr}(\phi)$ is either injective or surjective, so is ϕ (see Proposition 3 here). You can use all these easy facts without proof throughout this sheet.

6. Exercise For $A \in \mathrm{CNL}_{\mathcal{O}}$ let $t_A^* = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \varpi_{\mathcal{O}} A)$ be the (mod $\varpi_{\mathcal{O}}$) cotangent space.

(a) Show that a map in $\mathrm{CNL}_{\mathcal{O}}$ is surjective if and only if it induces a surjection on the cotangent spaces.
Hint. Assume $\phi : A \to B$ induces a surjection $t_A^* \to t_B^*$. First show that $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective and then use the associated graded rings.

(b) Let $\phi : A \to B$ be a morphism in $\mathrm{CNL}_{\mathcal{O}}$. Show that the induced map $\mathrm{Hom}(B, -) \to \mathrm{Hom}(A, -)$ is smooth if and only if B is a power series ring over A (as an A-algebra).
Hint. Choose a basis $x_1, ..., x_n \in B$ for $t_{B/A}^* = \mathfrak{m}_B/\mathfrak{m}_B^2 + B\mathfrak{m}_A$ and let $C = A[[X_1, ..., X_n]]$. Use the remark after proposition 9 from the lectures (with A playing the role of $\mathcal{O}$) to construct a map of local A-algebras:

$$B \to C/\mathfrak{m}_C^2 + C\mathfrak{m}_A$$

Then use the smoothness assumption to lift this to a map $B \to C/\mathfrak{m}_C^2$, then to $B \to C/\mathfrak{m}_C^3$, and so on.

(c) Show that the morphism $D_{\overline{\rho}}^{\square} \to D_{\overline{\rho}}$ which forgets the framing is smooth, for any representation $\overline{\rho}$.

7. Exercise Let $F : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ be a functor such that $F(\mathbb{F}) = \{*\}$ and the Mayer–Vietoris map $(*)_{T_F} : F(\mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon]) \to F(\mathbb{F}[\varepsilon]) \times F(\mathbb{F}[\varepsilon])$ is bijective. Show that $T_F = F(\mathbb{F}[\varepsilon])$ is an $\mathbb{F}$-vector space for the following actions:

(i) The map $\mathbb{F} \to \mathrm{End}_{\mathrm{Set}}(T_F), \alpha \mapsto F(m_\alpha)$ defining the scalar multiplication on T_F, is induced from

$$\mathbb{F} \to \mathrm{End}_{\mathrm{Ar}_{\mathcal{O}}}(\mathbb{F}[\varepsilon]), \alpha \mapsto m_\alpha \text{ with } m_\alpha(x + \varepsilon y) = x + \varepsilon \alpha y \quad \text{for } x, y \in \mathbb{F}.$$

(ii) The map $T_F \times T_F \to T_F$ is given by $F(+) \circ (*)_{T_F}^{-1}$ for

$$+ : \mathbb{F}[\varepsilon] \times_{\mathbb{F}} \mathbb{F}[\varepsilon] \to \mathbb{F}[\varepsilon], (x + \varepsilon y, x + \varepsilon y') \mapsto (x + \varepsilon(y + y')) \quad \text{for } x, y, y' \in \mathbb{F}.$$

8. Exercise Let $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation and fix a character $\mu : \Gamma \to \mathcal{O}^\times$ lifting $\det(\overline{\rho})$. Show that there are natural isomorphisms of k-vector spaces

(a) $Z^1(\Gamma, \mathrm{ad}^0(\overline{\rho})) \simeq D_{\overline{\rho}}^{\square,\mu}(\mathbb{F}[\epsilon])$.

(b) $H^1(\Gamma, \mathrm{ad}^0(\overline{\rho})) \simeq D^{\mu}_{\overline{\rho}}(\mathbb{F}[\epsilon])$.

For the following exercise, you may freely use the following **Theorem (Carayol)**.
Let $\rho : \Gamma \to \mathrm{GL}_n(A)$ be a continuous representation with Γ profinite and $(A, \mathfrak{m})$ a complete local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and such that the reduction $\overline{\rho} := \rho \otimes_A \mathbb{F}$ is absolutely irreducible. Let A^{Tr} be the closed $\mathcal{O}$-subalgebra generated by $\{\mathrm{Tr}(\rho(g)) : g \in \Gamma\}$. Then there exists a representation $\rho^{\mathrm{Tr}} : \Gamma \to \mathrm{GL}_n(A^{\mathrm{Tr}})$, unique up to isomorphism, such that $\rho \cong \rho^{\mathrm{Tr}} \otimes_{A^{\mathrm{Tr}}} A$.

9. Exercise Suppose that Γ satisfies the condition Φ_p and that $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ is an absolutely irreducible representation, and let $R^{\square}_{\overline{\rho}}$ in $\mathrm{CNL}_{\mathcal{O}}$ be an associated universal lifting ring. Show the following:

(a) One has $(R^{\square}_{\overline{\rho}})^{\mathrm{Tr}} = R^{\square}_{\overline{\rho}}$.

(b) Let A be a complete local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ and let $\rho : \Gamma \to \mathrm{GL}_n(A)$ be a lift of $\overline{\rho}$. Then A^{Tr} lies in $\mathrm{CNL}_{\mathcal{O}}$.

10. Exercise In this exercise we shall prove the existence of $R^{\square}_{\overline{\rho}}$ for $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\mathbb{F})$ when Γ satisfies the finiteness condition Φ_p. For an arbitrary group Γ (not necessarily profinite !), let $\mathrm{Rep}_\Gamma : O\text{–Alg} \to \mathrm{Set}$ be the functor which sends an O-algebra A to the set of representations $\rho : \Gamma \to \mathrm{GL}_n(A)$.
We proceed in the following steps:

(a) Let first assume that Γ be a discrete finitely generated free group on d generators. Show that Rep_Γ is representable by GL_n^d.

(b) Let now Γ be finite. Show that Rep_Γ is representable by a closed subscheme $\mathrm{Spec}\mathcal{R}$ of GL_n^d.
Hint. Represent Γ by finitely many generators, say d many, and finitely many relations. The relations can be interpreted as equations cutting out a subscheme of GL_n^d.

(c) Let again Γ be finite. Show that $\overline{\rho}$ induces an O-algebra homomorphism $\alpha_{\overline{\rho}} : \mathcal{R} \to \mathbb{F}$ for $\mathcal{R}$ from (b), and that prove that for $I = \ker \alpha_{\overline{\rho}}$ that I-adic completion of $\mathcal{R}$ represents $D^{\square}_{\overline{\rho}}$.

(d) Denote by $R^{\square}_{\Gamma,\overline{\rho}}$ the ring from (c). Let $\phi : \Gamma' \to \Gamma$ a morphism of finite groups. Interpreting ϕ as a map of deformation functors, show that ϕ induces a morphism $r_\phi : R^{\square}_{\Gamma',\overline{\rho}} \to R^{\square}_{\Gamma,\overline{\rho}}$.

(e) Let $H_0 = \ker \overline{\rho}$ and consider the restriction $\overline{\rho}_0 = \overline{\rho}|_{H_0}$. Show that $T_{R^{\square}_{\Gamma,\overline{\rho}}}$ is naturally a subspace of $T_{R^{\square}_{H_0,\overline{\rho}_0}}$, and that $T_{R^{\square}_{H_0,\overline{\rho}_0}} = \mathrm{Hom}_{\mathrm{Groups}}(H_0, \mathbb{F}_p) \otimes_{\mathbb{F}_p} M_n(\mathbb{F})$.

(f) Let now Γ be an arbitrary profinite group, let Γ_0 be its quotient $\Gamma/\ker\overline{\rho}$, and write Γ as a filtered inverse limit $\lim_{i\in I} \Gamma_i$ for finite groups Γ_i such that Γ_0 is a terminal objects. By (d) we have a corresponding inverse system of $R^{\square}_{\Gamma_i,\overline{\rho}}$, and we define $R^{\square}_{\Gamma,\overline{\rho}}$ as the complete $\mathcal{O}$-algebra $\lim_{i\in I} R^{\square}_{\Gamma_i,\overline{\rho}}$. Show that $R^{\square}_{\Gamma,\overline{\rho}}$ represents $\mathcal{D}_{\overline{\rho}}$.

(g) Suppose finally that Γ satisfies Φ_p. Show that the limit in (f) induces an isomorphism $T_{R^{\square}_{\Gamma,\overline{\rho}}} = \lim_{i\in I} T_{R^{\square}_{\Gamma_i,\overline{\rho}}}$ in which $\dim_{\mathbb{F}} T_{R^{\square}_{\Gamma_i,\overline{\rho}}}$ is bounded by $\mathrm{Hom}_{\mathrm{Groups}}(H_0, \mathbb{F}_p) \otimes_{\mathbb{F}_p} M_n(\mathbb{F})$, independently of i, and deduce that $\dim_{\mathbb{F}} T_{R^{\square}_{\Gamma,\overline{\rho}}}$ is finite.

(h) Suppose that Γ satisfies Φ_p. Show that $R^{\square}_{\Gamma,\overline{\rho}}$ lies in $\mathrm{CNL}_{\mathcal{O}}$ for the ring from (f).
Hint: Let $t = \dim_{\mathbb{F}} T_{R^{\square}_{\Gamma_i,\overline{\rho}}}$. Use (g) to show that for any fixed $m > 0$ the system of rings $(R^{\square}_{\Gamma_i,\overline{\rho}}/\mathfrak{m}_?^m)_i$ becomes stationary and is a quotient of $\mathcal{R}_t := \mathcal{O}[[X_1, \ldots, X_t]]$. The mod $\mathfrak{m}^m$ inverse limits form an inverse system converging to $R^{\square}_{\Gamma,\overline{\rho}}$, and this will prove (h).

11. Exercise Let $K/\mathbb{Q}$ be the splitting field of X^3-X+1. Then one easily sees that $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_3$. Since the discriminant of X^3-X+1 is equal to -23, the extension $K/\mathbb{Q}$ is unramified outside 23 and ∞. Recall representation theory of S_3 and consider the standard representation of S_3 over $\mathbb{F}_{23}$:

$$\mathrm{std} : S_3 \hookrightarrow \mathrm{GL}_2(\mathbb{F}_{23})$$

which is faithful and absolutely irreducible (note that $23 \nmid \sharp S_3$). One can easily see that $\mathrm{ad}(\mathrm{std}) \simeq \mathrm{std} \oplus 1 \oplus \mathrm{sign}$ and therefore $\mathrm{ad}(\mathrm{std})^\vee \simeq \mathrm{ad}(\mathrm{std})$. Also, notice that for $n > 0$ all higher cohomology groups $H^n(S_3, V)$ vanish for $V \in \mathrm{Rep}_{\mathbb{F}_{23}}(S_3)$, because they must be 6-torsion (or because $23 \nmid 6$ and hence all representations are semi-simple, all exact sequences split and therefore all higher Ext groups vanish).
Now this gives us an irreducible Galois representation:

$$\overline{\rho} : G_{\mathbb{Q},\{23,\infty\}} \to \mathrm{Gal}(K/\mathbb{Q}) \simeq S_3 \xrightarrow{\mathrm{std}} \mathrm{GL}_2(\mathbb{F}_{23})$$

(a) Show that $\overline{\rho}$ is odd.

(b) Compute the cohomological dimension expectation of the deformation ring of $\overline{\rho}$.

(c) Let $G := (G^{ab}_{K,\{23,\infty\}})_{23}$. One can use global class field theory to analyze this group and show that $G/([G,G]G^{23}) \simeq \mathrm{ad}(\overline{\rho})$ as a representation of S_3 (see Boston). Use this fact to show that $\overline{\rho}$ is unobstructed.
Hint. Let $S = \{23, \infty\}$. First use the global Euler characteristic formula to show that it is enough to prove $\dim H^1(G_{\mathbb{Q},S}, \mathrm{ad}(\overline{\rho})) = 3$. Then write the inflation restriction sequence to relate this to $H^1(G_{K,S}, \mathrm{ad}(\overline{\rho}))^{S_3}$. Then show the following sequence of isomorphisms:

$$H^1(G_{K,S}, \mathrm{ad}(\overline{\rho}))^{S_3} \simeq (H^1(G_{K,S}, \mathbb{F}_{23}) \otimes \mathrm{ad}(\overline{\rho}))^{S_3} \simeq (\mathrm{Hom}(G_{K,S}, \mathbb{F}_{23}) \otimes \mathrm{ad}(\overline{\rho}))^{S_3} \simeq \mathrm{Hom}_{S_3}(G^{ab}_{K,S}, \mathrm{ad}(\overline{\rho}))$$

and finish the proof.

(d) Deduce that $R_{\overline{\rho}} \simeq \mathbb{Z}_{23}[[T_1, T_2, T_3]]$.

12. Exercise Let K be a local or global field. A famous theorem of Tate states that if one considers the trivial action of Γ_K on $\mathbb{Q}/\mathbb{Z}$ then

$$H^2(\Gamma_K, \mathbb{Q}/\mathbb{Z}) = 0$$

You can use this without proof in this exercise.

(a) Show that this implies

$$H^2(\Gamma_K, \mathbb{C}^\times) = 0$$

Hint. Use the embedding $\iota : \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$ by the exponential map and show that $\mathrm{coker}\,\iota$ is a $\mathbb{Q}$-vector space.

Now let $\rho : \Gamma_K \to \mathrm{PGL}_n(\mathbb{C})$ be a (continuous) projective Galois representation. Recall that the image of ρ is finite since Γ_K is a profinite group. Choose a set theoretic section ϕ for the natural projection $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{PGL}_n(\mathbb{C})$.

(b) Show that for any $\sigma, \tau \in \Gamma_K$ one has

$$c(\sigma,\tau) := \phi(\rho(\sigma))\phi(\rho(\tau))\phi(\rho(\sigma\tau))^{-1} \in \mathbb{C}^\times,$$

and that $(\sigma,\tau) \mapsto c(\sigma,\tau)$ is a 2-cocycle.

(c) Show that one can lift ρ to a continuous honest representation

$$\tilde{\rho} : \Gamma_K \to \mathrm{GL}_n(\mathbb{C})$$

Remark: Using non-abelian cohomology, one can directly prove (c) from (a).

Let K be a number field, $S \subset \mathrm{Pl}_K$ a finite subset, and let $\overline{\rho} : G_{K,S} \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation. Let v be a place of K and set $\overline{\rho}_v = \overline{\rho}_{G_{K_v}}$.

13. Exercise A *local deformation collection* at v is defined to be a collection $\mathcal{C}_v$ of pairs (A, ρ_A) such that $(A, \mathfrak{m}_A)$ is in $\mathrm{CNL}_{\mathcal{O}}$, $\rho \in \mathcal{D}^{\square}_{\overline{\rho}_v}(A)$ and the following conditions hold:

1. $(\mathbb{F}, \overline{\rho}_v) \in \mathcal{C}_v$.
2. If $\phi : A \to A'$ is any morphism in $\mathrm{CNL}_{\mathcal{O}}$ and $(A, \rho_A) \in \mathcal{C}_v$, then $(A', \phi \circ \rho_A) \in \mathcal{C}_v$.
3. If $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ is a diagram in $\mathrm{Ar}_{\mathcal{O}}$ with α, β surjective and suppose that (A, ρ_A), (B, ρ_B) and (C, ρ_C) lie in $\mathcal{C}_v$ and satisfy $\rho_C = \alpha \circ \rho_A = \beta \circ \rho_B$. Then $(A \times_C B, (\rho_A \oplus \rho_B)|_{A \times_C B})$ lies in $\mathcal{C}_v$.
4. Suppose $(A_i, \rho_i)_{i \in I}$ is an inverse system in $\mathcal{C}_v$ with $A := \lim_I A_i \in \mathrm{CNL}_{\mathcal{O}}$. Then $(A, \lim_I \rho_i) \in \mathcal{C}_v$.
5. If (A, ρ) is in $\mathcal{C}_v$ and if ρ' is strictly equivalent to ρ, then $(A, \rho') \in \mathcal{C}_v$.
6. Suppose $\alpha : A' \to A$ is an injective map in $\mathrm{CNL}_{\mathcal{O}}$ and $\rho' \in \mathcal{D}^{\square}_{\overline{\rho}_v}(A')$. Then $(A, \alpha \circ \rho') \in \mathcal{C}_v \Rightarrow (A', \rho') \in \mathcal{C}_v$.

Given a local deformation collection $\mathcal{C}_v$ one obtains a functor $\mathcal{D}_{\mathcal{C}_v} : \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Set}$ by mapping $A \to \{\rho \in \mathcal{D}^{\square}_{\overline{\rho}_v} \mid (A, \rho) \in \mathcal{C}_v\}$ with maps on morphism using 2. The collection $\mathcal{C}_v$ also defines an ideal $I_{\mathcal{C}_v} \in R^{\square}_{\overline{\rho}_v}$ defined as

$$\bigcap_{(A,\rho) \in \mathcal{C}_v} \ker\left(R^{\square}_{\overline{\rho}_v} \xrightarrow{\alpha_\rho} A\right),$$

where α_ρ denotes the unique morphism induced from the universal property of $R^{\square}_{\overline{\rho}_v}$. Conversely, given a local deformation functor $\mathcal{D}_v$, one defines $\mathcal{C}_{\mathcal{D}_v}$ as the set of pairs (A, ρ) such that $\rho \in \mathcal{D}_v(A)$. Show that

(a) For a local deformation collection $\mathcal{C}_v$ at v, the functor $\mathcal{D}_{\mathcal{C}_v}$ is a local deformation functor represented by the quotient map $R^{\square}_{\overline{\rho}_v} \to R^{\square}_{\overline{\rho}_v} / I_{\mathcal{C}_v}$.

(b) For a local deformation functor $\mathcal{D}_v$, the collection $\mathcal{C}_{\mathcal{D}_v}$ is a local deformation collection at v.

(c) The assignments $\mathcal{C}_v \mapsto \mathcal{D}_{\mathcal{C}_v}$ and $\mathcal{D}_v \mapsto \mathcal{C}_{\mathcal{D}_v}$ are mutually inverse.

(d) For $v \in \mathrm{Pl}_K \backslash S$, the functor $\mathcal{D}^{\mathrm{unr}}_v$ mapping A to $\{\rho \in \mathcal{D}^{\square}_{\overline{\rho}_v}(A) \mid \rho \text{ is unramified}\}$ is a local deformation functor at v.