EXERCISES FOR THE 2025 AARHUS MINICOURSE

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The exercises marked $*$ in this document require more background to complete.

1. On the method of Godement and Jacquet

The primary reference for the material in these exercises is [GJ72].

Let F be a number field. The Schwartz space of $\mathfrak{gl}_n(\mathbb{A}_F)$ is the space

$$\mathcal{S}(\mathfrak{gl}_n(\mathbb{A}_F)) := \mathcal{S}(\mathfrak{gl}_n(F_\infty)) \otimes C_c^\infty(\mathfrak{gl}_n(\mathbb{A}_F^\infty)).$$

The group $\mathrm{GL}_n(\mathbb{A}_F) \times \mathrm{GL}_n(\mathbb{A}_F)$ acts on this space via

$$\mathcal{R}(g,h)f(X) = f(g^{-1}Xh)$$

Let $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ be a nontrivial character. Let

$$\mathcal{F}_\psi(f)(x) := \int_{\mathfrak{gl}_n(\mathbb{A}_F)} f(y)\psi(\mathrm{tr}(xy))dx$$

be the Fourier transform. Here the measure dx is the unique Haar measure on $\mathfrak{gl}_n(\mathbb{A}_F)$ such that $\mathcal{F}_{\psi\psi} \circ \mathcal{F}_{\overline{\psi}} = Id$.

The Poisson summation formula states that

$$\sum_{\gamma\in\mathfrak{gl}_n(F)} f(\gamma) = \sum_{\gamma\in\mathfrak{gl}_n(F)} \mathcal{F}_\psi(f)(\gamma)$$

where

(1) Prove that $\mathcal{F}_\psi \circ \mathcal{R}(g,h) = |\det gh^{-1}|^n \mathcal{R}(h,g) \circ f$.

Let $\langle\, ,\, \rangle$ denote the pairing on $L^2(A_{\mathrm{GL}_n}\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F))$.

Let π be a cuspidal automorphic representation of $A_{\mathrm{GL}_n}\backslash\mathrm{GL}_n(\mathbb{A}_F)$. For $(f,\varphi_1,\varphi_2) \in \mathcal{S}(\mathfrak{gl}_n(\mathbb{A}_F)) \times \pi \times \pi$ we can then form the zeta function

$$Z(f,\varphi_1,\varphi_2,s) := \int_{\mathrm{GL}_n(\mathbb{A}_F)} \langle f(g)\varphi_1(g),\varphi_2\rangle|\det g|^{s-(n-1)/2}dg.$$

(2) When $n=1$, prove that $Z(f,\varphi_1,\varphi_2,s)$ converges absolutely for $\mathrm{Re}(s)>1$.

(2*) Prove in general that $Z(f,\varphi_1,\varphi_2,s)$ converges for $\mathrm{Re}(s)$ sufficiently large (Hint: bounding φ by the trivial representation may help).

(3) Assume φ is fixed by $\mathrm{GL}_n(\widehat{\mathcal{O}}_F^S)$ and that $f = f_S \mathbb{1}_{\mathfrak{gl}_n(\widehat{\mathcal{O}}_F^S)}$. Prove that

$$Z(\varphi_1,\varphi_2,f,s) = \int_{\mathrm{GL}_n(F_S)} f_S(g_S)\langle\varphi_1(g_S)dg_S,\varphi_2\rangle \mathrm{tr}(\pi|\det|^{s-(n-1)/2})(\mathbb{1}_{\mathfrak{gl}_n(\widehat{\mathcal{O}}_F^S)}).$$

(4) When $n = 1$, prove that

$$\mathrm{tr}(\pi|\det|^{s-(n-1)/2})(\mathbb{1}_{\mathfrak{gl}_n(\widehat{\mathcal{O}}_F^S)}) = L(s,\pi^S) \tag{1.0.1}$$

(4*) Prove the same identity for all n.

For $a \in A_{\mathrm{GL}_n}$ let

$$Z_a(f,\varphi_1,\varphi_2,s) = \int_{\mathrm{GL}_n(\mathbb{A}_F)^1} \langle f(ag)\varphi_1(g),\varphi_2\rangle dg.$$

Thus

$$Z(f,\varphi_1,\varphi_2,s) = \int_{A_{\mathrm{GL}_n}} Z_a(f,\varphi_1,\varphi_2)|\det a|^{s-(n-1)/2}da.$$

(5) Prove that

$$\begin{aligned} Z_a(f,\varphi_1,\varphi_2) &= \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \sum_{\gamma\in \mathrm{GL}_n(F)} f(a\gamma g)\langle \varphi_1(g),\varphi_2\rangle dg \\ &= \int_{\Delta \mathrm{GL}_n(F)\backslash (\mathrm{GL}_n(\mathbb{A}_F)^1)^2} \sum_{\gamma\in \mathrm{GL}_n(F)} f(ah^{-1}\gamma g)\langle \varphi_1(g),\varphi_2(h)\rangle dg dh. \end{aligned}$$

(6) If $n = 1$ prove that

$$\int_{F^\times\backslash(\mathbb{A}_F^\times)^1} f(a0g)\langle\varphi_1(g),\varphi_2\rangle dg = 0 \tag{1.0.2}$$

unless π is trivial.

(6*) If $n > 1$ prove that

$$\int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)^1} \sum_{\gamma\in\mathfrak{gl}_n(F)-\mathrm{GL}_n(F)} f(a\gamma g)\langle\varphi_1(g),\varphi_2\rangle dg = 0.$$

(7) If $n > 1$, use Poisson summation to prove that

$$Z_a(f,\varphi_1,\varphi_2) = |a|^{-n} Z_{a^{-1}}(\mathcal{F}_\psi(f),\varphi_2,\varphi_1). \tag{1.0.3}$$

(7*) Formulate and prove an analogue of (1.0.3) when $n = 1$.

(8) Prove that $\int_{a:|\det a|\geq 1} Z_a(f,\varphi_1,\varphi_2)|a|^{s-(n-1)/2}da$ converges absolutely for all s.

(9) Using (1.0.3), prove that $Z(f,\varphi_1,\varphi_2,s)$ admits a meromorphic continuation to the plane and satisfies the functional equation

$$Z(f,\varphi_1,\varphi_2,s) = Z(\mathcal{F}_\psi(f),\varphi_2,\varphi_1,1-s). \tag{1.0.4}$$

The equality (1.0.4) together with (1.0.1) reduce the functional equation of the standard L-functions of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ to local considerations.

References

[GJ72] R. Godement and H. Jacquet. *Zeta functions of simple algebras.* Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972. 1

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