Introduction to Relative Trace Formulas
Problem set 1

Notation : K is a global field (number field or function field), $\mathbf{A}$ its ring of adeles decomposed as $\mathbf{A} = K_\infty \times \mathbf{A}_f$ (where $K_\infty = 1$ in the function field case), $|.| : \mathbf{A} \to \mathbf{R}_+$ the normalized absolute value, G a connected reductive group over K, $H_1, H_2 \subset G$ two closed algebraic subgroups and $\chi_i : [H_i] \to \mathbf{C}^\times$ automorphic characters.

We will also denote by A the standard split torus in PGL_2 :

$$A = \begin{pmatrix} \star & \\ & 1 \end{pmatrix} \subset \mathrm{PGL}_2 .$$

Exercise 1 *1. Show that if X is an quasiaffine[1] variety over K then $X(K) \subset X(\mathbf{A})$ is discrete. (If you wonder how to define properly the topology on $X(\mathbf{A})$, and even if you don't, you may check the wonderful notes of B. Conrad* `https://math.stanford.edu/~conrad/papers/adelictop.pdf` *).*

2. Deduce that if X is a quasiaffine G-variety, then for every $f \in C_c^\infty(X(\mathbf{A}))$ the theta series

$$\Theta_f^X(g) := \sum_{x \in X(K)} f(xg), \quad g \in G(\mathbf{A}),$$

converges.

3. Assume that G is semisimple, $X := H\backslash G$ is quasiaffine and K is a function field. Then, show that for every cusp form $\varphi \in A_{cusp}(G)$ the period integral

$$P_H(\varphi) = \int_{H(K)\backslash H(\mathbf{A})} \varphi(h)dh$$

converges. (We recall that in this situation cusp forms are compactly supported. The same statement holds over number fields but requires first showing that Θ_f^X is of moderate growth in a suitable sense, see `https://arxiv.org/abs/1602.06538` *Appendix A.)*

Exercise 2 *Let $f \in C_c^\infty(G(\mathbf{A}))$ and*

$$K_f(x,y) = \sum_{\gamma \in G(K)} f(x^{-1}\gamma y)$$

1. Recall that a quasiaffine variety is an open subset of an affine variety.

its automorphic kernel. Check that for every compact subset $C \subset [G]$*, the function*

$$(x, y) \in C \times [G] \mapsto K_f(x, y)$$

is compactly supported. Deduce that if $[H_1]$ *is compact, the expression defining the relative trace formula*

$$\mathrm{RTF}_{\mathfrak{X}}(f) = \int_{[H_1]\times[H_2]} K_f(h_1, h_2)\chi_1(h_1)\chi_2(h_2)dh_1dh_2$$

is absolutely convergent and show that it admits the geometric expansion

$$\mathrm{RTF}_{\mathfrak{X}}(f) = \sum_{\substack{\gamma \in H_1(K)\backslash G(K)/H_2(K) \\ \chi_1\otimes\chi_2|_{(H_1\times H_2)_\gamma}=1}} \mathrm{vol}([(H_1 \times H_2)_\gamma])\,\mathrm{Orb}(\gamma, f)$$

where

$$\mathrm{Orb}(\gamma, f) = \int_{(H_1\times H_2)_\gamma(\mathbf{A})\backslash(H_1\times H_2)(\mathbf{A})} f(h_1^{-1}\gamma h_2)\chi_1(h_1)\chi_2(h_2)dh_1dh_2$$

is the relative orbital integral.

Exercise 3 *Construct the indicated isomorphisms of GIT quotients and identify their semi-simple regular loci.*

(i) $\mathbb{A}^1//\mathbb{G}_m \simeq pt$ *for the action* $(t, x) \mapsto tx$ *;*

(ii) $\mathbb{A}^2//\mathbb{G}_m \simeq \mathbb{A}^1$ *for the action* $t \cdot (x, y) = (tx, t^{-1}y)$ *;*

(iii) $(A\backslash \mathrm{PGL}_2)//A \simeq \mathbb{A}^1$*.*

Exercise 4 *Let* $\chi_0, \chi_\infty : [\mathbb{G}_m] \to \mathbf{C}^\times$ *be ideles class characters. Define*

$$\mathcal{F}_{\chi_0,\chi_\infty}([\mathbb{G}_m])$$

to be the space of functions $f : [\mathbb{G}_m] \to \mathbf{C}$ *satisfying the following conditions :*

— *f is smooth i.e. it is right invariant by a compact-open subgroup of* $\mathbf{A}_f^\times$ *and for every* $t_f \in \mathbf{A}_f^\times$ *the function*

$$t_\infty \in K_\infty^\times \mapsto f(t_\infty t_f)$$

is C^∞ *in the usual sense.*

— *There exist constants* $c_0 = c_0(f)$ *and* $c_\infty = c_\infty(f) \in \mathbf{C}$ *such that for every invariant differential operator D on* $K_\infty^\times$ *and every* $N > 0$ *we have*

$$|D(f - c_\infty\chi_\infty)(t)| \ll_{D,N} |t|^{-N},\ \textit{for } |t| \geq 1,$$

$$|D(f - c_0\chi_0)(t)| \ll_{D,N} |t|^{N},\ \textit{for } |t| \leq 1.$$

The subspace of $f \in \mathcal{F}_{\chi_0,\chi_\infty}([\mathbb{G}_m])$ *with* $c_0 = c_\infty = 0$ *is the space* $\mathcal{S}([\mathbb{G}_m])$ *of rapidly deacaying (aka Schwartz) functions on* $[\mathbb{G}_m]$*, so that we have a short exact sequence of* $\mathbf{A}^\times$*-representations*

$$0 \to \mathcal{S}([\mathbb{G}_m]) \to \mathcal{F}_{\chi_0,\chi_\infty}([\mathbb{G}_m]) \to \mathbf{C}_{\chi_0} \oplus \mathbf{C}_{\chi_\infty} \to 0$$

whose last nonzero map if $f \mapsto (c_0(f), c_\infty(f))$*.*

(i) Show that if $\chi_0 \neq 1 \neq \chi_\infty$ then there exists a unique $\mathbf{A}^\times$-invariant linear form

$$\int_{[\mathbb{G}_m]}^{\mathrm{reg}} : \mathcal{F}_{\chi_0,\chi_\infty}([\mathbb{G}_m]) \to \mathbf{C}$$

whose restriction to $\mathcal{S}([\mathbb{G}_m])$ is the (convergent) integral

$$f \mapsto \int_{[\mathbb{G}_m]} f(t)dt.$$

(ii) Let $X_1 = \mathbb{A}^1$ equipped with the usual $\mathbb{G}_m$-action (by scaling). Show that for every $f \in \mathcal{S}(\mathbf{A})$ (the Schwartz-Bruhat space of $\mathbf{A}$), we have

$$\Theta_f^{X_1} \in \mathcal{F}_{|.|^{-1},1}([\mathbb{G}_m]).$$

Hint *: Use Poisson summation formula to control the behavior of $\Theta_f^{X_1}(t)$ when $|t| \to 0$.*

(iii) Let $X_2 = \mathbb{A}^2$ equipped with the $\mathbb{G}_m$-action given by $t \cdot (x,y) = (tx, t^{-1}y)$. Show that for every $f \in \mathcal{S}(\mathbb{A}^2)$, we have

$$\Theta_f^{X_2} \in \mathcal{F}_{|.|^{-1},|.|}([\mathbb{G}_m]).$$

(iv) Let $X_3 = A\backslash \mathrm{PGL}_2$ equipped with the $\mathbb{G}_m = A$-action by right translation. Show that for every $f \in C_c^\infty(A(\mathbb{A})\backslash \mathrm{PGL}_2(\mathbf{A}))$, we have

$$\Theta_f^{X_3} \in \mathcal{F}_{|.|^{-1},|.|}([\mathbb{G}_m]).$$

Hint *: Show first that there exists an open cover $X_3//\mathbb{G}_m = V_1 \cup V_2$ as well as an open subset $V \subset X_2//\mathbb{G}_m$ such that, setting $U_1 = \pi_{X_3}^{-1}(V_1)$, $U_2 = \pi_{X_3}^{-1}(V_2)$ and $U = \pi_{X_2}^{-1}(V)$, we have $\mathbb{G}_m$-equivariant isomorphisms $U_1 \simeq U_2 \simeq U$. Then, try to use a "partition of unity" kind of argument to reduce to point (iii).*

Exercise 5 *Set*

$$[\mathbb{G}_m]^1 = \{t \in [\mathbb{G}_m] \mid |t| = 1\}.$$

It is a compact subgroup of $[\mathbb{G}_m]$.

(i) Let χ_0, χ_∞ be characters of $[\mathbb{G}_m]$ whose restrictions to $[\mathbb{G}_m]^1$ are non trivial. Show that, for $f \in \mathcal{F}_{\chi_0,\chi_\infty}([\mathbb{G}_m])$, the function

$$a \in [\mathbb{G}_m] \mapsto \int_{[\mathbb{G}_m]^1} f(at)dt$$

belongs to $\mathcal{S}([\mathbb{G}_m])$, in particular is integrable, and that the regularized integral of f is given by

$$\int_{[\mathbb{G}_m]}^{\mathrm{reg}} f(a)da = \int_{[\mathbb{G}_m]} \int_{[\mathbb{G}_m]^1} f(at)dtda,$$

provided we normalize the Haar measure on $[\mathbb{G}_m]^1$ to have total mass one.

(ii) Let $X = \mathbb{A}^2$ with the $\mathbb{G}_m$-action $t \cdot (x, y) = (tx, t^{-1}y)$. Let $f \in \mathcal{S}(\mathbf{A})$ and $\eta : [\mathbb{G}_m] \to \mathbf{C}^\times$ a character with $\eta \neq |.|, |.|^{-1}$. Note that by (i) and (iii) of the previous exercise there is a well defined regularized integral

$$\int_{[\mathbb{G}_m]}^{\mathrm{reg}} \Theta_f^X(t)\eta(t)dt.$$

Show that it admits the following geometric expansion

$$\int_{[\mathbb{G}_m]}^{\mathrm{reg}} \Theta_f^X(t)\eta(t)dt = \sum_{a \in K} \mathrm{Orb}_a(f)$$

where

$$\mathrm{Orb}_a(f) = \int_{\mathbf{A}^\times} f(ta, t^{-1})\eta(t)dt$$

if $a \neq 0$, and

$$\mathrm{Orb}_0(f) = \left(\int_{\mathbf{A}^\times} f(t, 0)\eta(t)|t|^s dt + \int_{\mathbf{A}^\times} f(0, t)\eta(t)^{-1}|t|^{-s} dt \right)_{s=0}.$$

***Hint** : We recall the following basic result from Tate's thesis : for $\varphi \in \mathcal{S}(\mathbf{A})$ the integral*

$$\int_{\mathbf{A}^\times} \varphi(t)|t|^s dt$$

is convergent for $\Re(s) \gg 1$ and admits a meromorphic continuation which is regular at $s = 0$ if $\eta \neq 1, |.|$ and at most a simple pole at $s = 0$ when $\eta = 1$ with residue $\varphi(0)$.

Exercise 6 *Let H be a connected reductive group over a field k and Y an affine H-variety. We denote by $\pi : Y \to Y//H$ the canonical map.*

(i) Let $a \in (Y//H)_{rs}(k)$, $\mathcal{O} = \pi^{-1}(a)$. Recall that $\mathcal{O}(k^{sep})$ is a single $H(k^{sep})$-orbit. Assume that $\mathcal{O}(k) \neq \emptyset$, pick $y \in \mathcal{O}(k)$ and let H_y be its stabilizer. Show that the map

$$\mathcal{O}(k)/H(k) \to \mathrm{Ker}(H^1(k, H_y) \to H^1(k, H)),$$
$$[z] \mapsto \left(\sigma \in \Gamma_k \mapsto \sigma(h_z)h_z^{-1}\right),$$

where $h_z \in H(k^{sep})$ is any element such that $z = yh_z$, is well defined and a bijection.

(ii) Let $\alpha \in H^1(k, H)$, T_α the corresponding H-torsor over k (with H acting on the left), $H_\alpha = \mathrm{Aut}_H(T_\alpha)$ the corresponding Pure Inner Form of H (whose defining action on T_α we take to be a right action). Check that

$$Y_\alpha := Y \times^H T_\alpha,$$

where $Y \times^H T_\alpha$ (contracted product) stands for the quotient $(Y \times T_\alpha)//H$ by the diagonal action $(y, t) \cdot h := (yh, h^{-1}t)$, is an affine H_α-variety and that there is a natural identification of GIT quotients

$$Y_\alpha//H_\alpha \simeq Y//H.$$

(iii) Assume that $X = I \backslash H$ *is a homogeneous variety. Let* $\alpha \in H^1(k, H)$. *Show that* $Y_\alpha(k)$ *is nonempty if and only if* α *is in the image of the map* $H^1(k, I) \to H^1(k, H)$.

(iv) Assume that every $y \in Y_{rs}$ *has a trivial stabilizer. Show that the quotient maps* $Y_\alpha \to Y_\alpha//H_\alpha$ *induce a bijection*

$$\bigsqcup_{\alpha \in H^1(k,H)} Y_{\alpha,rs}(k)/H_\alpha(k) \simeq (Y//H)_{rs}(k).$$

Hint *: You can remark that for every* $a \in (Y//H)_{rs}(k)$, $\mathcal{O}_a = \pi^{-1}(a)$ *is a* H*-torsor.*