Introduction to Relative Trace Formulas
Problem set 2

Notation : k is a field of characteristic $\neq 2$, ℓ/k a separable quadratic extension, B a quaternion algebra over k containing a copy of ℓ, $G = \mathrm{PGL}_{2,k}$, $G^B = B^\times/k^\times$ (or rather the algebraic group whose k-points are given by this quotient) both containing a copy of the torus $T = \ell^\times/k^\times$ (same remark).

Exercise 1 *Let $\ell_B^- \subset B$ be the orthogonal complement of $\ell \subset B$ for the symmetric bilinear form associated to the reduced norm $N : B \to k$. Show that*

(i) The restriction of N to ℓ is the usual norm $N_{\ell/k} : \ell \to k$ and that $B = \ell \oplus \ell_B^-$. For $\delta \in B$, we will write $\delta = \delta^+ + \delta^-$ for the corresponding decomposition .

(ii) The map $\nu : G^B \to \mathbb{P}^1 \setminus \{1\}$, $\delta \mapsto -\frac{N(\delta^-)}{N(\delta^+)}$ induces an injection

$$T(k)\backslash G^B(k)/T(k) \hookrightarrow \mathbb{P}^1(k) \setminus \{1\}$$

with image $(-N(\ell_B^-) \setminus \{1\}) \sup\{\infty\}$.

(iii) For $\delta \in G^B(k)$, we have

$$(T \times T)_\delta = \begin{cases} 1 & \text{if } \nu(\delta) \neq 0, \infty \\ T^\Delta := \{(t,t), t \in T\} & \text{if } \nu(\delta) = 0 \\ T^{a\Delta} := \{(t,t^{-1}), t \in T\} & \text{if } \nu(\delta) = \infty. \end{cases}$$

(iv) There exists $c_B \in k^\times$ such that $N\mid_{\ell_B^-} \sim c_B N_{\ell/k}$ and we have a bijection

$$\{\text{quaternion alg } B/k \text{ with } \ell \subset B\}/iso \simeq k^\times/N_{\ell/k}(\ell^\times),$$

$$B \mapsto [c_B].$$

(v) Deduce that

$$\bigsqcup_{\text{quaternion alg } B/k \text{ with } \ell\subset B} \nu(G^B_{rs}(k)) = k^\times \setminus \{1\},$$

where $G^B_{rs} := \nu^{-1}(\mathbb{G}_m \setminus \{1\})$.

(vi) Reprove this using Exercise 6 (iv) from the first sheet.

Exercise 2 *Let* $\mu : G \to \mathbb{P}^1 \setminus \{1\}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto -\frac{bc}{ad}$. *Show that*

(i) For $x \in k^\times \setminus \{1\}$, $\mu_k^{-1}(x) = A(k)\gamma_x A(k)$ *with* $\gamma_x = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$.

(ii) $\mu_k^{-1}(0) = A(k) \sqcup A(k)\gamma_0^+ A(k) \sqcup A(k)\gamma_0^- A(k)$ *where* $\gamma_0^+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\gamma_0^- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(iii) $\mu_k^{-1}(\infty) = A(k)w \sqcup A(k)\gamma_\infty^+ A(k) \sqcup A(k)\gamma_\infty^- A(k)$ *where* $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $\gamma_\infty^\pm = w\gamma_0^\pm$.

Exercise 3 *Assume that* $k = K$ *is a number field and let* $\eta : \mathbf{A}^\times / K^\times \to \{\pm 1\}$ *be the quadratic character associated to the quadratic extension* $L := \ell$. *We fix representatives of the cosets* $A(K)\backslash G(K)/A(K)$ *as in the previous exercise.*

1. *Check that for* $x \in K^\times \setminus \{1\}$, *the coset* $A(\mathbf{A})\gamma_x A(\mathbf{A})$ *is closed in* $G(\mathbf{A})$ *and deduce that the relative orbital integral*

$$\mathrm{Orb}_{\gamma_x}(f) = \int_{A(\mathbf{A})\times A(\mathbf{A})} f(a_1\gamma_x a_2)\eta(a_2)da_1da_2, \quad f \in C_c^\infty(G(\mathbf{A})),$$

is convergent.

2. *Let* $f \in C_c^\infty(G(\mathbf{A}))$ *and, for* $s \in \mathbf{C}$, *set*

$$Z_{\gamma_0^+}(f,s) := \int_{A(\mathbf{A})\times A(\mathbf{A})} f(a_1\gamma_0^+ a_2)\eta(a_2)|a_2|^{-s}da_2.$$

Let $\varphi \in C_c^\infty(\mathbf{A})$ *be defined by*

$$\varphi(x) = \int_{A(\mathbf{A})} f(a\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})da$$

and let

$$Z^{\mathrm{Tate}}(\varphi,\eta,s) := \int_{\mathbf{A}^\times} \varphi(t)\eta(t)|t|^s dt$$

be the correpsonding Tate's Zeta integral. Show that for $\Re(s) > 1$ *both Zeta integrals converge and that we have* $Z_{\gamma_0^+}(f,s) = Z^{\mathrm{Tate}}(\varphi,\eta,s)$. *Deduce from that and Tate's thesis that* $Z_{\gamma_0^+}(f,s)$ *extends to an entire function on* $\mathbf{C}$. *We then defined the* regularized *orbital integral at* γ_0^+ *to be*

$$\mathrm{Orb}^{\mathrm{reg}}_{\gamma_0^+}(f) := \Big(Z_{\gamma_0^+}(f,s)\Big)_{s=0}.$$

Propose a similar definition of all the orbital integrals $\mathrm{Orb}^{\mathrm{reg}}_{\gamma_x^\pm}(f)$, $x \in \{0,\infty\}$.

3. *Recall from Exercise 4 of the previous sheet that the integral defining* $\mathrm{RTF}_{A\backslash G/A,\eta}$*. Show that this reguarization admits the following geometric expansion*

$$\mathrm{RTF}_{A\backslash G/A,\eta}(f)=\sum_{x\in\mathbb{P}^1(K)} O_x^\eta(f),\quad f\in C_c^\infty(G(\mathbf{A})),$$

where

$$O_x^\eta(f)=\begin{cases}\mathrm{Orb}_{\gamma_x}(f) & \text{if } x\notin\{0,\infty\},\\ \mathrm{Orb}^{\mathrm{reg}}_{\gamma_x^+}(f)+\mathrm{Orb}^{\mathrm{reg}}_{\gamma_x^-}(f) & \text{if } x\in\{0,\infty\}.\end{cases}$$

Moreover show that, for a given $f\in C_c^\infty(G(\mathbf{A}))$*, all except finitely many of the terms* $O_x^\eta(f)$ *are zero.*

Hint *: Use Exercise 5 from the previous problem set giving an expicit description of the regularization of* $\mathrm{RTF}_{A\backslash G/A,\eta}$*.*

Exercise 4 *Let* E/F *be a (separable) quadratic extension fo local fields,* $N_{E/F}:E^\times\to F^\times$ *be the norm mapping and* $\eta: F^\times/N_{E/F}(E^\times)\simeq\{\pm1\}$ *be the character associated to this extension by local class field theory.*

1. *Let* $X=F^2$ *equipped with the* $A=F^\times$*-action* $t\cdot(x,y)=(tx,t^{-1}y)$*. For* $f\in C_c^\infty(X)=C_c^\infty(F^2)$ *and* $x\in F^\times$ *we define the* ***local orbital integral***

$$O_x^\eta(f):=\int_A f(tx,t^{-1})\eta(t)d^\times t.$$

(Note that the integrand is compactly supported hence the integral converges.) Let

$$\mathrm{Orb}(X/A):=\{x\in F^\times\mapsto O_x^\eta(f)\mid f\in C_c^\infty(F^2)\}$$

be the space of orbital integral functions for X/A*. Show that*

$$\mathrm{Orb}(X/A)=C_c^\infty(F)\mid_{F^\times}+\eta C_c^\infty(F)\mid_{F^\times}$$

(where we identify the two spaces on the right with function spaces on $F^\times$ *by restriction.)*

2. *Let* $Y=E$ *equipped with the scaling action of* $T=\mathrm{Ker}\,N_{E/F}$*. For* $f\in C_c^\infty(Y)=C_c^\infty(E)$ *and* $x\in N_{E/F}(E^\times)$ *we define the local orbital integral*

$$O_x(f)=\int_T f(tz_x)dt$$

where $z_x\in E^\times$ *is any element with* $N_{E/F}(z_x)=x$*. Show that the space of orbital integral functions*

$$\mathrm{Orb}(Y/T):=\{x\in N_{E/F}(E^\times)\mapsto O_x(f)\mid f\in C_c^\infty(E)\}$$

is $C_c^\infty(F)\mid_{N_{E/F}(E^\times)}$*.*

3. *Deduce the following transfer result. Pick* $\epsilon \in F^\times \setminus N_{E/F}(E^\times)$. *We say that a function* $f \in C_c^\infty(X)$ *and a pair of functions* $(f_+, f_-) \in C_c^\infty(Y) \oplus C_c^\infty(Y)$ ***match*** *if for every* $x \in N_{E/F}(E^\times)$ *we have*

$$O_x^\eta(f) = O_x(f_+) \text{ and } O_{\epsilon x}^\eta(f) = O_x(f_-).$$

Show that for every f *we can find a pair of matching functions* (f_+, f_-) *and, conversely, that for every pair* (f_+, f_-) *we can find a function* f *matching it.*