Cusp forms of weight 1/2 and pairs of quadratic forms

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I will speak about a spectral summation formula for the product of $four$ Fourier coefficients of cusp forms of weight $1/2$.

But first I recall briefly my recent result on the Hyperbolic Circle Problem, and a step of its proof:

Let $\mathbb{H}$ be the upper half plane. For $z, w \in \mathbb{H}$ let $u(z,w) = \frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}$.
For $z \in \mathbb{H}$ and $X > 2$ define

$$N(z,X) := \left|\left\{\gamma \in PSL_2(\mathbf{Z}) : \ 4u(\gamma z, z) + 2 \leq X\right\}\right|.$$

A special case of an unpublished theorem of Selberg states that

$$N(z,X) = 3X + O_z\left(X^{\frac{2}{3}}\right) = 3X + O_z\left(X^{\frac{1}{2}+\frac{1}{6}}\right).$$

Let $\mathcal{F}$ be a fundamental domain of $SL_2(\mathbf{Z})$ in $\mathbb{H}$ and $d\mu_z = \frac{dxdy}{y^2}$.

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$$\left(\int_\Omega (N(z,X) - 3X)^2 \, d\mu_z\right)^{1/2} = O_{\Omega,\epsilon}\left(X^{\frac{9}{14}+\epsilon}\right) = O_{\Omega,\epsilon}\left(X^{\frac{1}{2}+\frac{1}{7}}\right).$$

An important part of the proof was the following observation: if $N_h(z, X)$ is the contribution of the hyperbolic $\gamma \in \Gamma$ to $N(z, X)$, then we can give an expression for the inner product

$$\int_{\mathcal{F}} N_h(z, X_1) N_h(z, X_2) \, d\mu_z$$

whose most essential part is a sum of type

$$\sum_{t_1 > 2} \sum_{t_2 > 2} \sum_{f^2 \neq \left(t_1^2 - 4\right)\left(t_2^2 - 4\right)} h\left(t_1^2 - 4, t_2^2 - 4, f\right) S_{t_1, t_2, f, X_1, X_2}$$

where t_1, t_2, f run over integers, $S_{t_1, t_2, f, X_1, X_2}$ is an analytic expression, and $h\left(t_1^2 - 4, t_2^2 - 4, f\right)$ has the following arithmetic meaning:

If $\delta_1, \delta_2, f \in \mathbf{Z}$, then $h(\delta_1, \delta_2, f)$ is the number of $SL_2(\mathbf{Z})$-equivalence classes of pairs (Q_1, Q_2) of quadratic forms with integer coefficients

$$Q_i(X, Y) = A_i X^2 + B_i XY + C_i Y^2$$

such that the discriminant of Q_i is δ_i, and for the codiscriminant of Q_1 and Q_2 we have

$$B_1 B_2 - 2A_1 C_2 - 2A_2 C_1 = f.$$

Explicit formulas for $h(\delta_1, \delta_2, f)$ were proved earlier under some conditions by $\mathrm{Hardy-Williams}$ and $\mathrm{Morales}$. E.g. Hardy and $\mathrm{Williams}$ proved that if $\delta_i < 0$ are fundamental discriminants, $(\delta_1 \delta_2, f) = 1$, $f^2 - \delta_1 \delta_2 \neq 0$ and $\delta_1 \delta_2 f$ is odd, then

$$h(\delta_1, \delta_2, f) = \sum_{e \mid (f^2 - \delta_1 \delta_2)/4} \left(\frac{d}{e} \right).$$

This inner product formula was the starting point of our new formula which expresses a spectral sum of the product of four Fourier coefficients of cusp forms of weight $1/2$ with generalizations of the class numbers $h(\delta_1, \delta_2, f)$:

We will take weighted sums over the above $SL_2(\mathbf{Z})$-equivalence classes of pairs of quadratic forms. We will call these sums *generalized class numbers.*

We now turn to this spectral summation formula.

The weight 0 Kuznetsov formula for $SL_2(\mathbf{Z})$:

Let $\{u_j(z) : \ j \geq 1\}$ be a complete orthonormal system of cusp forms of weight 0 for $SL_2(\mathbf{Z})$ with $\Delta_0 u_j = \left(-\frac{1}{4} - t_j^2\right) u_j$ having Fourier expansion

$$u_j(z) = \sum_{m \neq 0} a_j(m) W_{0,it_j}\left(4\pi \left|m\right| y\right) e\left(mx\right).$$

Similarly, let $\phi\left(m,s\right)$ be the mth Fourier coefficient of the Eisenstein series $E\left(z,s\right)$. Then for integers $mn \neq 0$ and for a nice test function h one has:

$$\sum_{j\geq 1} \overline{a_j(m)} a_j(n) h\left(t_j\right) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\phi\left(m, \frac{1}{2}+ir\right)} \phi\left(n, \frac{1}{2}+ir\right) h\left(r\right) dr =$$

$$= \delta_{mn} h_0 + \sum_{c\geq 1} \frac{S\left(m,n,c\right)}{c} h^{\pm}\left(\frac{\sqrt{|mn|}}{c}\right).$$

Here h_0 is a number and $h^{\pm}$ are functions depending on h. We take h^+ if $mn > 0$ and h^- if $mn < 0$.

Let H_n be the nth Hecke operator and $H_n u_j = \lambda_j(n) u_j$. For $n > 0$ we have

$$n^{1/2} a_j(n) = a_j(1) \lambda_j(n),$$

so for $m, n > 0$ we have

$$m^{1/2} n^{1/2} \overline{a_j(m)} a_j(n) = |a_j(1)|^2 \lambda_j(m) \lambda_j(n).$$

Hence the Kuznetsov formula gives an expression for a spectral average of

$$|a_j(1)|^2 \lambda_j(m) \lambda_j(n).$$

By the multiplicativity of the Hecke eigenvalues we get an expression for a spectral average of products

$$|a_j(1)|^2 \lambda_j(m_1) \lambda_j(m_2) \ldots . \lambda_j(m_k).$$

So we automatically get a formula for a sum involving the product of more factors.

Recall the idea of the proof of the Kuznetsov formula:

For $z \in \mathbb{H}$ define the Poincare series

$$P_m(z, F) := \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbf{Z})} F(\operatorname{Im} \gamma z)\, e(m \mathrm{Re} \gamma z),$$

where F is a function on the positive real axis and $m \neq 0$ is an integer. This is an automorphic function defined in an explicit way.

Then

$$\int_{\mathcal{F}} P_m(z, F)\, \overline{u_j(z)} d\mu_z = \overline{a_j(m)} h_{m,F}(t_j)$$

with a function $h_{m,F}$. By the spectral theorem we see that

$$\int_{\mathcal{F}} P_m(z, F_1)\, \overline{P_n(z, F_2)} d\mu_z$$

is a spectral average of the products $\overline{a_j(m)} a_j(n)$. So it remains to give an elementary expression for this inner product, and it is possible to express it as a sum of Kloosterman sums.

A function f on $\mathbb{H}$ is called an automorphic function of weight $1/2$ if

$$f(\gamma z) = \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{1/2} f(z)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, where $j_\gamma(z) := cz+d$ and $\nu(\gamma) := \left(\frac{c}{d}\right)\overline{\epsilon_d}$, where $\epsilon_d = 1 \text{ for } d \equiv 1(4)$ and $\epsilon_d = i, \text{ if } d \equiv -1(4)$.

Let V be the subspace of automorphic function of weight $1/2$ which are square integrable on a fundamental domain of $\Gamma_0(4)$ and cuspidal.

The hyperbolic Laplace operator of weight $1/2$ is

$$\Delta_{1/2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{i}{2} y \frac{\partial}{\partial x}.$$

A function $F \in V$ is a cusp form of weight $1/2$ if it is an eigenfunction of $\Delta_{1/2}$. If $\Delta_{1/2} F = \left(-\frac{1}{4} - t^2\right) F$, then one has the Fourier expansion

$$F(z) = \sum_{m \neq 0} \rho_F(m) W_{\frac{1}{4}\operatorname{sgn}(m), it} \left(4\pi \left|m\right| y\right) e\left(mx\right).$$

For $F \in V$ define the operators τ_2, σ and $L := \tau_2\sigma$ in this way:

$$\tau_2 F(z) := e\left(\frac{1}{8}\right)\left(\frac{z}{|z|}\right)^{-1/2} F\left(-\frac{1}{4z}\right),$$

$$\sigma F(z) := \frac{\sqrt{2}}{4} \sum_{\nu \bmod 4} F\left(\frac{z+\nu}{4}\right).$$

One can see that τ_2 and σ map V into V, hence L is also a $V \to V$ operator. It is known that L satisfies $(L-1)\left(L+\frac{1}{2}\right) = 0$.

Let V^+ be the subspace of V with L-eigenvalue 1. This space is called Kohnen's subspace. It is known that a cusp form F of weight $1/2$ for $\Gamma_0(4)$ belongs to V^+ if and only if $\rho_F(m) = 0$ for every integer $m \equiv 2, 3(4)$.

For automorphic forms F of weight $\frac{1}{2}$ for $\Gamma_0(4)$ and for an odd prime p one can define the Hecke operator T_{p^2}. The operators T_{p^2} and L form a commouting family of self-adjoint operators $V^+ \to V^+$, and each of these operators commute with $\Delta_{1/2}$.

Let F_j $(j = 1, 2, \ldots)$ be an orthonormal basis of V^+ consisting of common eigenfunctions of $\Delta_{\frac{1}{2}}$ and the Hecke operators T_{p^2}. Let

$$\Delta_{1/2} F_j = \left(-\frac{1}{4} - r_j^2\right) F_j$$

for $j \geq 1$. Denote the Fourier coefficients of F_j by $b_j(m)$, i.e.

$$b_j(m) = \rho_{F_j}(m).$$

It is possible to generalize the Kuznetsov formula for sums of the form

$$\sum_{j \geq 1} \overline{b_j(m)} b_j(n) h(r_j) + \text{Eisenstein part}.$$

The other side contains generalized Kloosterman sums for $4|c$:

$$\sum_{d \bmod c,\ (d,c)=1} \epsilon_d \left(\frac{c}{d}\right) e\left(\frac{md + n\overline{d}}{c}\right).$$

The Kuznetsov formula has been generalized to arbitrary weights by Proskurin. The proof uses the same idea sketched above, but uses Poincare series of nonzero weight.

It is a further step to restrict the spectral sum to Kohnen's subspace. This can be done computing the action of L on the Fourier expansions at various cusps, and combining the Proskurin formula for different cusps.

This was discussed in several papers by $\mathrm{Biró}$, $\mathrm{Ahlgren-Andersen}$, $\mathrm{Andersen-Duke}$ and $\mathrm{Blomer-Corbett}$.

However, since in weight $1/2$ we have Hecke operators only for squares, this does not give at once a formula for a sum involving the product of more similar factors.

We show now how to give a formula for a weighted spectral sum containing the product of four Fourier coefficients $b_j(m)$.

A summation formula of a different shape for the product of four half-integral weight coefficients, in the case when two of the factors are first coefficients, was proved by $\mathrm{Blomer-Corbett}$.

The idea is the following:

Instead of the Poincare series, we use another automorphic function defined in an explicit way. Namely:

Let n, t be integers, $n, t > 0$, and for $\delta := t^2 - 4n$ assume $\delta \neq 0$. Let

$$\Gamma_{n,t} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbf{Z}, \ ad - bc = n, \ a + d = t \right\}.$$

The group $SL_2(\mathbf{Z})$ acts on this set by conjugation.

If m is a function on $[0, \infty)$ and $z, w \in \mathbb{H}$, let

$$m(z, w) = m\left(\frac{|z - w|^2}{4\mathrm{Im}z\mathrm{Im}w} \right),$$

and let

$$M_{t,n,m}(z) := \sum_{\gamma \in \Gamma_{n,t}} m\left(z, \gamma z\right).$$

This is an automorphic function with respect to $SL_2(\mathbf{Z})$.

More generally: If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n,t}$, let

$$Q_\gamma(X,Y) = cX^2 + (d-a)XY - bY^2.$$

This is a one-to-one correspondence between $\Gamma_{n,t}$ and

$$\mathrm{Q}_\delta := \left\{AX^2 + BXY + CY^2 : \ A,B,C \in \mathbf{Z},\ B^2 - 4AC = \delta\right\}$$

with $\delta = t^2 - 4n$.
If $Q \in \mathrm{Q}_\delta$ with $Q(X,Y) = aX^2 + bXY + cY^2$, D is a fundamental discriminant with $D|\delta$ and $\delta/D \equiv 0, 1 \pmod 4$, define

$$\omega_D(Q) = \begin{cases} 0 & \text{if} \quad (a,b,c,D) > 1, \\ \left(\frac{D}{r}\right) & \text{if} \quad (a,b,c,D) = 1, \end{cases}$$

where r is any number represented by Q with $(r, D) = 1$.

If n, t are integers, $n, t > 0$, and for $\delta := t^2 - 4n$ we have $\delta \neq 0$, D is a fundamental discriminant with $D|\delta$ and $\delta/D \equiv 0, 1 \pmod 4$, m is a function on $[0, \infty)$ define

$$M_{t,n,D,m}(z) := \sum_{\gamma \in \Gamma_{n,t}} \omega_D(\gamma)\, m(z, \gamma z),$$

where $\omega_D(\gamma) := \omega_D(Q_\gamma)$. This is also an automorphic function with respect to $SL_2(\mathbf{Z})$.

Similarly to the proof of the Kuznetsov formula, we have to compute the following quantities:

1., The spectral coefficients:

$$\int_{\mathcal{F}} M_{t,n,D,m}(z) u_j(z)\, d\mu_z,$$

2., The inner products:

$$\int_{\mathcal{F}} M_{t_1,n_1,D_1,m_1} M_{t_2,n_2,D_2,m_2} d\mu_z.$$

We start with

$$\int_{\mathcal{F}} M_{t_1,n_1,D_1,m_1} M_{t_2,n_2,D_2,m_2} d\mu_z .$$

This is a summation over pairs

$$(\gamma_1, \gamma_2) \in \Gamma_{n_1,t_1} \times \Gamma_{n_2,t_2} =: H\left(n_1, t_1, n_2, t_2\right).$$

$SL_2(\mathbf{Z})$ acts on this set by conjugation, and we consider conjugacy classes of pairs, similarly as in the proof of the Selberg Trace Formula.

E.g. if $\delta_1 > 0$, $\delta_2 > 0$, and $H^*\left(n_1, t_1, n_2, t_2\right)$ is the set of such pairs $(\gamma_1, \gamma_2) \in \Gamma_{n_1,t_1} \times \Gamma_{n_2,t_2}$ for which the set of fixed points of γ_1 do not coincide with the set of fixed points of γ_2, then

$$\sum_{(\gamma_1,\gamma_2) \in H^*(n_1,t_1,n_2,t_2)} \int_{\mathcal{F}} m_1\left(z, \gamma_1 z\right) m_2\left(z, \gamma_2 z\right) d\mu_z$$

equals

$$\sum_{f \in \mathbf{Z}, f^2 \neq \delta_1 \delta_2} h_{D_1,D_2}\left(\delta_1, \delta_2, f\right) I\left(\delta_1, \delta_2, \frac{f}{\sqrt{\delta_1 \delta_2}}, m_1, m_2\right),$$

where $I\left(\delta_1, \delta_2, \frac{f}{\sqrt{\delta_1 \delta_2}}, m_1, m_2\right)$ is an analytic expression, and the arithmetic part is the following generalized class number:

$$h_{D_1,D_2}\left(\delta_1, \delta_2, f\right) := \sum_{(Q_1,Q_2)} \omega_{D_1}\left(Q_1\right) \omega_{D_2}\left(Q_2\right),$$

the summation is over the $SL_2(\mathbf{Z})$-equivalence classes of pairs (Q_1, Q_2) with $Q_i\left(X, Y\right) = A_i X^2 + B_i XY + C_i Y^2$ such that the discriminant of Q_i is δ_i, and for the codiscriminant of Q_1 and Q_2 we have

$$B_1 B_2 - 2A_1 C_2 - 2A_2 C_1 = f.$$

For $D_1 = D_2 = 1$ this is the class number $h\left(\delta_1, \delta_2, f\right)$.

We now turn to the spectral coefficients

$$\int_{\mathcal{F}} M_{t,n,D,m}(z) u_j\left(z\right) d\mu_z.$$

For this we have to introduce the Shimura correspondence.

If $F \in V^+$ satisfies $\Delta_{1/2} F = \left(-\frac{1}{4} - t^2\right) F$ and has the Fourier expansion

$$F(z) = \sum_{m \neq 0,\, m \equiv 0,1(4)} b_F(m) \mathrm{W}_{\frac{1}{4}\mathrm{sgn}(m), it} \left(4\pi \left|m\right| y\right) \mathrm{e}\left(mx\right),$$

then for every fundamental discriminant d let

$$Sh_d F\left(z\right) := \sum_{k \neq 0} a_{Sh_d F}\left(k\right) W_{0,2it}(4\pi \left|k\right| y) e(kx),$$

where

$$a_{Sh_d F}\left(k\right) := \sum_{PQ=k, P>0} \frac{\left|Q\right|^{\frac{1}{2}}}{P} \left(\frac{d}{P}\right) b_F\left(dQ^2\right).$$

There is a d such that $b_j\left(d\right) \neq 0$, and with such d we define

$$\mathrm{Shim} F_j\left(z\right) := \frac{1}{b_j\left(d\right)} Sh_d F_j\left(z\right).$$

It can be shown that we get the same function using any fundamental discriminant d such that $b_j(d) \neq 0$, and that $\mathrm{Shim} F_j$ is an even Hecke normalized Maass-Hecke cusp form of weight 0 for $SL_2(\mathbf{Z})$.

We know that

$$\Delta_0 \left(\mathrm{Shim} F_j\right) = \left(-\frac{1}{4} - 4r_j^2\right) \mathrm{Shim} F_j,$$

for any prime $p > 2$ we have that

$$H_p \left(\mathrm{Shim} F_j\right) = \mathrm{Shim}\left(T_{p^2} F_j\right),$$

and the map $j \to \mathrm{Shim} F_j$ gives a bijection between the positive integers and the even Hecke normalized Maass-Hecke cusp forms of weight 0 for $SL_2(\mathbf{Z})$.

This follows mainly from the work of $\mathrm{Baruch} - \mathrm{Mao}$.

Using the Shimura lift, we can express the spectral coefficients as products of two weight $1/2$ Fourier coefficients in the case $D > 0$.

Proposition. *Let* n, t *be positive integers such that* $t^2 - 4n = \delta \neq 0$, *let* $D > 0$ *be a fundamental discriminant with* $D|\delta$ *and* $\delta/D \equiv 0, 1 \pmod 4$. *Let* $u = \mathrm{Shim} F_j$ *for some* $j \geq 1$ *and let* $\Delta_0 u = \lambda u$, $\lambda < 0$. *Write* $(u, u) := \int_{\mathcal{F}} |u(z)|^2 \, d\mu_z$. *Then for a nice test functon* m *we have*

$$\frac{1}{(u,u)} \int_{\mathcal{F}} M_{t,n,D,m}(z) u(z) d\mu_z = \delta^{3/4} \overline{b_j\left(D\right)} b_j \left(\frac{\delta}{D}\right) f_{\delta,n,m}\left(\lambda\right),$$

where $f_{\delta,n,m}$ *can be explicitly given in terms of* δ, n *and* m.

This follows from formulas proved for the left-hand side in my earlier papers and from a Katok-Sarnak type formula.

The $\delta < 0$ case of this Proposition is enough for that case of our spectral summation formula where there are two positive and two negative coefficeients.

Note that this is not the hardest case but the only one what I can precisely state at the moment.

Notations:

1., If $\delta \neq 0$ with $\delta \equiv 0, 1 \pmod 4$, the Zagier L-series times $|\delta|^{s/2}$ is:

$$L^*(s,\delta) := |\delta|^{s/2} \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{1}{q^s} \left(\sum_{r \bmod 2q,\ r^2 \equiv \delta(4q)} 1 \right).$$

2., For $i = 1, 2$ let $\delta_i < 0$ be integers. Let $D_i > 0$ be fundamental discriminants for $i = 1, 2$ with $D_i | \delta_i$ and $\delta_i / D_i \equiv 0, 1$ (mod 4). Let

$$E_{\delta_1,\delta_2,D_1,D_2} := \sum_{(Q_1,Q_2)} \frac{\omega_{D_1}(Q_1)\,\omega_{D_2}(Q_2)}{M(Q_1)},$$

where the summation is over the $SL_2(\mathbf{Z})$-equivalence classes of pairs (Q_1, Q_2) of quadratic forms such that the discriminant of Q_i is δ_i, $Q_1 = \lambda Q_2$ with some $\lambda \in \mathbf{Q}$. Here $M(Q_1)$ is the number of automorphs of Q_1.

3., If f $\chi(z)$ is a function for $z \geq 0$, let $T_\chi(y)$ denote

$$\frac{1}{288\pi^2} \int_0^{\infty} \left| \frac{\Gamma\left(\frac{1}{4} + iz\right) \Gamma\left(\frac{3}{4} + iz\right)}{\Gamma(2iz)} \right|^2 F\left(\frac{1}{4} - iz, \frac{1}{4} + iz, 1, -y\right) \chi(z)\, dz.$$

THEOREM 2 (B, 2025). *For $i = 1, 2$ let $\delta_i < 0$ be integers. Let $D_i > 0$ be fundamental discriminants for $i = 1, 2$ with $D_i | \delta_i$ and $\delta_i / D_i \equiv 0, 1 \pmod 4$. If χ is a nice test function which is holomorphic on a large enough strip containing the real line, then the sum of*

$$\frac{1}{12\pi^3} \delta_{1,D_1} \delta_{1,D_2} L^* (1, \delta_1) L^* (1, \delta_2) \chi \left(\frac{i}{4} \right),$$

$$|\delta_1 \delta_2|^{3/4} \sum_{j=1}^{\infty} (\mathrm{Shim} F_j, \mathrm{Shim} F_j) b_j (D_1) \overline{b_j \left(\frac{\delta_1}{D_1} \right)} b_j \left(\frac{\delta_2}{D_2} \right) \overline{b_j (D_2)} \chi (r_j)$$

and

$$\frac{1}{2\pi i} \int_{\left(\frac{1}{2}\right)} \frac{L^* (\overline{s}, D_1) L^* \left(\overline{s}, \frac{\delta_1}{D_1} \right) L^* (s, D_2) L^* \left(s, \frac{\delta_2}{D_2} \right) \chi \left(i \frac{s - \frac{1}{2}}{2} \right)}{144 \zeta (2s) \zeta (2 - 2s)} ds$$

equals

$$E_{\delta_1, \delta_2, D_1, D_2} T_\chi (0) + \sum_{f \in \mathbf{Z}, f^2 > \delta_1 \delta_2)} h_{D_1, D_2} (\delta_1, \delta_2, f) T_\chi \left(\frac{f^2}{\delta_1 \delta_2)} - 1 \right).$$

In this theorem we have two positive and two negative coefficients, and this comes from

$$\int_{\mathcal{F}} M_{t_1,n_1,D_1,m_1} M_{t_2,n_2,D_2,m_2} d\mu_z$$

with $D_1, D_2 > 0$, $\delta_1, \delta_2 < 0$.
The cases when we have more positive than negative Fourier coefficients can be handled in the same way: we still take $D_1, D_2 > 0$, and:

when we have four positive Fourier coefficients, we take $\delta_1, \delta_2 > 0$, if we have three positive Fourier coefficients, we take $\delta_1 > 0$, $\delta_2 < 0$.

But if there are more negative than positive Fourier coefficients, we have to take $D_1 < 0$ or $D_2 < 0$.

However for $D < 0$ the terms $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathrm{adj}\,(\gamma) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ cancel out, so

$$M_{t,n,D,m}(z) = \sum_{\gamma \in \Gamma_{n,t}} \omega_D\,(\gamma)\, m\,(z, \gamma z) = 0.$$

So for $D < 0$ we modify $M_{t,n,D,m}$ in this way: for $z, w \in \mathbb{H}$ let

$$h(z,w) := (z - \overline{w})^2 \left| z - \overline{w} \right|^{-2}.$$

Let n, t be integers, $n, t > 0$, and let $\delta := t^2 - 4n > 0$. Let $D < 0$ be a fundamental discriminant with $D|\delta$ and $\delta/D \equiv 0, 1 \pmod 4$, and

$$N_{t,n,D,m}(z) := \sum_{\gamma \in \Gamma_{n,t}} \omega_D(\gamma)\, m(z, \gamma z)\, h(\gamma z, z) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^2.$$

THEOREM 3 (B, 2025). *Let $\delta > 0$, let $D < 0$ be a fundamental discriminant with $D|\delta$ and $\delta/D \equiv 0, 1 \pmod 4$. Let $n, t > 0$ be integers such that $t^2 - 4n = \delta$. Let $u = \mathrm{Shim} F_j$ for some $j \geq 1$ and let $\Delta_0 u = \lambda u$, $\lambda < 0$. Then for any nice test functon m we have*

$$\frac{1}{(u,u)} \int_{\mathcal{F}} N_{t,n,D,m}(z) u(z) d\mu_z = \delta^{3/4} \overline{b_j(D)} b_j\left(\frac{\delta}{D} \right) F_{\delta,n,m}(\lambda),$$

where $F_{\delta,n,m}$ can be explicitly given in terms of δ, n and m.

For the proof Theorem 3 the extension of the Katok-Sarnak formula for the case of two negative Fourier coefficients is important. This extension was proved relatively recently by Duke – Imamoglu – Tóth and Imamoglu – Lägeler – Tóth.