Dirac Cohomology, Twisted Index and Unitary Representations

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- Earlier and current results with Pavle Pandžić.
- Joint with Chao-Ping-Dong, and Daniel Wong for complex classical groups and spin groups.
- More recent joint with Daniel Wong on E_8

One main goal is to classify the unitary representations with nontrivial Dirac cohomology.
In this talk I want to touch on an application, realizing these representations as automorphic forms, discrete components ocurring in $L^2(\Gamma\backslash G)$ for $\Gamma \subset G$ an arithmetic subgroup.
Still very much in progress. See the two sets of references at the end.
First, some background. I am not a specialist, there are many accounts in the Physics literature. I mainly took information from Wikipedia.

Original Dirac Operator I

One of the simplest versions of the Dirac operator is

$$D = \sum \partial_i \epsilon_i$$

with the property that it is a *formal square root* of the Laplacian, i.e.

$$D^2 = \Delta = \sum (\epsilon_i \epsilon_j + \epsilon_j \epsilon_i) \partial_{ij} = 2 \sum \partial_i^2 .$$

This forces $\epsilon_i \epsilon_j + \epsilon_j \epsilon_i = 2\delta_{ij}$, which makes sense in the Clifford algebra. According to Wikipedia, the original version of the Dirac equation, which he found staring into the fireplace, is

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}\partial_t \right) \psi = \kappa \psi$$

A, B, C, D are 4×4 matrices, formed out of the 2×2 Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac operator from Physics I

Look for D such that $D^2 = -\sum \partial_i^2$. (Or $D^2 = \sum \pm\partial_i^2$.)

If $D = \sum e_i \partial_i$, get

$$e_i^2 = -1; \qquad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

The original Dirac equation was motivated by trying to give a relativistic version of the Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{c^2}\right)\psi = \frac{m^2 c^2}{h^2}\psi.$$

Dirac replaced $\nabla^2 - \frac{1}{c^2}$ by $\left(A\partial_x + B\partial_y + C\partial_x + \frac{i}{c}D\partial_t\right)^2$. The requirements are

$$\begin{aligned} A^2 = B^2 = C^2 = D^2 = 1, \\ AB + BA = 0 \ldots \end{aligned}$$

This leads to the Clifford algebra.

Dirac operator from Physics II

Similarly, the relativistic version of Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2m}\Delta\Psi \qquad h := 1$$

requires a square root of Δ. This also leads to the Dirac equation.

("Anonymous Quote")

In particle physics, the Dirac equation is a relativistic wave equation derived by British physicist Paul Dirac in 1928. In its free form, or including electromagnetic interactions, it describes all $spin - 1/2$ massive particles such as electrons and quarks for which parity is a symmetry. It is consistent with both the principles of quantum mechanics and the theory of special relativity, reference [D], and was the first theory to account fully for special relativity in the context of quantum mechanics. It was validated by

Dirac operator from Physics III

accounting for the fine details of the hydrogen spectrum in a completely rigorous way.
The equation also implied the existence of a new form of matter, antimatter, previously unsuspected and unobserved and which was experimentally confirmed several years later. It also provided a theoretical justification for the introduction of several component wave functions in Pauli's phenomenological theory of spin.

Many in the audience are far more expert.

Background I

If a Lie group G acts on a manifold X, then it also induces a representation on functions on X, via

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

Typically there is a G-invariant measure dx on X.
For example:

$$C^\infty(X) \text{ is a smooth representation of } G$$

$$L^2(X) \text{ is a unitary representation of } G$$

A representation of G is a complex topological vector space V, typically complete, with a continuous G-action by linear operators.
Harmonic analysis: *"decompose such representations into irreducible representations."*

Irreducible Representations: those with no closed invariant subspace.

Background II

Example: $G = \mathbb{T}$, the circle group. The irreducible modules are 1-dimensional, spanned by functions $f_n : e^{it} \mapsto e^{int}$ on $\mathbb{T}$, $n \in \mathbb{Z}$, and

$$L^2(\mathbb{T}) = \widehat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}\, f_n$$

(Fourier series).
Similarly, for $G = \mathbb{R}$, the irreducible unitary representations are 1-dimensional, spanned by the functions $f_n : t \mapsto e^{ixt}$ on $\mathbb{R}$, $x \in \mathbb{R}$, and

$$L^2(\mathbb{R}) = \int_{\chi\in\mathbb{R}}^{\oplus} \mathbb{C}\, f_\chi \, d\chi$$

(Fourier transformation).

Connection with differential equations

Let Δ be a $G-$invariant differential operator on X.
Then any eigenspace of Δ is $G-$invariant.

Conversely, (by some version of Schur's Lemma) Δ acts by scalars on irreducible G-subspaces.

So in the presence of such an operator, decomposing the representation is related to finding Δ-eigenspaces.

The representation of G gives extra structure to the eigenspace.

Real reductive groups

G: a real reductive Lie group (often assumed connected).

Main examples: closed (Lie) subgroups of $GL(n,\mathbb{C})$, stable under the Cartan involution $\Theta(g) = {}^t\bar{g}^{-1}$.

E.g., $SL(n,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$, $O(p,q)$.

$K = G^\Theta$: maximal compact subgroup

E.g., $SO(n) \subset SL(n,\mathbb{R})$; $U(p) \times U(q) \subset U(p,q)$;
$U(n) \subset Sp(2n,\mathbb{R})$, $O(p) \times O(q) \subset O(p,q)$
and the various real forms of the exceptional groups.
An important set of examples are the complex groups viewed as real groups.

Cohomology of Discrete Groups I

- G the real points of a linear algebraic reductive connected group.
- $\mathfrak{g}_0 := Lie(G)$, θ a Cartan involution, $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}}$, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$, K the maximal compact subgroup, $\mathfrak{k}_0 := Lie(K)$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.
- A $(\mathfrak{g}, K)$ module $(\pi, \mathcal{H})$ is called unitary, if $\mathcal{H}$ admits a $\mathfrak{g}-$invariant positive hermitian form.
- $\Gamma \subset G$ a discrete cocompact subgroup. The theory of automorphic forms deals with the decomposition $L^2(\Gamma\backslash G) = \bigoplus m_\pi \pi$. Let $X := \Gamma\backslash G/K$. Then

$$H^i(\Gamma) = H^i(X) = \bigoplus m_\pi H^i(\mathfrak{g}, K, \pi).$$

Cohomology of Discrete Groups II

The multiplicities m_π are very hard to compute. In order to get information about Γ, one approach is to study $H^i(\mathfrak{g}, K, \pi)$ for π unitary.

Problem: Classify all unitary representations with nontrivial $(\mathfrak{g}, K)-$cohomology.

For complex groups, this was solved by Enright, and then generalized to real groups by Vogan-Zuckerman.
In the real case the answer is that $\pi = \mathcal{R}^s_{\mathfrak{q}}(\mathbb{C}_\lambda)$, where

- $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a $\theta-$stable parabolic subalgebra, $s = \dim \mathfrak{u} \cap \mathfrak{p}$,
- $\mathcal{R}^i_{\mathfrak{q}}$ is cohomological induction introduced by Parthasarathy and Zuckerman.
- $\mathbb{C}_\lambda$ is a unitary character such that $\mathcal{R}^s_{\mathfrak{q}}(\mathbb{C}_\lambda)$ has infinitesimal character the same as the trivial representation.

$H^i(\mathfrak{g}, K, \pi) = \mathrm{Hom}_K[\wedge^i \mathfrak{p}, \pi]$ is computable explicitly for such modules.

The index of the Dirac operator I

We review a basic tool to construct discrete series, the index of the Dirac operator. Atiyah-Schmid, Schmid, and Parthasarthy used it to construct Discrete Series.
It is always easier to study representations of the Lie algebra, and then derive properties of the representations of the Lie group. For real reductive groups, these are the $(\mathfrak{g}, \mathbf{K})-$**modules.**

Following Harish-Chandra, one associates a $(\mathfrak{g}, K)-$module to each representation of the group. Let V be an admissible representation V of G, i.e., $\dim \mathrm{Hom}(V_\delta, V) < \infty$ for all irreducible K-representations V_δ.

Let V_K be the space of K-finite vectors in V. These vectors are smooth i.e. one can differentiate the group action to get an action of the Lie algebra. $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$, the complexification of the real Lie algebra acts automatically.

Definition

A $(\mathfrak{g}, K)-$module is a vector space V, with a Lie algebra action of $\mathfrak{g}$ and a locally finite action of K, which are compatible, i.e., induce the same action of $\mathfrak{k}_0 := Lie(K)$. (If K is disconnected, one requires also that the action $\mathfrak{g} \otimes V \to V$ is $K-$equivariant). Such a V can be decomposed under K as

$$V = \bigoplus_{\delta \in \hat{K}} m_\delta V_\delta.$$

V is called a Harish-Chandra module if it is finitely generated and all $m_\delta < \infty$.

Casimir element and Infinitesimal Character

The Casimir Element, $\mathrm{Cas}_{\mathfrak{g}}$, in the center of the enveloping algebra $U(\mathfrak{g})$, is defined as follows:
Fix a nondegenerate invariant symmetric bilinear form B on $\mathfrak{g}$.
Take dual bases b_i, d_i of $\mathfrak{g}$ with respect to B. Write

$$\mathrm{Cas}_{\mathfrak{g}} = \sum b_i d_i.$$

This is an element of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ which act as scalars on irreducible modules. This defines the infinitesimal character of a module M, $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$.

Harish-Chandra proved that $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)^W$, so infinitesimal characters correspond to $\mathfrak{h}^*/W$.

($\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$; in examples, the diagonal matrices. W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$; it is a finite reflection group.)

The Clifford algebra for G

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition.

($\mathfrak{k}$ and $\mathfrak{p}$ are the ± 1 eigenspaces of the Cartan involution;

$\mathfrak{k}$ is the complexified Lie algebra of the maximal compact subgroup K of G.)

Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B:

the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx + 2B(x, y) = 0.$$

The Dirac operator for G

Let b_i be any basis of $\mathfrak{p}$; let d_i be the dual basis with respect to B.
Dirac operator:

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

D is independent of b_i and K-invariant.

$(\mathfrak{g}, K)-$cohomology I

There is extensive work to realize representations in the cuspidal spectrrum of $L^2(\Gamma\backslash G)$ using $(\mathfrak{g}, K)-$cohomology or the index of the Dirac operator. A few names:

Clozel, DeGeorge-Wallach, Labesse, Borel-Labesse-Schwermer, Speh, Rohlfs-Speh, Wallach ... B–Speh also proved a version.

$(\mathfrak{g}, K)-$cohomology II

The main idea is to use the Arthur-Selberg trace formula. This is an equality between a RHS and a LHS akin to the Poisson summation formula; a RHS, sum of traces $Tr\pi(f)$ equals a LHS which is a sum of orbital integrals.
For the case when $\Gamma\backslash G$ is compact,

$$\sum m_\pi \Theta_\pi(f) = \sum vol(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(g\gamma g^{-1})\, dg.$$

Theorem

Let τ be a finite order automorphism such that G^τ is equal rank. Then there exist irreducible representations with nontrivial Lefschetz numbers of $G \ltimes \langle\tau\rangle$ which are cuspidal automorphic. □

The automorphism τ induces an automorphism of $H^i(\mathfrak{g}, K)$ and the Lefschetz number is the Euler characteristic of the trace of this element.

$(\mathfrak{g}, K)-$cohomology III

This result holds in greater generality, B-Pandzic, namely there exist such representations with nontrivial Dirac cohomology.
The results about $(\mathfrak{g}, K)-$cohomology can be deduced as a consequence of the ones about Dirac cohomology.
The group need no be equal rank, and the infinitesimal character does not need to be integral or even regular.
The Arthur version of the trace formula is valid in the general setting of $vol(\Gamma\backslash G)$ finite. The "*simple version of the twisted trace formula*" as in Borel-Labesse-Schwermer and Kottwitz provides a crucial simplifications of the right hand side of the AS-trace formula. At least formally it coincides with the cocompact case, but many more terms are zero when evaluated on certain functions called pseudo-coefficients.
We construct "*pseudo-coefficients*" for which $\mathrm{Tr}\,\pi(f) \neq 0$ only for representations with nonzero twisted index, and for which the right hand side is computable and shown to be nonzero.

Dirac cohomology

Motivated by the Dirac inequality (see below) and its uses to compute spectral gaps, Vogan introduced the notion of Dirac Cohomology.

Let M be an admissible $(\mathfrak{g}, K)$-module. Let S be a Spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ maximal isotropic.

Then D acts on $M \otimes S$.

Dirac cohomology of M:

$$H_D(M) = \mathrm{Ker}\, D / (\mathrm{Im}\, D \cap \mathrm{Ker}\, D)$$

$H_D(M)$ is a module for the spin double cover $\widetilde{K}$ of K. It is finite-dimensional if M is of finite length.

If M is unitary, then D is self adjoint *w.r.t.* an inner product. So

$$H_D(M) = \mathrm{Ker}\, D = \mathrm{Ker}\, D^2,$$

and $D^2 \geq 0$ (Dirac inequality).

Dirac Inequality I

The adjoint representation of K on $\mathfrak{p}$ lifts to $\mathrm{Ad} : \widetilde{K} \longrightarrow Spin(\mathfrak{p}_0)$, where $\widetilde{K}$ is the spin double cover of K. The Dirac operator is defined as

$$D = \sum_i b_i \otimes d_i \quad \in U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

where $C(\mathfrak{p})$ denotes the Clifford algebra of $\mathfrak{p}$ with respect to the form B, b_i is a basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B. D is independent of the choice of the basis b_i and $K-$invariant. It satisfies

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho_{\mathfrak{g}}\|^2) + (\Delta(\mathrm{Cas}_{\mathfrak{k}}) + \|\rho_{\mathfrak{k}}\|^2).$$

In this formula, due to Parthasarathy [P1],

- $\mathrm{Cas}_{\mathfrak{g}}$ and $\mathrm{Cas}_{\mathfrak{k}}$ are the Casimir operators for $\mathfrak{g}$ and $\mathfrak{k}$ respectively,
- $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ is a fundamental θ-stable Cartan subalgebra with compatible systems of positive roots for $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$,

Dirac Inequality II

- $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{k}}$ are the corresponding half sums of positive roots,
- $\Delta : \mathfrak{k} \to U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is given by $\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$, where α is the action map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ composed with the usual identifications $\mathfrak{so}(\mathfrak{p}) \cong \bigwedge^2(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.

If π is a $(\mathfrak{g}, K)-$module, then D induces an operator

$$D = D_\pi : \pi \otimes Spin \longrightarrow \pi \otimes Spin,$$

where $Spin$ is a spin module for $C(\mathfrak{p})$. If π is unitary, then $\pi \otimes Spin$ admits a $K-$invariant inner product $\langle \; , \; \rangle$ such that D is self adjoint with respect to this inner product. It follows that $D^2 \geq 0$ on $\pi \otimes Spin$. Using the above formula for D^2, we find that

$$Cas_{\mathfrak{g}} + \|\rho_{\mathfrak{g}}\|^2 \leq \Delta(Cas_{\mathfrak{k}}) + \|\rho_{\mathfrak{k}}\|^2$$

on any $K-$type τ occurring in $\pi \otimes Spin$.

Dirac Inequality III

Another way of putting this is

$$\|\Lambda\|^2 \leq \|\tau + \rho_{\mathfrak{k}}\|^2, \tag{1}$$

for any τ occurring in $\pi \otimes Spin$, where Λ is the infinitesimal character of π.
In the case of an equal rank group ($rankG = rankK$), $Spin(\mathfrak{p}) = S^+ \oplus S^-$. The Dirac operator satisfies

$$D : X \otimes Spin^{\pm} \longrightarrow X \otimes Spin^{\mp}.$$

The index is $I(D, X) = \mathrm{Ker}\, D^+ - \mathrm{Coker}\, D^+$. This is zero in the unequal rank case.
The aim is to find a framework that gives nontrivial results in the unequal rank case. A main example are the complex groups viewed as real groups.

Vogan's Conjecture

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. View $\mathfrak{t}^* \subset \mathfrak{h}^*$ via extension by 0 over $\mathfrak{a}$.

The following was conjectured by Vogan in 1997, and proved by Huang-Pandžić in 2002.

Theorem

Assume M has an infinitesimal character, and $H_D(M)$ contains a $\widetilde{K}$-type E_τ of highest weight $\tau \in \mathfrak{t}^*$. Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be a fundamental $\theta-$stable Cartan subalgebra. The infinitesimal character is a $W-$orbit of a semisimple element $\Lambda \in \mathfrak{h}^*$.

Then there is $w \in W$ such that $w\Lambda \mid_{\mathfrak{t}} = \tau + \rho_{\mathfrak{k}}$, and $w\Lambda \mid_{\mathfrak{a}} = 0$.

Motivation

- unitarity: Dirac inequality and its improvements.
- irreducible unitary M with $H_D \neq 0$ are interesting (discrete series, $A_{\mathfrak{q}}(\lambda)$ modules, unitary highest weight modules, some unipotent representations...) They should form a nice part of the unitary dual.
- H_D is related to classical topics like generalized Weyl character formula, generalized Bott-Borel-Weil Theorem, construction of discrete series, multiplicities of automorphic forms
- There are nice constructions of representations with $H_D \neq 0$; e.g., Parthasarthy and Atiyah-Schmid constructed the discrete series representations using spin bundles on G/K.

Twisted Dirac Index I

We need a cover of K,

$$K^{\dagger} := \{(k, g) \in K \times Pin(\mathfrak{p}) \mid \mathrm{Ad}(k)\mid_{\mathfrak{p}} = p(g)\}$$

for the usual projection $p : Pin(\mathfrak{p}) \longrightarrow O(\mathfrak{p})$. Then $X \otimes Spin$ is an $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), K^{\dagger})-$module via:

$$(u \otimes c) \cdot (x \otimes c) = ux \otimes cs,$$
$$(k, g) \cdot (x \otimes s) = kx \otimes gsg^{-1}.$$

Let γ be an automorphism of $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), K^{\dagger})$ i.e.

1. γ consists of an automorphism $\gamma^{\mathfrak{g}}$ of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ and an automorphism γ^{K} of $K^{\dagger}$;

Twisted Dirac Index II

2. γ is compatible with the action of $K^\dagger$ on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ in the sense that

$$\gamma^{\mathfrak{g}}((k,g)(u \otimes c)) = \gamma^K(k,g)\gamma^{\mathfrak{g}}(u \otimes c)$$

for $(k,g) \in K^\dagger$ and $u \otimes c \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$;

3. the differential of γ^K coincides with the restriction of $\gamma^{\mathfrak{g}}$ to $\mathfrak{k}_\Delta$.

We assume that $X \otimes S$ has a compatible action of γ, i.e. there is an operator $\pi(\gamma)$ on $X \otimes S$ such that

$$\pi(\gamma)\pi(u \otimes c)\pi(\gamma)^{-1} = \pi(\gamma^{\mathfrak{g}}(u \otimes c)), \quad u \otimes c \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$
$$\pi(\gamma)\pi(k)\pi(\gamma)^{-1} = \pi(\gamma^K(k)), \quad k \in K^\dagger.$$

We assume that γ satisfies

$$\gamma(D) = -D,$$

Twisted Dirac Index III

so $\pi(\gamma)$ and $\pi(D)$ anticommute. Then γ preserves $H_D(X)$. If we denote the fixed points of γ in $K^\dagger$ by $K_\gamma^\dagger$, then $K_\gamma^\dagger$ preserves the above decomposition, and we define the $\gamma-$index of D on $X \otimes S$ as the function

$$\chi_\gamma^X(k) = \mathrm{tr}(\gamma k; H_D(X)) \qquad k \in K_\gamma^\dagger. \tag{2}$$

Theorem

$$Tr(k\gamma : H_D(X)) = Tr(k\gamma : X \otimes S)$$

Sketch of Proof I

We may consider $H_D(X)$ as a $K^\diamond := K^\dagger \ltimes \langle \tau \rangle-$submodule of $X \otimes S$; namely, $\operatorname{Ker} D$ and $\operatorname{Ker} D \cap \operatorname{Im} D$ are $K^\diamond$-submodules of $X \otimes S$, and we identify $H_D(X)$ with a $K^\diamond-$ invariant direct complement of $\operatorname{Ker} D \cap \operatorname{Im} D$ in $\operatorname{Ker} D$. Recall that $H_D(X)$ is finite dimensional; see the paragraph below Definition 3.2.3 in [HP2]. We decompose $X \otimes S$ into the direct sum of finite-dimensional eigenspaces $(X \otimes S)_\lambda$ for D^2. Each of these eigenspaces is invariant under the action of $k\gamma$, since D^2 commutes with $k\gamma$. If $\lambda \neq 0$, then $D : (X \otimes S)_\lambda \to (X \otimes S)_\lambda$ is a $K^\dagger-$isomorphism. So if $E \subseteq (X \otimes S)_\lambda$ is an irreducible $K^\dagger-$submodule, then $F = D(E)$ is another irreducible $K^\dagger-$submodule of $(X \otimes S)_\lambda$. There are two cases: either $F = E$, or $F \neq E$; in the latter case $E \cap F = 0$. Since $K^\diamond = K^\dagger \ltimes \langle \tau \rangle$ is compact, E and F are unitary with respect to an appropriate inner product. So we can decompose E and F into eigenspaces for $k\gamma$:

$$E = \bigoplus E_\mu; \qquad F = \bigoplus F_\mu,$$

Sketch of Proof II

for $\mu \in \mathbb{C}$ satisfying $|\mu| = 1$. In particular, $\mu \neq 0$, and $E_\mu \cap E_{-\mu} = 0$ for every μ that appears.
Since D anticommutes with $k\gamma$, it must send E_μ isomorphically onto $F_{-\mu}$ for each μ. It follows that

$$\mathrm{tr}(k\gamma;\, F) = -\,\mathrm{tr}(k\gamma;\, E). \tag{3}$$

Thus,

- If $E \neq F$, (3) implies that the trace of $k\gamma$ is 0 on $E \oplus F \subset X \otimes S$.
- If $E = F$, (3) implies that the trace of $k\gamma$ is 0 on E.

In conclusion, the trace of $k\gamma$ is 0 on $(X \otimes S)_\lambda$ for any $\lambda \neq 0$.
The eigenspace $(X \otimes S)_0 = \mathrm{Ker}\, D^2$ can be decomposed as

$$\mathrm{Ker}\, D^2 = H_D(X) \oplus (\mathrm{Ker}\, D \cap \mathrm{Im}\, D) \oplus Y,$$

with D sending Y isomorphically to $\mathrm{Ker}\, D \cap \mathrm{Im}\, D$. Thus the trace of $k\gamma$ is 0 on $(\mathrm{Ker}\, D \cap \mathrm{Im}\, D) \oplus Y$, and the proposition follows.

Examples I

Equal Rank Case: This is the ordinary Dirac index in the equal rank case. Let $\mathfrak{h}_0 = \mathfrak{t}_0$ be the compact Cartan subalgebra in $\mathfrak{g}_0$. In this case $\dim \mathfrak{p}$ is even, so there is only one spin module S, and it is a graded module for $C(\mathfrak{p}) = C^0(\mathfrak{p}) \oplus C^1(\mathfrak{p})$, i.e., $S = S^+ \oplus S^-$, with $S^\pm$ preserved by $C^0(\mathfrak{p})$ and interchanged by $C^1(\mathfrak{p})$. (Recall that S can be constructed as $\bigwedge \mathfrak{p}^+$ with $\mathfrak{p}^+$ a maximal isotropic subspace of $\mathfrak{p}$, and that one can take $S^+ = \bigwedge^{\text{even}} \mathfrak{p}^+$ and $S^- = \bigwedge^{\text{odd}} \mathfrak{p}^+$.)

Recall that θ denotes the Cartan involution of $\mathfrak{g}$. It induces $-\operatorname{Id} \in O(\mathfrak{p}_0)$, and so gives rise to two elements in $\operatorname{Pin}(\mathfrak{p}_0)$. It is easy to see that these elements are

$$\pm Z_1 Z_2 \dots Z_s \in C(\mathfrak{p}_0),$$

where $Z_1, \dots, Z_s$ is any orthonormal basis of $\mathfrak{p}_0$. We fix one of these two elements, and call it again θ. In this way θ acts on S, and one easily checks that $S = S^+ \oplus S^-$ is the decomposition into

Examples II

eigenspaces of θ. Moreover, we can make the choice of θ compatible with the choice of $S^{\pm}$, so that θ is 1 on S^{+} and -1 on S^{-}. Furthermore, we can extend the automorphism $\theta = -\operatorname{Id}$ of $\mathfrak{p}_0$ to an automorphism of $C(\mathfrak{p})$, and this automorphism is exactly the conjugation by the element $\theta \in C(\mathfrak{p})$. (This automorphism is in fact equal to the sign automorphism of $C(\mathfrak{p})$.)

We now consider the automorphism γ of $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), K^{\dagger})$ constructed from the automorphisms $\gamma_1 = \operatorname{Id}$ of $(\mathfrak{g}, K)$ and $\gamma_2 = \theta$ of $C(\mathfrak{p})$. We conclude

$$I(X) = X \otimes S^{+} - X \otimes S^{-}. \tag{4}$$

If $K^{\dagger}_{\gamma} = K^{\dagger}$, i.e. the natural map from $K^{\dagger}$ to $\operatorname{Pin}(\mathfrak{p}_0)$ maps $K^{\dagger}$ into $\operatorname{Spin}(\mathfrak{p}_0)$, then $I(X)$ is the usual index as in [P1] and the work of Hecht-Schmid and Atiyah-Schmid.

Examples III

Unequal Rank Case: If $\mathfrak{g}$ and $\mathfrak{k}$ do not have equal rank, then the above usual notion of index is trivial. Instead, we consider the extended group

$$G^+ = G \rtimes \{1, \theta\},$$

with θ acting on G by the Cartan involution, and with $\theta^2 = 1 \in G$. The maximal compact subgroup of G^+ is

$$K^+ = K \times \{1, \theta\}.$$

A $(\mathfrak{g}, K^+)-$module (π, X) can be thought of as a $(\mathfrak{g}, K)-$module with an additional action of θ by $\pi(\theta)$, which satisfies

$$\begin{aligned} \pi(\theta)\pi(k)\pi(\theta) &= \pi(k), & k &\in K; \\ \pi(\theta)\pi(\xi)\pi(\theta) &= \pi(\theta(\xi)), & \xi &\in \mathfrak{g}. \end{aligned} \tag{5}$$

We now consider the automorphism γ of $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), K^\dagger)$ built from the automorphisms $\gamma_1 = \theta$ of $(\mathfrak{g}, K)$ and $\gamma_2 = \mathrm{Id}$ of $C(\mathfrak{p})$.

Examples IV

Here $K^\dagger$ still denotes the Pin double cover of K, not of K^+. The compatibility condition is now trivial, and so is the fact that γ is an involution satisfying $\gamma(D) = -D$. It is also clear that in this case $K^\dagger_\gamma = K^\dagger$. Moreover, γ acts on $X \otimes S$ whenever X is a $(\mathfrak{g}, K^+)-$module.
We can now consider the $\gamma-$index of D on $X \otimes S$, which we denote by $I_\theta(X)$ in the present case, and call the twisted Dirac index of X. In particular, we have the following equality of virtual $K^\dagger-$modules

$$I_\theta(X) = X^+ \otimes S - X^- \otimes S, \tag{6}$$

where $X^\pm$ denote the ± 1 eigenspaces of θ on X.

This setting makes sense in the equal rank case as well. Since $\theta = \mathrm{Ad}\, k_0$ is inner, any $(\mathfrak{g}, K)-$module extends naturally to an $G^+ = G \rtimes \{1, \theta\}-$module via $\pi(\theta) = \pi(k_0)$. The resulting twisted index is not substantially different from the usual notion of index.

Examples V

Namely let $\tilde{k}_0 = (k_0, \theta) \in K^\dagger$, where $\theta \in \mathrm{Spin}(\mathfrak{p}_0)$ is the top degree element acting by ± 1 on $S^\pm$. Then $\tilde{k}_0$ is in $K^\dagger_\gamma$.

Let χ_1 (respectively χ_2) be the function defined by (2) for the ordinary (respectively twisted) Dirac index. These functions are both defined for any $k \in K^\dagger_\gamma$. Since $\tilde{k}_0^2$ acts as the identity on S, we have

$$\begin{aligned}\chi_1(\tilde{k}_0 k) &= \mathrm{tr}(\tilde{k}_0 k; X)\,\mathrm{tr}(\tilde{k}_0 k\theta; S) = \mathrm{tr}(\tilde{k}_0 k; X)\,\mathrm{tr}(\tilde{k}_0^2 k; S) = \\ &= \mathrm{tr}(\tilde{k}_0 k; X)\,\mathrm{tr}(k; S) = \chi_2(k).\end{aligned}$$

So we see that the twisted Dirac index χ_2 is the same as the ordinary Dirac index χ_1 with the argument translated by $\tilde{k}_0$.

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