Tatuzawa's theorem for Rankin–Selberg L-functions

(joint work with Jesse Thorner)

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11 August 2025
Aarhus Automorphic Forms Conference
Aarhus University

Standard L-functions and Rankin–Selberg L-functions

Cuspidal representations

Let $\mathfrak{F}_n$ be the set of unitary cuspidal automorphic representations of GL_n over a fixed number field F.
Let $\mathfrak{F}_n^* \subset \mathfrak{F}_n$ be the subset of representations in $\mathfrak{F}_n$ whose central character is trivial on the diagonally embedded positive reals.

Each $\pi \in \mathfrak{F}_n$ gives rise to a standard L-function $L(s, \pi)$, which has similar properties as the product of n Hecke L-functions (over F). In fact the product of n Hecke L-functions is the L-function of an isobaric automorphic representation of GL_n over F.

Each $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$ gives rise to a Rankin–Selberg L-function $L(s, \pi \times \rho)$, which has similar properties as the product of nm Hecke L-functions. Langlands functoriality predicts that $L(s, \pi \times \rho)$ is a product of standard L-functions. Hoffstein–Ramakrishnan (1995) used this hypothesis to prove the non-existence of Landau–Siegel zeros other than those of Hecke L-functions.

Twisting and normalizing cuspidal representations

GL_1-twists

$\mathfrak{F}_1$ is the abelian group of unitary Hecke characters acting on $\mathfrak{F}_n$ as follows. For each $\pi \in \mathfrak{F}_n$ and $\chi \in \mathfrak{F}_1$, we denote by $\pi \otimes \chi \in \mathfrak{F}_n$ the representation $g \mapsto \pi(g)\chi(\det g)$ embedded into the cuspidal subspace of $L^2(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F))$ in the usual way.

A special case of this action results in the shifting of the L-function by purely imaginary numbers it $(t \in \mathbb{R})$:

$$L(s+it,\pi) = L(s,\pi \otimes |\cdot|^{it}),$$
$$L(s+it,\pi\times\rho) = L(s,\pi\times(\rho\otimes|\cdot|^{it})).$$

There is a unique decomposition $\pi = \pi^* \otimes |\cdot|^{it_\pi}$ with $\pi^* \in \mathfrak{F}_n^*$ and $t_\pi \in \mathbb{R}$, and similarly for ρ. It follows that

$$L(s,\pi) = L(s+it_\pi,\pi^*),$$
$$L(s,\pi\times\rho) = L(s+it_\pi+it_\rho,\pi^*\times\rho^*).$$

Nonvanishing of Rankin–Selberg L-functions

Establishing zero-free regions and lower bounds for automorphic L-functions has a venerable history: Dirichlet (1837), Riemann (1859), Hadamard (1896), de la Vallée Poussin (1896 & 1899), Gronwall (1913), Landau (1918), Titchmarsh (1930), Page (1935), **Siegel (1935)**, **Tatuzawa (1951)**, Jacquet–Shalika (1976), **Shahidi (1981)**, Moreno (1985), **Hoffstein–Lockhart (1994)**, **Goldfeld–Hoffstein–Lieman (1994)**, **Hoffstein–Ramakrishnan (1995)**, Banks (1997), Ramakrishnan–Wang (2003), Iwaniec–Kowalski (2004), Sarnak (2004), Gelbart–Lapid (2006), Goldfeld–Li (2018), Humphries (2019), Jiang–Lü–Thorner–Wang (2021), Luo (2023), Zhang (2023), Wattanawanichkul (2025).

Theorem (Brumley 2006–2019, Humphries–Thorner 2022)

There exists $c_1 = c_1(n, m, [F : \mathbb{Q}]) > 0$ with the following property. If $(\pi, \rho) \in \mathfrak{F}_n^* \times \mathfrak{F}_m^*$, then $L(\sigma + it, \pi \times \rho)$ has no zero in the region

$$\sigma \geqslant 1 - c_1(C(\pi)C(\rho))^{-n-m}(|t| + 1)^{-nm}.$$

Moreover, if $\pi = \widetilde{\pi}$ or $\rho = \widetilde{\rho}$ or $\rho = \widetilde{\pi}$, then $L(\sigma + it, \pi \times \rho)$ has at most one zero (necessarily real and simple) in the region

$$\sigma \geqslant 1 - c_1/\log(C(\pi)C(\rho)(|t| + 3)).$$

If the exceptional zero exists, then $(\pi, \rho) = (\widetilde{\pi}, \widetilde{\rho})$ or $\rho = \widetilde{\pi}$.

A new zero-free region

We extended the celebrated lower bound of Siegel (1935) to all GL_1-twists of general $\mathrm{GL}_n \times \mathrm{GL}_m$ Rankin–Selberg L-functions.

Theorem (Harcos–Thorner 2025)

Let $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$. For all $\varepsilon > 0$, there exists an ineffective constant $c_2 = c_2(\pi, \rho, \varepsilon) > 0$ such that if $\chi \in \mathfrak{F}_1$, then

$$|L(\sigma, \pi \times (\rho \otimes \chi))| \geqslant c_2 C(\chi)^{-\varepsilon}, \qquad \sigma \geqslant 1 - c_2 C(\chi)^{-\varepsilon}.$$

Remark

It follows that in fact $|L(\sigma + it, \pi \times (\rho \otimes \chi))| \geqslant c_3 C(\chi)^{-\varepsilon}(|t| + 1)^{-\varepsilon}$ for $\sigma \geqslant 1 - c_3 C(\chi)^{-\varepsilon}(|t| + 1)^{-\varepsilon}$, with some $c_3 = c_3(\pi, \rho, \varepsilon) > 0$.

The proof relies on the group structure of $\mathfrak{F}_1$, and it utilizes an auxiliary L-function with nonnegative coefficients that extends the constructions of de la Vallée Poussin (1899) and Siegel (1935).

An analogue of the Siegel–Walfisz theorem

The new zero-free region allowed us to prove an analogue of the Siegel–Walfisz theorem for Rankin–Selberg L-functions. Here is a particular case over the rational field $F = \mathbb{Q}$ for simplicity.

Theorem (Harcos–Thorner 2025)

For $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$, let $\Lambda_{\pi\times\rho}(k)$ denote the k-th Dirichlet coefficient of $-L'(s, \pi \times \rho)/L(s, \pi \times \rho)$. Moreover, let

$$\mathcal{M}_{\pi\times\rho}(x) = \begin{cases} x^{1-it}/(1-it), & \rho = \widetilde{\pi} \otimes |\cdot|^{it} \\ 0, & \textit{otherwise} \end{cases}$$

Let $A > 0$ be arbitrary. Let $q \leqslant (\log x)^A$ be a positive integer coprime to the conductors of π and ρ, and let $a \pmod{q}$ be a reduced residue class modulo q. Then

$$\sum_{\substack{k\leqslant x \\ k\equiv a \pmod{q}}} \Lambda_{\pi\times\rho}(k) = \frac{\mathcal{M}_{\pi\times\rho}(x)}{\varphi(q)} + O_{\pi,\rho,A}\left(\frac{x}{(\log x)^A}\right).$$

A generalization of Tatuzawa's theorem (1 of 2)

The theorem of Siegel (1935) can be made "almost effective":

Theorem (Tatuzawa 1951)

For every $\varepsilon > 0$, there exists a primitive quadratic Dirichlet character ψ such that if $\chi \neq \psi$ is any other primitive quadratic Dirichlet character, then

$$L(1,\chi) > \frac{\varepsilon}{10} C(\chi)^{-\varepsilon}.$$

Inspired by this result, Jesse Thorner and I proved that among all the GL_1-twists of a given Rankin–Selberg L-function, all but one admits a good effective zero-free interval on the real axis.

Moreover, if the exceptional GL_1-twist exists, then it has at most one exceptional zero (necessarily simple) on the real axis.

A generalization of Tatuzawa's theorem (2 of 2)

Theorem (Harcos–Thorner 2025+)

Let $(\pi, \rho, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_m \times \mathfrak{F}_1$ and $\varepsilon > 0$. There exist an effectively computable constant $c_4 = c_4(n, m, [F : \mathbb{Q}], \varepsilon) > 0$ and a character $\psi = \psi_{\pi,\rho,\varepsilon} \in \mathfrak{F}_1$ such that if $L(s, \pi \times (\rho \otimes \chi))$ differs from $L(s, \pi \times (\rho \otimes \psi))$, then

$$L(\sigma, \pi \times (\rho \otimes \chi)) \neq 0, \qquad \sigma \geqslant 1 - c_4(C(\pi)C(\rho)C(\chi))^{-\varepsilon}.$$

Moreover, $L(s, \pi \times (\rho \otimes \psi))$ has at most one zero (necessarily simple) in the interval $\sigma \geqslant 1 - c_4(C(\pi)C(\rho)C(\psi))^{-\varepsilon}$.

Corollary

If $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$ and $\varepsilon > 0$, then $L(\sigma + it, \pi \times \rho)$ has at most one zero (necessarily simple) in the region

$$\sigma \geqslant 1 - c_4(C(\pi)C(\rho)D_F(|t| + 3)^{[F:\mathbb{Q}]})^{-\varepsilon}.$$

The Key Proposition

The proof relies on the observation that the desired zero-free interval can be established under some auxiliary assumptions.

Key Proposition

Let $(\pi, \rho, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_m \times \mathfrak{F}_1$ *and* $\varepsilon \in (0, 1)$. *Put*

$$Q = Q(\pi, \rho, \chi) = (C(\pi)C(\rho))^{2(n+m)} C(\chi)^{(n+m)^2}.$$

Assume that $L(s, \pi \times (\rho \otimes \chi))$ *is entire, and one of the following two conditions holds true:*

1. $L(s, \pi \times \widetilde{\pi})$ *has a zero in the interval* $[1 - \varepsilon/16, 1)$.
2. $L(s, \pi \times \rho)$ *is entire and has a zero in the interval* $[1 - \varepsilon/16, 1)$. *Furthermore, if* $\pi \otimes \chi^* = \pi$ *or* $\rho \otimes \chi^* = \rho$, *then* $|t_\chi| \geqslant Q^{-\varepsilon/64}$.

There exists an effectively computable constant $c_5 = c_5(n, m, [F : \mathbb{Q}], \varepsilon) > 0$ *such that*

$$L(\sigma, \pi \times (\rho \otimes \chi)) \neq 0, \qquad \sigma \geqslant 1 - c_5 Q^{-\varepsilon}.$$

Outline of the proof (1 of 3)

For ease of notation, we shall seek zero-free intervals of the form

$$\sigma \geqslant 1 - c_6(C(\pi)C(\rho)C(\chi)^{2n+2m})^{-\varepsilon},$$

where $\varepsilon \in (0,1)$. We shall apply the Key Proposition with

$$\varepsilon' = \frac{\varepsilon}{2(n+m)}$$

in the role of ε, and possibly some (ρ', χ') in the role of (ρ, χ).

If $L(s, \pi \times \widetilde{\pi})$ has a zero in the interval $[1 - \varepsilon'/16, 1)$, then we are done by the Key Proposition and the standard zero-free region for $L(s, \pi \times \widetilde{\pi})$ established by Humphries–Thorner (2022). Otherwise, $L(s, \pi \times \widetilde{\pi})$ has no exceptional zero, and we can focus on $\chi \in \mathfrak{F}_1$ such that $L(s, \pi \times (\rho \otimes \chi))$ is entire. Let S be the set of such χ.

If, for all $\chi \in S$, the L-function $L(s, \pi \times (\rho \otimes \chi))$ has no zero in $[1 - \varepsilon'/16, 1)$, then we are done. Otherwise, we can fix $\lambda \in S$ with minimal analytic conductor such that $L(s, \pi \times (\rho \otimes \lambda))$ has a zero in $[1 - \varepsilon'/16, 1)$.

Outline of the proof (2 of 3)

After fixing $\lambda \in S$ as above, we can assume that $C(\chi) \geqslant C(\lambda)$, for otherwise we are done. We make the change of variables

$$\rho' = \rho \otimes \lambda, \qquad \chi' = \chi\bar{\lambda}.$$

Note that $L(s, \pi \times \rho')$ is entire and has a zero in $[1 - \varepsilon'/16, 1)$. Moreover, $L(s, \pi \times (\rho' \otimes \chi')) = L(s, \pi \times (\rho \otimes \chi))$ is entire.

We apply the Key Proposition with $(\rho', \chi', \varepsilon')$ in place of $(\rho, \chi, \varepsilon)$. Accordingly, we work with $Q = Q(\pi, \rho', \chi')$, which satisfies

$$(C(\pi)C(\rho\otimes\lambda)C(\rho\otimes\chi))^{n+m} < Q < (C(\pi)C(\rho))^{2(n+m)}C(\chi)^{4(n+m)^2}.$$

If the twist equivalence condition of the Key Proposition is satisfied for (ρ', χ') in place of (ρ, χ), then we are done. Otherwise,

$$L(s, \pi \times (\rho \otimes \chi)) = L(s + it_\chi - it_\lambda, \pi \times (\rho \otimes \lambda)),$$

where $|t_\chi - t_\lambda| < (C(\pi)C(\rho \otimes \lambda)C(\rho \otimes \chi))^{-\varepsilon/128}$.

Outline of the proof (3 of 3)

So every zero of $L(s, \pi \times (\rho \otimes \chi))$ in the interval

$$\sigma \geqslant 1 - c_6 (C(\pi) C(\rho) C(\chi)^{2n+2m})^{-\varepsilon}$$

yields a zero $\sigma + i(t_\chi - t_\lambda)$ very close to 1 of $L(s, \pi \times (\rho \otimes \lambda))$.

Moreover, we can recover the twist $L(s, \pi \times (\rho \otimes \chi))$ and its zero $\sigma \approx 1$ from the zero $\sigma + i(t_\chi - t_\lambda) \approx 1$ of $L(s, \pi \times (\rho \otimes \lambda))$.

However, by a Goldfeld–Hoffstein–Lieman type argument, $L(s, \pi \times (\rho \otimes \lambda))$ has at most 1 such zero, with multiplicity:

Lemma

There exists a constant $c_7 = c_7(n, m) > 0$ such that the product

$$L(s, \pi \times \widetilde{\pi}) L(s, \rho \times \widetilde{\rho}) L(s, \pi \times (\rho \otimes \lambda)) L(s, \widetilde{\pi} \times (\widetilde{\rho} \otimes \overline{\lambda}))$$

has at most 2 zeros (counted with multiplicity) in the region

$$\sigma \geqslant 1 - \frac{c_7}{\log(C(\pi) C(\rho \otimes \lambda))} \quad \textit{and} \quad |t| \leqslant \frac{c_7}{\log(C(\pi) C(\rho \otimes \lambda))}.$$

Hoffstein–Ramakrishnan & Goldfeld–Hoffstein–Lieman

Let $\Pi = \pi_1 \boxplus \cdots \boxplus \pi_\ell$ be an isobaric sum of unitary cuspidal automorphic representations.

Lemma (Hoffstein–Ramakrishnan 1995)

The logarithm of $L(s, \Pi \times \widetilde{\Pi})$ has nonnegative Dirichlet coefficients.

Lemma (after Goldfeld–Hoffstein–Lieman 1994)

There exists an absolute and effectively computable constant $c_8 > 0$ with the following property. Let r be the order of the pole of $L(s, \Pi \times \widetilde{\Pi})$ at $s = 1$. Let $E(\Pi \times \widetilde{\Pi})$ be the logarithm of $C(\Pi \times \widetilde{\Pi})$ plus the sum of reciprocal imaginary parts of the poles of $L(s, \Pi \times \widetilde{\Pi})$ in the upper half-plane, counted with multiplicity. Then $L(s, \Pi \times \widetilde{\Pi})$ has at most r zeros (counted with multiplicity) in the region

$$\sigma \geqslant 1 - \frac{c_8}{rE(\Pi \times \widetilde{\Pi})} \qquad \text{and} \qquad |t| \leqslant \frac{c_8}{\sqrt{r}E(\Pi \times \widetilde{\Pi})}.$$

Proof of the Key Proposition, generic case (1 of 2)

Generic case: $L(s, \pi \times (\rho \otimes \chi^2))$ is entire in condition ❷.

We use the auxiliary L-function $L(s, \Pi \times \widetilde{\Pi})$, where Π is one of:

❶ $\Pi = \pi \boxplus \widetilde{\rho} \otimes \overline{\chi}$

❷ $\Pi = \pi \boxplus \pi \otimes \chi \boxplus \widetilde{\rho} \boxplus \widetilde{\rho} \otimes \overline{\chi}$

Hence $L(s, \Pi \times \widetilde{\Pi})$ has nonnegative Dirichlet coefficients, and it has a zero $\beta \in [1 - \varepsilon/16, 1)$. We apply Perron's formula and the Residue Theorem to the meromorphic function

$$s \mapsto L(s, \Pi \times \widetilde{\Pi}) x^{s-\beta} \Gamma(s - \beta),$$

where $x \geqslant 1$ is a parameter. Each residue can be bounded in terms of $|L(1, \pi \times (\rho \otimes \chi))|$, and after optimizing $x \geqslant 1$, we conclude that

$$|L(1, \pi \times (\rho \otimes \chi))| \gg_{n,m,[F:\mathbb{Q}],\varepsilon} (1 - \beta)^4 Q^{-\varepsilon/2}.$$

Of course, this bound is not useful when β is very close to 1.

Proof of the Key Proposition, generic case (2 of 2)

Our Goldfeld–Hoffstein–Lieman type lemma actually implies the existence of an effectively computable constant $c_9 = c_9(n, m, \varepsilon) > 0$ such that either $L(s, \pi \times (\rho \otimes \chi))$ has no zero in $[1 - c_9 Q^{-\varepsilon/64}, 1)$, or $\beta < 1 - c_9 Q^{-\varepsilon/64}$.

If $L(s, \pi \times (\rho \otimes \chi))$ has no zero in $[1 - c_9 Q^{-\varepsilon/64}, 1)$, then we are done of course. Otherwise, writing β_χ for the largest zero of $L(s, \pi \times (\rho \otimes \chi))$, we are done by the following inequality:

$$Q^{-2\varepsilon/3} \ll_{n,m,[F:\mathbb{Q}],\varepsilon} |L(1, \pi \times (\rho \otimes \chi))| \ll_{n,m,[F:\mathbb{Q}],\varepsilon} (1 - \beta_\chi) Q^{\varepsilon/3}.$$

Proof of the Key Proposition, non-generic case

Non-generic case: $L(s, \pi \times (\rho \otimes \chi^2))$ has a pole in condition ❷.

In this case, $\rho \otimes \chi^2 = \widetilde{\pi} \otimes |\cdot|^{it}$ holds with a unique $t \in \mathbb{R}$.
We calculate $t = t_\pi + t_\rho + 2t_\chi$. Consider the Hecke character $\kappa = \overline{\chi}|\cdot|^{it}$, which satisfies $\widetilde{\pi} \otimes \kappa = \rho \otimes \chi$ and

$$C(\kappa) \leqslant C(\chi)(|t| + 3)^{[F:\mathbb{Q}]} < (C(\pi)C(\rho)C(\chi)^3)^{[F:\mathbb{Q}]}.$$

The L-functions

$$L(s, \pi \times (\widetilde{\pi} \otimes \kappa)) = L(s, \pi \times (\rho \otimes \chi))$$
$$L(s, \pi \times (\widetilde{\pi} \otimes \kappa^2)) = L(s + it, \pi \times \rho)$$

are entire by assumption. Therefore, applying our Goldfeld–Hoffstein–Lieman type lemma to

$$\Pi = \pi \boxplus \pi \otimes \kappa \boxplus \pi \boxplus \pi \otimes \overline{\kappa},$$

we obtain an absolute, effective constant $c_{10} > 0$ such that

$$L(\sigma, \pi \times (\widetilde{\pi} \otimes \kappa)) \neq 0, \qquad \sigma \geqslant 1 - c_{10}/\log(C(\pi)^n C(\kappa)^{n^2}).$$

This is stronger than the desired conclusion.