Algebraicity Results for Special Values of L-functions on Exceptional Groups

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A result of Shimura

- Let f and g be holomorphic modular forms with Fourier expansions

$$f = \sum_{n=0}^{\infty} a_n e^{2\pi inz}, g = \sum_{n=0}^{\infty} b_n e^{2\pi inz}$$

- Define the product L-function

$$D(s, f, g) = \sum_{n=0}^{\infty} a_n \overline{b_n} n^{-s}$$

Theorem (Shimura)

Let f be a Hecke eigenform of weight k and g a holomorphic modular form of weight $n < k$. Then, when s is an integer with $\frac{1}{2}(n+k-2) < s < k$,

$$\pi^{-k} \frac{D(s, f, g)}{< f, f >} \in \mathbb{Q}(f)\mathbb{Q}(g)$$

A result of Shimura

- Proof of theorem: Integral representation, control of Fourier coefficients, Maass-Shimura operators
- Integral representation (Rankin, Selberg):

$$< f(z), g(z) \cdot E_m(z, s) > \approx D(s + k - 1)$$

where

$$E_m(z, s) = \frac{1}{2} \sum_{gcd(c,d)=1} (cz + d)^{-m} \frac{y^s}{|cz + d|^{2s}}$$

and $m = k - n$.
- When $s = 0$, $E_m(z, 0)$ is a holomorphic Eisenstein series of weight m. So

$$< f, g \cdot E_m(z, 0) > \approx D(k - 1),$$

which implies

$$\pi^{-k} < f, f >^{-1} D(k - 1, f, g) \in \mathbb{Q}(f)\mathbb{Q}(g)$$

after keeping track of extra factors.

A result of Shimura

- To get algebraicity results for critical values to the left of $k-1$, use Maass-Shimura differential operators

$$\delta_m = \frac{1}{2\pi i}\left(\frac{m}{2iy}+\frac{\partial}{\partial z}\right), \delta_m^{(r)} = \delta_{m+2r-2}\circ\cdots\circ\delta_{m+2}\circ\delta_m$$

- Then

$$E_{k-n}(z,-r) \approx \delta^{(r)}_{k-n-2r}E_{k-n-2r}(z,0)$$

and

$$< f, g\cdot\delta^{(r)}_{k-n-2r}E_{k-n-2r}(z,0) > \approx < f, g\cdot E_{k-n}(z,-r) > \approx D(k-1-r)$$

- Conclusion: algebraicity of $\pi^{-k} < f,f >^{-1} D(k-1-r)$
- Remark: algebraicity for (holomorphic projection of) $g\cdot\delta^{(r)}_m E_m(z,0)$ can be derived from analysis of branching problem for holomoprhic discrete series of $SL_2(\mathbb{R})$ embedded diagonally in $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ (Harris)

Quaternionic Modular Forms

- Let $G_2/\mathbb{Q}$ be the automorphism group of the split octonions
- $G_2(\mathbb{R})$ has maximal compact $K_{\mathbb{R}} \cong (SU(2)^{long} \times SU(2)^{short})/\mu_2$
- $G_2(\mathbb{R})$ has non-holomorphic discrete series representations π_n with K-types

$$\pi_n = \bigoplus_{r \geqslant 0} \mathrm{Sym}^{2n+r}(V_2^{long}) \boxtimes \mathrm{Sym}^r(\mathrm{Sym}^3(V_2^{short}))$$

Definition (Gan-Gross-Savin, A. Pollack)

A quaternionic modular form on G_2 of weight n and level 1 is a function $\Phi : G_2(\mathbb{Z})\backslash G_2(\mathbb{R}) \to \mathrm{Sym}^{2n}(V_2^{long}) \boxtimes \mathbf{1}$ satisfying:

1. $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in G_2(\mathbb{Z})$ and $g \in G_2(\mathbb{R})$
2. $\Phi(gk) = k^{-1}\Phi(g)$ for all $k \in K_{\mathbb{R}}$ and $g \in G_2(\mathbb{R})$
3. $D_n\Phi = 0$ for a certain differential operator D_n
4. Φ is nice (“smooth", “moderate growth")

Quaternionic Modular Forms

Theorem (A. Pollack)

For all $x \in W$ and $m \in M$,

$$\Phi_{[N,N]}(n(x)m) = \Phi_N(m) + \sum_{\omega \in 2\pi W(\mathbb{Q}), \omega \geqslant 0} a_\Phi(\omega) e^{-i\langle \omega, x \rangle} \mathcal{W}_\omega(m)$$

for some completely explicit functions $\mathcal{W}_\omega$.

- G_2 has a Heisenberg parabolic $P = MN$
- $M \cong GL_2$ and $W := N/[N,N] \leftrightarrow$ {binary cubic forms}
- Example of a QMF: The degenerate Heisenberg Eisenstein series

$$E(g, f_n, s) = \sum_{\gamma \in P \backslash G} f_n(\gamma g, s), \text{ where } f_n(g, s) \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(|\nu|^s) \otimes \mathrm{Sym}^{2n}(V_2^{long})$$

is a QMF of weight n when n is even and $s = n + 1$. Write $E_n(g) := E(g, f_n, s)$.

Adjoint L-Function

- From now on let $G = G_2$. There is a subgroup $H \subseteq G$ isomorphic to $SU(2,1)$. One can define QMFs for H, and they also have a good theory of Fourier expansion (Koseki-Oda, Hilado-McGlade-Yan).
- Let Π be a generic cuspidal automorphic representation for H, with Π_∞ quaternionic of weight n. Then, for $\varphi \in \Pi$,

$$< \varphi(h), E(h, f_n, s) >_H \approx L(s-1, \Pi, Ad)$$

  at least at unramified finite places (J. Hundley) and the archimedean place.
- We want to talk about algebraicity of our L-function at critical points (in the sense of Deligne), i.e. $L(s)$ for $s = n, n-2, n-4, \ldots, 2$

Algebraicity Results

- Ingredients of integral representation:

$$< \varphi(h), E(h, f_n, s) >_H \approx L(s-1, \Pi, Ad)$$

  are a cuspidal QMF on $H = SU(2,1)$, degenerate Heisenberg Eisenstein series on $G = G_2$.
- Assumption: There is a basis for the space of cuspidal QMFs on H all of whose Fourier coefficients are algeraic numbers.
- Assumption: $E_n(g) = E(g, f_n, s = n+1)$ can be normalized to have algebraic Fourier Coefficients ($n = 4$ W.T. Gan, $n > 4$ ongoing joint work with J. Johnson-Leung, F. McGlade, A. Pollack, M. Roy)
- Then taking $s = n + 1$ shows

$$\frac{L(n, \Pi, Ad)}{< \varphi, \varphi >} \in \pi^{\mathbb{Z}} \times \overline{\mathbb{Q}}$$

Remark: There is a basis for the space of cuspidal QMFs on the exceptional groups G_2, F_4, E_6, E_7, E_8, all of whose Fourier Coefficients are algebraic numbers (A. Pollack)

Algebraicity Results

Theorem (in progress)

Let $r \geqslant 0$. There exist completely explicit differential operators D_r such that, if Φ is a QMF on G of weight n, then $D_r\Phi|_H$ is a QMF on H of weight $n+r$.

Furthermore, the Fourier coefficients of $D_r\Phi|_H$ are $\overline{\mathbb{Q}}$-linear combinations of the Fourier coefficients of Φ.

Proof:

- Analysis of branching laws for quaternionic discrete series representations (H.Y. Loke) to find operators.
- Relate $D_r\mathcal{W}_\omega$ to Whittaker functions for H.

Corollary

For m an even integer with $4 \leqslant m \leqslant n$,

$$\frac{L(m, \Pi, \mathrm{Ad})}{< \varphi, \varphi >} \in \pi^{\mathbb{Z}} \times \overline{\mathbb{Q}}$$

Thank you!