Surfaces Associated to Zeros of Automorphic L-functions

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Montgomery's Pair Correlation Conjecture

- Consider the gaps $\gamma - \gamma'$, where γ, γ' are imaginary parts of non-trivial zeros of $\zeta(s)$.

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- For $\alpha \in \mathbb{R}$, let

$$F(\alpha) := F(\alpha, T) = \frac{1}{N(T)} \sum_{0<\gamma'\leqslant T} \sum_{0<\gamma\leqslant T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'),$$

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Montgomery (1973)

Assume RH. Then uniformly for $0 \leqslant \alpha \leqslant 1$,

$$F(\alpha, T) \sim T^{-2\alpha} \log T + \alpha, \quad \text{as} \quad T \to \infty.$$

Montgomery's Pair Correlation Conjecture

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For $0 < \beta_1 < \beta_2 < \infty$,

$$\frac{1}{N(T)} \sum^{*}_{\substack{0<\gamma,\gamma'\leqslant T \\ \beta_1\leqslant\gamma-\gamma'\leqslant\beta_2}} 1 \sim \int_{\beta_1}^{\beta_2} 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 du.$$

Dyson's Observation and Work of Rudnick–Sarnak

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- Rudnick and Sarnak established higher level correlations of non-trivial zeros of general automorphic L-functions under certain natural restrictions. Their work shows that local fluctuations of zeros match the predictions of the GUE model suggested by Dyson.

From a Statistical View Point

- $F(\alpha, T)$ can be viewed as the mean of

$$S\left(X, T, \rho'\right) = \operatorname{Re} \sum_{|\operatorname{Im} \rho| \leqslant T} X^{(\rho-\rho')} w\left(\rho - \rho'\right),$$

as ρ' runs over non-trivial zeros of $\zeta(s)$. Here $X = T^{\alpha}$.

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- Similar questions addressed by work of Miller and Hughes on the distribution of low-lying zeros across a family of L-functions.

A Gaussian Distribution led along $F(\alpha)$

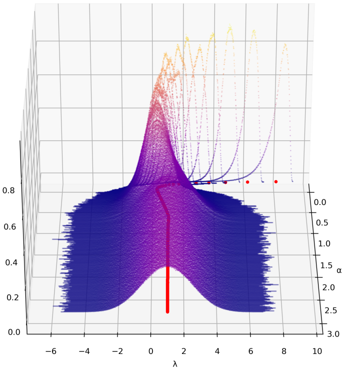

Figure: Surface plot for the PDF associated with the set $\{\operatorname{Re} S(X, T, \rho_n) \colon 1 \leqslant n \leqslant N\}$ for $\zeta(s)$, $N = 10^6$, $X = T^\alpha$ and $\alpha \in (0, 3]$ in a discrete equi-spaced manner.

New Results

Theorem (B.–Castillo–Zaharescu)

Let π be an irreducible cuspidal automorphic representation of GL_m over $\mathbb{Q}$ with unitary central character. Assume RH and some *mild* hypothesis on the coefficients of $L(s,\pi)$. Let $r \in \mathbb{N}$ and

$$0 < \alpha < \frac{1}{mr(1 + \frac{4}{3}\theta_m)},$$

where θ_m is an admissible exponent towards GRC. Then

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\widetilde{S}_\pi(X, T, \gamma_\pi))^r = \mu_r + o(1).$$

Here $X = T^{\alpha m}$ and μ_r are the Gaussian moments.

A Pair Correlation Surface

PCS Associated to Zeros of $L(s, \pi)$

Fix $\alpha \in \mathbb{R} \setminus \{0\}, \lambda \in \mathbb{R}$, and let

$$g_\pi(\alpha, \lambda) := \lim_{\delta \to 0} \frac{1}{2\delta} \lim_{T \to \infty} \frac{1}{N_\pi(T)} D_\pi(\lambda, \delta, T)$$

where $D_\pi(\lambda, \delta, T) := \# \left\{ |\operatorname{Im}(\rho'_\pi)| < T : \widetilde{S}_\pi(T^{|\alpha| m}, T, \rho'_\pi) \in [\lambda - \delta, \lambda + \delta] \right\}.$

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Conjecture (Existence of PCS)

We have

$$g_\pi(\alpha, \lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2}.$$

A Pair Correlation Surface

PCS for $\zeta(s)$

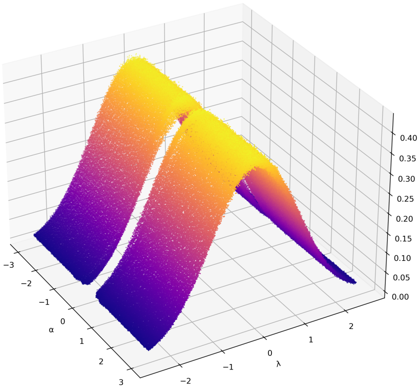


Figure: Pair Correlation Surface for $\zeta(s)$.

A Pair Correlation Surface

PCS for $L(s, \Delta)$ and $L(s, E)$

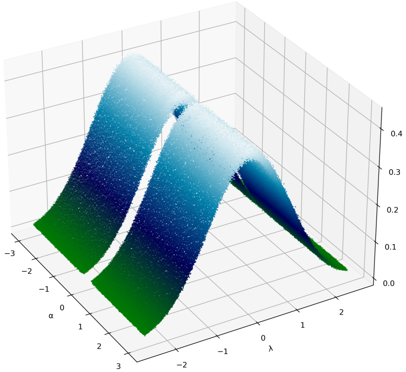
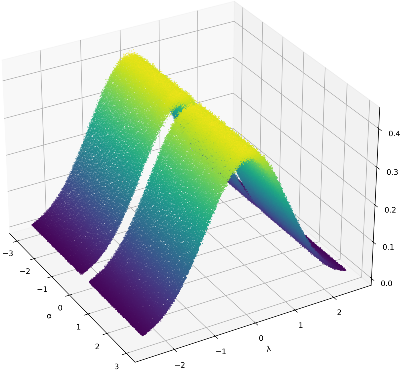


Figure: PCS for zeros of $L(s, \Delta)$, $|\alpha| > 0.2$



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Additional Remarks

- We also establish unconditional results using zero density estimates, albeit on much weaker ranges.

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- For triple correlation, there are two parameters: α_1 and α_2. The moments in the case are *not* Gaussian.
- There is an unexpected phase transition as we move from $\alpha_1 \neq \alpha_2$ to $\alpha_1 = \alpha_2$. The distribution changes from Laplace to Chi-Squared.

A Triple Correlation Surface

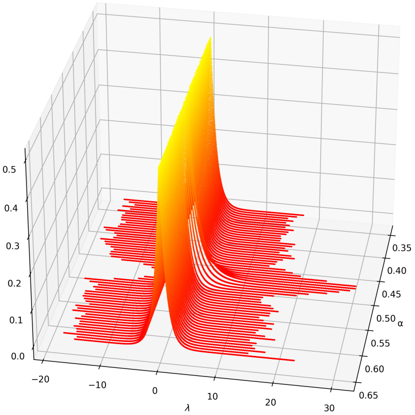

Figure: Triple Correlation Surface for $\zeta(s)$.

Thank you!