Quantum unique ergodicity for Hilbert modular forms on shrinking sets

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$SL(2,\mathcal{O}_k)$ and $\mathcal{H}^n$

- k totally real number field.
- k has narrow class number 1.
- $n=[k:\mathbb{Q}]$, D discriminant of k, R regulator of k, d volume of $\mathcal{O}_k$.
- σ $\mathbb{Q}$-morphism $\Rightarrow \sigma : k \hookrightarrow \mathbb{R}$ thus $\sigma : SL(2,\mathcal{O}_k) \hookrightarrow SL(2,\mathbb{R})$.

Consider $\mathcal{H}=\{z\in\mathbb{C} \mid \Im(z)>0\}$. Then $SL(2,\mathbb{R})$ acts on $\mathcal{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

Thus $SL(2,\mathbb{R})^n$ acts on $\mathcal{H}^n$. There are n $\mathbb{Q}$-morphisms:

$$SL(2,\mathcal{O}_k) \hookrightarrow SL(2,\mathbb{R})^n \circlearrowright \mathcal{H}^n$$

$SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n$ as a Riemannian manifold

We may endow $SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n$ with the hyperbolic metric:

$$g = \sum_{j=1}^{n} \frac{dx_j^2 + dy_j^2}{y_j^2}$$

The geometry defined by g gives us the following concepts:

- A Laplace-Beltrami operator $\Delta = \sum_{j=1}^{n} \Delta_j$ where $\Delta_j = y_j^2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$.
- A hyperbolic measure $\mu = \frac{\lambda_{2n}}{\prod_{j=1}^{n} y_j^2}$ where λ_{2n} is the Lebesgue measure on $\mathbb{R}^{2n}$.

Hilbert modular forms

A function f defined on $\mathcal{H}^n$ is a Hilbert modular form if:

1. f is of moderate growth.
2. $\forall j \in [\![1, n]\!], \exists a \in \mathbb{C}, \Delta_j f = af$.
3. f is $SL(2, \mathcal{O}_k)$-invariant.

The simplest Hilbert modular forms we can construct are Eisenstein series, for $z \in \mathcal{H}^n$, $\Re(s) > 1$ and $m \in \mathbb{Z}^{n-1}$:

$$E(z, s, m) = \sum_{[\gamma] \in Stab(\infty) \backslash SL(2, \mathcal{O}_k)} \prod_{j=1}^{n} \Im(\gamma_j z_j)^{s + i\rho_j(m)}$$

$\rho_j(m)$ is such that $\prod_{j=1}^{n} y_j^{s+i\rho_j(m)}$ is $Stab(\infty)$ invariant.

Quantum unique ergodicity

- X Riemannian manifold.
- f_n normalized eigenfunctions of Δ for the eigenvalues α_n.
- $\forall n \in \mathbb{N}, \alpha_n \leq \alpha_{n+1}$

For $A \subset X$, we would like to prove:

$$\frac{1}{\mu(A)} \int_A |f_n(x)|^2 \, d\mu(x) \underset{n \to +\infty}{\longrightarrow} \frac{1}{\mu(X)}$$

- $\Gamma < SL(2, \mathbb{R})$, $\Gamma \setminus \mathcal{H}$ compact: Lindenstrauss (2006)
- $\Gamma < SL(2, \mathbb{R})$, $\Gamma \setminus \mathcal{H}$: Soundararajan (2010)

Our question

Let us denote $E_{t,m} = E\left(\cdot, \frac{1}{2} + it, m\right)$, $\mu_{t,m} = \left|E_{t,m}\right|^2 \mu$. For shrinking sets $A(t)$ and $B(t)$, we wish to prove that:

$$\frac{\mu_{t,m}(A(t))}{\mu_{t,m}(B(t))} \underset{t\to\infty}{\sim} \frac{\mu(A(t))}{\mu(B(t))}$$

- Truelsen (2007): Fixed sets.
- Young (2013): $k = \mathbb{Q}$

In what follows we will adapt Young's approach to our setting.

Involving functions of compact support

Let $A(t)$ be a family of shrinking sets whose radii goes to 0, f_t smooth such that $f_t \approx 1_{A(t)}$.

$$\begin{aligned}\mu_{t,m}(A(t)) &= \int_{SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n} 1_{A(t)}(z)\,|E_{t,m}(z)|^2\,d\mu(z)\\ &\approx \int_{SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n} f_t(z)\,|E_{t,m}(z)|^2\,d\mu(z)\\ &= \left\langle |E_{t,m}|^2, f_t\right\rangle\end{aligned}$$

Thus our goal is to estimate $\left\langle |E_{t,m}|^2, f_t\right\rangle$.

Spectral decomposition of $\mathcal{L}^2\left(SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n\right)$

By studying the operator Δ we obtain the following Parseval formula, for $f, g \in \mathcal{L}^2\left(SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n\right)$ we have:

$$\langle f, g\rangle = \frac{1}{\mu\left(SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n\right)}\langle f, 1\rangle\langle 1, g\rangle + \sum_{u\in S}\langle f, u\rangle\langle u, g\rangle + \frac{1}{2^{n+1}\pi dR}\sum_{k\in\mathbb{Z}^{n-1}}\int_{\mathbb{R}}\langle f, E_{l,k}\rangle\langle E_{l,k}, g\rangle\, d\lambda(l)$$

We would like to apply this to $|E_{t,m}|^2$ and f_t but $|E_{t,m}|^2$ is not in $\mathcal{L}^2\left(SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n\right)$...

Zagier's renormalization theory

- We adapted Zagier's results (1981).
- $\forall \eta \neq 0, \exists \xi_\eta$ explicit linear combination of Eisenstein series, such that $E_{t+\eta,m}E_{-t,-m} - \xi_\eta \in \mathcal{L}^2\left(SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n\right)$.

Applying Parseval's formula to $E_{t+\eta,m}E_{-t,-m} - \xi_\eta$ and f_t:

$$\left\langle \left|E_{t,m}\right|^2, f_t \right\rangle$$

$$= \lim_{\eta\to 0} \langle \xi_\eta, f_t \rangle + \sum_{u\in S} \left\langle \left|E_{t,m}\right|^2, u \right\rangle \langle u, f_t \rangle$$

$$+ \frac{1}{2^{n+1}\pi dR} \sum_{k\in\mathbb{Z}^{n-1}} \int_{\mathbb{R}} \lim_{\eta\to 0} \langle E_{t+\eta,m}E_{-t,-m} - \xi_\eta, E_{l,k} \rangle \langle E_{l,k}, f_t \rangle \, d\lambda(l)$$

Estimating the discrete and continuous terms

1. $\left\langle |E_{t,m}|^2, u \right\rangle =$ product of L-functions related to u and Gamma functions. We are able to estimate for any N:

$$\sum_{u \in S} \left\langle |E_{t,m}|^2, u \right\rangle \langle u, f_t \rangle \ll \|f_t\|_2 \, a(t)^{\frac{n}{2}} \log(t)^{\frac{4}{3}} (\log\log(t))^{\frac{2}{3}} t^{-\delta} + \|f_t\|_1 \, t^{-N}$$

2. $\langle E_{t+\eta,m} E_{-t,-m} - \xi_\eta, E_{l,k} \rangle =$ product of L-functions and Gamma functions. We should have (ongoing work):

$$\sum_{k \in \mathbb{Z}^{n-1}} \int_{\mathbb{R}} \lim_{\eta \to 0} \langle E_{t+\eta,m} E_{-t,-m} - \xi_\eta, E_{l,k} \rangle \langle E_{l,k}, f_t \rangle \, d\lambda(l) \ll \|f_t\|_2 \, t^{-\frac{n}{6}+\varepsilon}$$

Bounding the main term

By using the explicit expression of ξ_η and the residue of $E(z, \cdot, 0)$ at 1 we are able to estimate:

$$\lim_{\eta \to 0} \langle \xi_\eta, f_t \rangle = \frac{n\pi^n 2^{n-2} R}{D\zeta_k(2)} \log\left(\frac{1}{4} + t^2\right) \int_{SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n} f_t(z) d\mu(z)$$

$$+ \mathcal{O}(\|f_t\|_1 \log(t)^{\frac{2}{3}} \log(\log(t))^{\frac{1}{3}}) + \mathcal{O}(t^{-N})$$

Here ζ_k is the Dedekind zeta function of k.

Conclusion

Combining the previous estimates we obtain:

$$\left\langle |E_{t,m}|^2, f_t \right\rangle = \frac{n\pi^n 2^{n-2} R}{D\zeta_k(2)} \log\left(\frac{1}{4} + t^2\right) \int\limits_{SL(2,\mathcal{O}_k)\backslash\mathcal{H}^n} f_t(z) d\mu(z)$$
$$+ \mathcal{O}(\|f_t\|_2 \, a(t)^{\frac{n}{2}} \log(t)^{\frac{4}{3}} (\log\log(t))^{\frac{2}{3}} t^{-\delta})$$
$$+ \mathcal{O}(\|f_t\|_1 \log(t)^{\frac{2}{3}} \log(\log(t))^{\frac{1}{3}}) + \mathcal{O}(t^{-N})$$

The first term is the dominating term $\Rightarrow$ QUE.
We need to choose appropriate functions f_t.

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Thank you for your attention!