# $\mathfrak{n}$ cohomology of limits of discrete series and GGP type branching laws

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# Introduction

$G :=$ a semisimple linear $\mathbb{Q}$ algebraic group

$\mathbb{A}_{\mathbb{Q}} :=$ adeles of $\mathbb{Q}$

$T \leq G$ anisotropic maximal torus ($T(\mathbb{R})$ compact)

$$\begin{array}{ccccccc} G(\mathbb{R}) & \geq & K(\mathbb{R}) & \geq & T(\mathbb{R}) & & \\ \mathfrak{g}_{\mathbb{R}} & \geq & \mathfrak{k}_{\mathbb{R}} & \geq & \mathfrak{t}_{\mathbb{R}} & & \\ \mathfrak{g} & \geq & \mathfrak{k} & \geq & \mathfrak{t} & & (\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) \end{array}$$

| | | |
|---|---|---|
| $\Phi$ | $= \Phi(\mathfrak{g}, \mathfrak{t}) \subset \mathfrak{t}^\vee$ | roots |
| $\Phi^+$ | $\subset \Phi$ | positive roots |
| $\rho$ | $= \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ | half sum of positive roots |
| $C$ | | positive Weyl chamber for $\Phi^+$ |
| $\Phi_K$ | | compact roots $\mathfrak{k} = \bigoplus_{\alpha \in \Phi_K} \mathfrak{g}_\alpha$ |
| $\Phi_K^+$ | $= \Phi_K \cap \Phi_K^+$ | positive compact roots |

$\pi = \otimes'_v \pi_v =$ cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character

$$\chi_{\pi_\infty} = \chi_{\lambda+\rho}$$

for $\lambda \in \mathfrak{t}^\vee$ an integral weight.

$\pi$ or $\pi_\infty$ are called

- discrete series
  if $\lambda + \rho \in C$
- limits of discrete series
  if $\lambda + \rho \in \overline{C}$
- totally degenerate limits of discrete series (TDLDS)
  if $\lambda + \rho$ fixed by Weyl group (most degenerate case!)

# n cohomology

# $\mathfrak{n}$ cohomology

From $\Phi^+$ get
$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ with
$\mathfrak{n} =$ negative root space

$$\rightsquigarrow H^*(\mathfrak{n}, \pi_\infty) = \mathfrak{n} \text{ cohomology}$$

Then $\mathfrak{t} \curvearrowright \mathfrak{n}$ and $\mathfrak{t} \subset \mathfrak{g} \curvearrowright \pi_\infty$

$$\rightsquigarrow \mathfrak{t} \curvearrowright H^*(\mathfrak{n}, \pi_\infty)$$

Denote $H^*(\mathfrak{n}, \pi_\infty)_\mu$ to be the $\mu \in \mathfrak{t}^\vee$ weight space

# $\mathfrak{n}$ cohomology

Why $\mathfrak{n}$ cohomology?

When $\pi_\infty$ is a limit of discrete series,

$\mathfrak{n}$ cohomology captures what $(\mathfrak{g}, K)$ cohomology cannot.

# n cohomology

If $\pi_\infty$ is a limit of discrete series then

$$\pi_\infty = \pi(\chi_{\lambda+\rho}, \Psi^+)$$

($\Psi^+$ a choice of positive roots)

**Proposition (L.)**

*If* $\pi_\infty = \pi(\chi_{\lambda+\rho}, \Phi^+)$ *is a nonzero TDLDS then*

$$H^{\#\Phi_K^+}(\mathfrak{n}, \pi_\infty)_{-\lambda} \neq 0$$

We then get a second nonvanishing statement by Serre duality:

**Proposition (L.)**

*If* $\pi_\infty = \pi(\chi_{\lambda+\rho}, \Phi^+)$ *is a nonzero TDLDS then*

$$H^{\#(\Phi^+ - \Phi_K^+)}(\mathfrak{n}, \pi_\infty^\vee)_{\lambda+2\rho} \neq 0$$

# n cohomology

Idea of proof: $\pi(\chi_{\lambda+\rho}, \Phi^+) \neq 0$

$\implies$ none of the $\Phi^+$ simple roots are compact

$\implies$ Vanishing differentials in complex used by Soergel

# Local period integral

# Local period integral

$G = U_n$ quasi split unitary group

$G' = U_{n+1}$ is also quasi split

$G \hookrightarrow G'$

After making suitable choices, define

$$\mathfrak{n} \subset \mathfrak{n}' \text{ as before for } G, G'$$

Let $\pi, \pi'$ be TDLDS representations for $G, G'$ respectively.

# Local period integral

Given inner products

$(,)$ for $\pi_\infty$, and

$(,)'$ for $\pi'_\infty$,

Get local period that occurs in GGP

$$\int_{G(\mathbb{R})} (g\phi', \psi')'(g\phi, \psi) dg$$

for $\phi', \psi' \in \pi'_\infty$ and $\phi, \psi \in \pi_\infty$

This induces a $G(\mathbb{R})$ intertwining map

$$\pi' \to \pi^\vee$$

# Local period integral

From $\pi' \to \pi^\vee$ and $\mathfrak{n} \subset \mathfrak{n}'$, obtain

$$H^*(\mathfrak{n}', \pi'_\infty) \to H^*(\mathfrak{n}, \pi^\vee_\infty)$$

For TDLDS, the main results imply

$$H^q(\mathfrak{n}', \pi'_\infty) \text{ and } H^q(\mathfrak{n}, \pi^\vee_\infty) \neq 0$$

at $q =$ number of noncompact positive roots of $G$.

# Local period integral

Question: is the nonvanishing of the map

$$H^q(\mathfrak{n}', \pi'_\infty) \to H^q(\mathfrak{n}, \pi^\vee_\infty)$$

equivalent to nonvanishing of the local period

$$\int_{G(\mathbb{R})} (g\phi', \psi')'(g\phi, \psi)dg$$

?