# On Shifted Convolution Sums of $\mathrm{GL}(3)$ Fourier Coefficients with Average over Shifts

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# Shifted Convolution Problem

$a(m)$ and $b(n)$ be two “interesting” arithmetic functions.

## Shifted Convolution Problem (SCP)

$$\sum_{N<n\leq 2N} a(n)b(n+h) = \text{Main term} + \text{Error term}$$

Appears in

- Moment of $L$-functions
- Sub-convexity bound of $L$-functions
- Quantum unique ergodicitiy (QUE) problem

Let $V(x)$ be a dyadic smooth function supported in $[1,2]$ satisfying $V^j(x) \ll_j 1$. We may also consider the following variant of the SCP

$$\sum_n a(n)b(n+h)V(n/N).$$

# History

- Let $d(n)$ be the divisor function.

## Shifted Divisor Problem

Ingham ('27), Estermann ('31), Deshouillers-Iwaniec ('82)

$$\sum_{n \leq N} d(n)d(n+h) = \text{Main term} \;+\; \text{Error term}$$

- Let $\lambda_f(n)$ be the Hecke eigenvalues of $f$, a holomorphic/ Maaß Hecke cusp form for $SL(2, \mathbb{Z})$.

## $\mathrm{GL}(2)$ Shifted Convolution Problem

Good ('83), Duke-Friedlander-Iwaniec ('93), Jutila ('96), Blomer (2004), Blomer-Harcos (2008)

$$\sum_{n} \lambda_f(n)\lambda_f(n+h)V(n/N) \ll_{\varepsilon} N^{\eta+\varepsilon}, \quad \eta < 1.$$

# History

- Let $\lambda_f(n)$ be the Hecke eigenvalues of $f$, a holomorphic/ Maaß Hecke cusp form for $SL(2, \mathbb{Z})$.
- $F$ be a Hecke-Maaß cusp form for $SL(3, \mathbb{Z})$ and $A(m, n)$ be its normalized $(A(1,1) = 1)$ Fourier coefficients.
- $d_3(n)$ be the triple divisor function

## $d_3(n) \times \mathrm{GL}(2)$ Shifted Convolution Problem

Pitt('95)

$$\sum_{n \leq N} d_3(n)\lambda_f(rn-1) \ll_\varepsilon N^{71/72+\varepsilon}$$

## $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Shifted Convolution Problem

Munshi (2013)

$$\sum_{n} A(1,n)\lambda_f(n+h)V(n/N) \ll_\varepsilon N^{19/20+\varepsilon}$$

# GL(3) Shifted Convolution Sum

- It is a hard open problem to show cancellation in the Shifted Convolution problem

$$\sum_{n \le N} d_3(n) d_3(n+h) \text{ or } \sum_n A(1,n)A(1,n+h)V\left(\frac{n}{N}\right).$$

- So, one tries to show cancellation with an extra averaging over the shift $h$:

$$\sum_{h \le H} \left( \sum_{n \le N} d_3(n) d_3(n+h) - \text{Main term} \right)$$

or

$$\sum_h \sum_n A(1,n)A(1,n+h)V\left(\frac{h}{H}\right)V\left(\frac{n}{N}\right).$$

# Shifted Convolution sum for $d_3(n)$

## Theorem 1 (Baier-Browning-Marasingha-Zhao, 2012)

*Let $d_3(n)$ be the triple divisor function. Then,*

$$\sum_{h\leq H}\left(\sum_{N<n\leq 2N} d_3(n)d_3(n+h) - \textit{Main term}\right) = O(N^{\varepsilon}(H^2+\sqrt{H}N^{13/12}))$$

- Theorem 1 provides a power saving error term when $N^{1/6+\varepsilon} \leq H \leq N^{1-\varepsilon}$.
- The proof uses results involving the moments of Riemann zeta function which is unavailable for the cuspidal case.

## Theorem 2 (Matomaki-Radziwill-Tao, 2019)

*For* $\log^{10000k\log k} N \leq H \leq N$,

$$\sum_{0<|h|\ll H}\left|\sum_{N<n\leq 2N} d_k(n)d_\ell(n+h) - \textit{Main term}\right| = o\left(HN(\log N)^{k+\ell-2}\right).$$

# GL(3) Shifted Convolution Sum

**Theorem 3 (Harun-Singh, 2024)**

*For* $N^{1/2+\varepsilon} \leq H \leq N$ *and some* $\delta > 0$,

$$\sum_h \sum_n A(1,n)A(1,n+h)V\left(\frac{h}{H}\right)V\left(\frac{n}{N}\right) \ll_\varepsilon N^{1-\delta}H.$$

**Theorem 4 (Dasgupta-Leung-Young, 2024)**

*For* $1 \leq H \leq N^{1/2-\varepsilon}$,

$$\sum_h \sum_n A(1,n)A(1,n+h)V\left(\frac{h}{H}\right)V\left(\frac{n}{N}\right) \ll_\varepsilon \frac{N^{4/3+\varepsilon}}{H^{1/3}} + \sqrt{H}N^{1+\varepsilon}.$$

- Upper bound of Theorem 4 is nontrivial for $H \gg N^{1/4+\varepsilon}$.

# Statement of the Result

- $F$ be a Hecke-Maaß cusp form for $SL(3,\mathbb{Z})$ and $A(m,n)$ be its normalized $(A(1,1)=1)$ Fourier coefficients.

**Theorem 5 (Pal-P. 2025+)**

*For $1 \leq H \leq N^{1/2-\varepsilon}$,*

$$\sum_{h} \sum_{N} A(1,n)A(1,n+h)V\left(\frac{h}{H}\right)V\left(\frac{n}{N}\right) \ll_{\varepsilon} N^{\varepsilon}\left(N^{1/4}H^{5/2} + H^{5/8}N^{17/16}\right).$$

- This upper bound is nontrivial for $H \gg N^{1/6+\varepsilon}$.
- This improves upon the range of Dasgupta-Leung-Young $(H \gg N^{1/4+\varepsilon})$ and matches the range of $H$ of the non-cuspidal case (Baier et al.), for which we can obtain power saving error term.

# Proof

- There are two popular avenues to attack the Shifted Convolution Problem: *Delta Method* and *Spectral Theory*. We have followed the first route.
- We have built up on the work of Aggarwal-Leung-Munshi.

**Tools used**

- Delta method of Duke-Friedlander-Iwaniec
- Voronoi type summation formula for $SL(3, \mathbb{Z})$
- "Linearization" of the phase function
- Dulaity Principle
- Cauchy's inequality ad infinitum

*Thank You*