Analytic orders for Tate–Shafarevich groups and regulators of lower-rank quadratic twisted elliptic curves

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Outline

Definition

Elliptic curve

An elliptic curve E over field K is the set of soultions of an equation of the form

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0,$$

and all coefficients are in K.

Weierstrass form

After linear changing of variables, E can be written in the form

$$y^2 = x^3 + ax + b,$$

with $a, b \in K$. This type of equation is called a Weierstrass normal form.

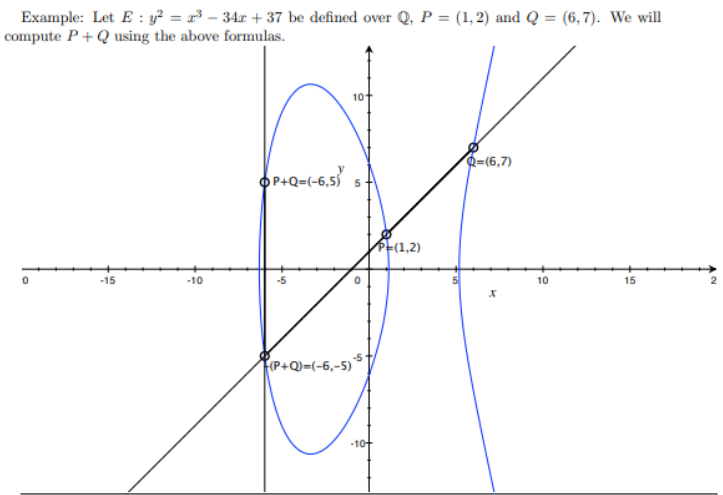
Here we let $\Delta(E) = -16(4a^3 + 27b^2) \neq 0$, then the elliptic curve has curve genus 1.

Group law

Take 2 points P and Q. Now "draw" a straight line through them and compute the third point of intersection R, set infinity be the identity point,

$$P + Q + R = 0,$$

and the inverse of $R = (x, y)$ is $-R = (x, -y)$.



(Image produced by Ashley Neal)

Mordell–Weil group

Mordell–Weil theorem

Let K be an finitely extension field of $\mathbb{Q}$. The group $E(K)$ of K-rational points is a finitely–generated abelian group, called the Mordell–Weil group.

$$E(K) \cong E(K)_{\text{tors}} \times \mathbb{Z}^r,$$

where $E(K)_{\text{tors}}$ is the torsion subgroup, in which every element has finite order, and r is the (algebraic) rank of $E(K)$.

Mazur: $E_{\text{tors}}(\mathbb{Q})$ is one of 15 possible groups
Kenku–Momose, Kamienny: for quadratic field K, $E_{\text{tors}}(K)$ is one of 18 possible groups
The maximal free subgroup of $E(K)$ is a lattice, with the volume $\mathrm{Reg}(E/K)$, with some kind of "canonical height" (Néron–Tate height).

Local-Global obstruction

Global→ Local

For index set I, rational equations

$$f_i(x_1, \ldots, x_n) = 0, \quad (i \in I)$$

have K solutions only if they have solutions in every (∞ or finite) places of K.

Could us go back (Local→ Global)?

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Tate–Shafarevich group

$$Ш(E(\mathbb{Q})) := \cap_v \ker(\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), E) \to \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), E),$$

is the quotient from local to global.

Tate–Shafarevich group

Questions

How large it is? How it looks like?

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Tate–Shafarevich conjecture

For any K and $E(K)$, it's finite group.

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Cassels's pairing

There is an alternating bilinear pairing

$$\langle \cdot, \cdot \rangle : Ш(E) \times Ш(E) \to \mathbb{Q}/\mathbb{Z},$$

which is non-degenerate assuming $Ш(E)$ is finite. Then there exists an isotropic subgroup $H(E) \subseteq Ш(E)$, meaning that $\langle x, y \rangle = 0$ for all $x, y \in H(E)$, with $\#H(E)^2 = \#Ш(E)$, and such subgroups generate $Ш(E)$.

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Gross–Zagier 1986, Kolyvagin 1990

For modular elliptic curves with $\mathbb{Q}$-rational coefficients and analytic rank at most 1, its Tate–Shafarevich group a finite group.

Hasse–Weil L-function of E

Take $a_{\mathfrak{p}} = \mathrm{N}(\mathfrak{p}) + 1 - \#E(\mathcal{O}_K/\mathfrak{p})$, the Hasse–Weil L-function of E over K with central point at 1 is

$$L(s, E/K) = \prod_{\mathfrak{p}} \frac{1}{1 - a_{\mathfrak{p}} \mathrm{N}(\mathfrak{p})^{-s} + \mathrm{N}(\mathfrak{p})^{1-2s}}, \quad \mathrm{Re}(s) \gg 1,$$

Thanks to the modularity theorem (Wiles 1995, Diamond 1996, Conrad–Diamond–Taylor 1999, Breuil–Conrad–Diamond–Taylor 2001), we have the functional equation and analytic continuation for Hasse–Weil L-function for elliptic curves with $\mathbb{Q}$-rational coefficients.

Analytic rank

The vanishing order of $L(s, E/K)$ at central point is the so-called analytic rank.

Gross–Zagier 1986, Kolyvagin 1990

Let $K = \mathbb{Q}$, when analytic rank ≤ 1, we have algebraic rank equals to analytic rank.

Birch and Swinnerton-Dyer conjecture

For E over number field K,

$$\lim_{s \to \text{central}} \frac{L(s, E/K)}{(s - \text{central})^r} = \frac{\#Ш(E/K)\,\mathrm{Reg}(E/K)\Omega_{E/K} \prod_{v \text{ finite}} \mathrm{Tam}_v(E(K))}{(\#E(K)_{\text{tors}})^2},$$

where:

- $r = \mathrm{ord}_{\text{central point } s} L(E/K, s)$ is the analytic rank of E,
- $Ш(E/K)$ is the Shafarevich–Tate group,
- $\Omega_{E/K} = \prod_{v|\infty} \int_{E(K_v)^0} |\omega|$ is the product of the real or complex periods at the archimedean places,
- $\mathrm{Tam}_v(E(K))$ are the Tamagawa numbers at the finite places v of K,
- $E(K)_{\text{tors}}$ is the torsion subgroup of $E(K)$.

Quadratic twists of E

Let $K = \mathbb{Q}$. Let E be an elliptic curve of the form

$$y^2 = x^3 + ax + b.$$

Given square-free d, the quadratic twist of E is the curve $E^{(d)}$, defined by the equation

$$y^2 = x^3 + ad^2x + bd^3.$$

For $\mathrm{Re}(s) \gg 1$

$$L(s, E) = L(s, E/\mathbb{Q}) = \sum_{n=1}^{\infty} a(n)n^{-s} \Rightarrow L(s, E^{(d)}) = \sum_{n=1}^{\infty} a(n)\chi_d(n)n^{-s}$$

For $d < 0$, we have base change L-function

$$L(s, E/\mathbb{Q}(\sqrt{d})) = L(s, E)L(s, E^{(d)}).$$

We normalize the central point to $s = \frac{1}{2}$.

Analytic rank 0: Admissible congruence classes

Let

$$\Omega := \{d : \mu(d)^2 = 1,\ (d, 2N) = 1, \epsilon_{E^{(d)}} = \epsilon_E \chi_d(-N) = 1\}.$$

Let $N_0 = [8, N]$. Let $\sigma \in \{\pm 1\}$ and $a \pmod{N_0}$ denote a residue class with $a \equiv 1 \pmod 4$. We assume that σ and a are such that for any fundamental discriminant d of sign σ with $d \equiv a \pmod{N_0}$, the root number $\epsilon_{E^{(d)}} = 1$. Define

$$\Omega(a, \sigma) := \{d \in \Omega :\ d \equiv a \pmod{N_0},\ \sigma d > 0\}.$$

Analytic rank 0: Analytic order of $Ш(E^{(d)})$

For $d \in \Omega$, then

$$S(E^{(d)}) = L\left(\frac{1}{2}, E_{E^{(d)}}\right) \frac{|E^{(d)}(\mathbb{Q})_{tors}|^2}{\Omega(E^{(d)})\,\mathrm{Tam}(E^{(d)})}.$$

If $L(\frac{1}{2}, E^{(d)}) \neq 0$, then the Birch and Swinnerton-Dyer conjecture predicts that $S(E^{(d)}) = Ш(E^{(d)})$.

Order of real period (Pal 2012)

For square-free $(d, 2N) = 1$, the real period $\Omega(E^{(d)})$ satisfies

$$\Omega(E^{(d)}) = \frac{\tilde{u}}{\sqrt{|d|}}\Omega(E)$$

with some $\tilde{u} \in \frac{1}{2}\mathbb{Z}$ only decided by E.

Radziwiłł–Soundararajan 2015 & 2024

Assuming the BSD conjecture (and the GRH), we have one-sided (two-sided) central limit theorem for $\log\left(Ш(E^{(d)})\,\mathrm{Tam}(E^{(d)})\right)$ with $d \in \Omega(a, \sigma)$.

A barrier

We have $\mathrm{Tam}_p(d) = 1$ for $p \nmid dN_0$, $\mathrm{Tam}_p(d)$ is fixed for $p \mid N_0$ even if d changes, and $\mathrm{Tam}_p(d) \in \{1, 2, 4\}$ for $p \mid d$.

trivial bound

$\tau(d)$ has the extremal large order as

$$\exp(\log 2 \frac{\log d}{\log \log d}).$$

Conjecture (Farmer–Gonek–Hughes 2007)

$$\max_{d \le D} |L(\frac{1}{2}, E^{(d)})| = \exp((1 + o(1))\sqrt{\log D \log \log D}).$$

Extreme analytic order of $Ш(E^{(d)})$

H.–Huang (arXiv: 2212.13360)

For any fixed $W \geq 20$ and any odd a coprime to the conductor of E, we have

$$\max_{\substack{d \in \Omega(a,\sigma) \\ \frac{D}{2} \leq |d| \leq \frac{5D}{2} \\ \omega(d) \leq W}} S(Ш(E^{(d)}) \geq \sqrt{D} \exp\left(\left(2\sqrt{\frac{W - 19.73}{22W + 12}} + o(1) \right) \frac{\sqrt{\log D}}{\sqrt{\log\log D}} \right),$$

as $D \to \infty$.

Reference:

Hua, Shenghao; Huang, Bingrong. Extreme central L-values of almost prime quadratic twists of elliptic curves. *Sci. China Math.* 66 (2023), no. 12, 2755–2766. (Dedicated to the 50th anniversary of Jing-run Chen's Theorem $(1 + 2)$ on the Goldbach Conjecture)

Decorrelation of analytic order of isotropy subgroups of $Ш(E^{(d)})$

H. (arXiv this week)

Let $1 \leq i \leq m$, and suppose each conductor N_i elliptic curve E_i is associate to a weight two Hecke eigenform f_i with root number ϵ_{f_i}. Fix $\sigma = \pm 1$. Let $N_0 = [8, N_1, \ldots, N_m]$. Let $a \bmod N_0$ denote a residue class with $a \equiv 1 \pmod 4$ and $(a, N_0) = 1$ and for every fundamental discriminant $\sigma d > 0$ with $d \equiv a \pmod{N_0}$, the root numbers satisfy $\epsilon_{f_i}(d) = \epsilon_{f_i}\chi_d(-N_i) = 1$, for all $i = 1, \ldots, m$.
Assume the GRH holds for $L(s, \mathrm{sym}^2 f_i)$ and $L(s, f_i \times \chi_d)$ for all f_i and for all $D \leq \sigma d \leq 2D$ with $d \equiv a \pmod{N_0}$, and the BSD conjecture holds for $E_i^{(d)}$. For any $\varepsilon > 0$, we have

$$\frac{1}{D} \sum_{\substack{D \leq \sigma d \leq 2D \\ \mu(|d|)^2 = 1 \\ d \equiv a \pmod{N_0}}} \prod_{i=1}^{m} \frac{\delta_{r(E_i^{(d)}=0)} \# H(E_i^{(d)})}{|d|^{1/4}} \ll_{E_1, \ldots, E_m, \varepsilon} (\log D)^{-\frac{m}{8} + \varepsilon}.$$

Decorrelation of mixed moments for quadratic twisted L-Functions

H. (arXiv this week)

- Decorrelation of quadratic twisted central L-values of even-weight Hecke eigenform or a Hecke–Maass form satisfying the Generalized Ramanujan Conjecture (GRC).
- Decorrelation of automorphic periods averaged over imaginary quadratic fields.
- Decorrelation of Fourier coefficients of half-integral weight modular forms.

Decorrelation of mixed moments for orthogonal families of L-Functions

Keating–Snaith conjecture / Katz–Sarnak philosophy

For orthogonal families of L-functions, the moments of central values in the range between order 0 and 1 (excluding the endpoints) contribute a negative power of $\log$ in their leading-order asymptotics.

Multiple automorphic forms share the same symmetry type

Chandee's work on shifted moments of the Riemann zeta function, and later the work of Milinovich and Turnage-Butterbaugh on integral moments of product of L-functions.

Applications

- Lester–Radziwiłł proved quantum unique ergodicity for half-integral weight automorphic forms.
- Huang–Lester investigated the quantum variance of dihedral Maass forms.
- Blomer–Brumley–Khayutin (newly) proved the joint equidistribution conjecture of Michel and Venkatesh.
- Blomer–Brumley proved arithmetic quotients orbits joint equidistribution.
- Jääsaari–Lester–Saha established sign changes for coefficients of Siegel cusp forms of degree 2.
- Jääsaari–Lester–Saha showed that the mass of Saito–Kurokawa lifted holomorphic cuspidal Hecke eigenforms for $\mathrm{Sp}_4(\mathbb{Z})$ equidistributes on the Siegel modular variety as the weight tends to infinity.
- **H.**–Huang–Li proof a case of our joint Gaussian moment conjecture.
- Huang established holomorphic version.
- Chatzakos–Cherubini–Lester–Risager obtained a logarithmic improvement on Selberg's longstanding bound for the error term in the hyperbolic circle problem over Heegner points with varying discriminants.
- **H.** demonstrated that for $0 < p < 2$, the ℓ^p-norm of some quadratic forms in holomorphic Hecke cusp forms tends to zero asymptotically.

Analytic rank 1: $\mathrm{Reg}(E/\mathbb{Q}(\sqrt{d}))$ and $\mathrm{Reg}(E^{(d)}/\mathbb{Q})$

Leibniz rule for imaginary quadratic field

$$L'(\frac{1}{2}, E/\mathbb{Q}(\sqrt{d})) = L(\frac{1}{2}, E)L'(\frac{1}{2}, E^{(d)}) + L'(\frac{1}{2}, E)L(\frac{1}{2}, E^{(d)})$$

Gross–Zagier formulas

Let E non-CM.
Heegner condition: $d < 0$, $d \equiv 1 \pmod 4$, and $(\frac{d}{q}) = 1$ for all $q \mid N$.
Then for $r(E) + r(E^{(d)}) = 1$, we have

$$L'(\frac{1}{2}, E/\mathbb{Q}(\sqrt{d})) = \frac{32\pi^2 \|f_E\|^2}{|\mathcal{O}^\times_{\mathbb{Q}(\sqrt{d})}|^2 \sqrt{|d|} \cdot \deg \phi_E} \cdot \widehat{h}_{\mathbb{Q}(\sqrt{d})}(P_{\mathbb{Q}(\sqrt{d})}),$$

and for $r(E) = 0$, $r(E^{(d)}) = 1$, we have

$$L'(\frac{1}{2}, E^{(d)}) = \frac{32\pi^2 \|f_E\|^2}{|\mathcal{O}^\times_{\mathbb{Q}(\sqrt{d})}|^2 \sqrt{|d|} \cdot \deg \phi_E} \cdot \widehat{h}_{\mathbb{Q}}(P_d).$$

Analytic rank 1: $\mathrm{Reg}(E/\mathbb{Q}(\sqrt{d}))$ and $\mathrm{Reg}(E^{(d)}/\mathbb{Q})$

H. (arXiv: 2507.20297)

For E non-CM, and d satisfies the Heegner condition.

- Handling some log contribution to get one-sided central limit theory for $\log \mathrm{Reg}(E/\mathbb{Q}(\sqrt{d}))$ and $\log \mathrm{Reg}(E^{(d)}/\mathbb{Q})$.
- Using Hölder's inequality to show lots of d such that $\mathrm{Reg}(E/\mathbb{Q}(\sqrt{d}))$ and $\mathrm{Reg}(E^{(d)}/\mathbb{Q})$ have size $\gg \sqrt{|d|}$.

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$$\Omega_{E/\mathbb{Q}(\sqrt{d})} = \Omega_E^+ \cdot \Omega_{E^{(d)}}^+.$$

Order of imaginary period (Pal 2012)

For negative square-free $(d, 2N) = 1$,

$$\Omega_{E^{(d)}}^+ = \frac{\tilde{u}(d)}{\sqrt{|d|}} c_\infty(E^{(d)}) \Omega_E^-$$

with some $\tilde{u} \in \frac{1}{2}\mathbb{Z}$ decided by E, and $c_\infty(E^{(d)})$ denotes the number of connected components of $E^{(d)}(\mathbb{R})$, which is either 1 or 2.

Uniform boundedness theorem

H. (arXiv: 2507.20297)

For d satisfies the Heegner condition, assuming the BSD conjecture, there exist two constants C_1, C_2 depending on E such that

$$C_1 \leq \#Ш(E/\mathbb{Q}(\sqrt{d})) \prod_{v \text{ finite}} c_v(E/\mathbb{Q}(\sqrt{d})) \leq C_2.$$

Moreover, when $L(\frac{1}{2}, E) \neq 0$, this result also applies to $\#Ш(E^{(d)}/\mathbb{Q}) \prod_{p \text{ prime}} c_p(E^{(d)}/\mathbb{Q})$.

Thank you !