Classification of exceptional hyperspherical data and their duals

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August 12, 2025

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Hypersphercal data

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F: algebraically closed field of characteristic 0.
G: reductive group over F.

The philosophy of Ben-Zvi, Sakellaridis, Venkatesh [BSV24] duality:

$$\begin{array}{ccc} G \circlearrowright M & & M^\vee \circlearrowleft G^\vee \\ \Delta = (G, H, \iota, S) & \longleftrightarrow & \Delta^\vee = (G^\vee, G^\vee_\Delta, \iota^\vee_\Delta, S^\vee_\Delta) \\ \text{Period integrals} & & L\text{-functions} \end{array}$$

Examples: Rankin-Selberg integral, Shalika model, Gan-Gross-Prasad, Trilinear model.

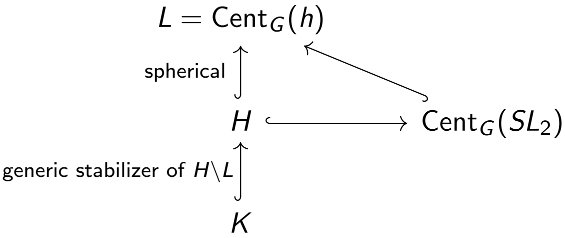
Hyperspherical data

The BZSV quadruple $\Delta = (G, H, \iota, S)$ is the following data:

1. Reductive groups $H \subset G$.
2. An embedding $\iota : SL_2 \to G$ whose image commutes with H.
3. A symplectic H-repn. S.

Hyperspherical data

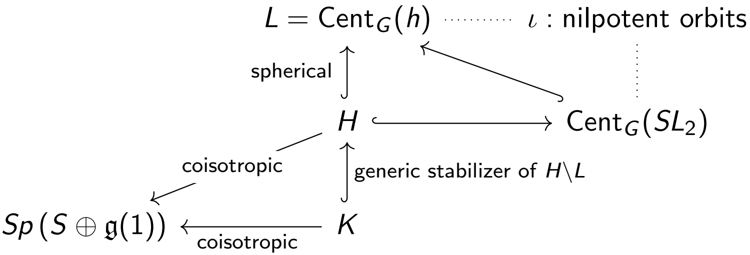
The structure of hyperspherical data $\Delta = (G, H, \iota, S)$:



Here h is a semisimple element in $Lie(\iota(SL_2))$.

Hyperspherical data

The structure of hyperspherical data $\Delta = (G, H, \iota, S)$:



1. h is the semisimple element in the $\mathfrak{sl}_2$-triple.
2. $\mathfrak{g}(1) \subset \mathfrak{g}$ is the weight 1 subspace under the action $Ad(h)$.
3. $S \oplus \mathfrak{g}(1)$ is a "coisotropic" and "anomaly-free" repn. of H and K.

The conjecture of BZSV

[BSV24] expects that the duality:

$$\Delta = (G, H, \iota, S) \longleftrightarrow \Delta^\vee = (G^\vee, G^\vee_\Delta, \iota^\vee_\Delta, S^\vee_\Delta)$$

preserves hypersphericality and anomaly-freeness.

Our work: Fix a group G, find all hyperspherical data and their duals.

Most of the duality statements are conjectural.

Approach 1 - Whittaker induction

[BSV24] reduces the dual of Δ to that of Δ_{red}:

$$\begin{array}{ccc} \Delta = (G, H, \iota, S) & \longleftrightarrow & \Delta^\vee = (G^\vee, G^\vee_\Delta, \iota^\vee_\Delta, S^\vee_\Delta) \\ \downarrow & & \uparrow \\ \Delta_{red} = (L, H, 1, S) & \longleftrightarrow & \Delta^\vee_{red} = (L^\vee, L^\vee_{\Delta_{red}}, \iota^\vee_{\Delta_{red}}, S^\vee_{\Delta_{red}}) \end{array}$$

1. $G^\vee_\Delta = \langle L^\vee_{\Delta_{red}}, SL_{2,\alpha} \mid \alpha \in \Delta_G \setminus \Delta_L \rangle$.
2. $\iota^\vee_\Delta = \iota^\vee_{\Delta_{red}}$.
3. $S^\vee_\Delta$ is generated by highest weights of $S^\vee_{\Delta_{red}}$ via the action of $G^\vee_\Delta$.

Approach 1 - Whittaker induction

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1. $G^\vee_\Delta = \langle L^\vee_{\Delta_{red}}, SL_{2,\alpha} \mid \alpha \in \Delta_G \setminus \Delta_L \rangle$.
2. $\iota^\vee_\Delta = \iota^\vee_{\Delta_{red}}$.
3. $S^\vee_\Delta$ is generated by highest weights of $S^\vee_{\Delta_{red}}$ via the action of $G^\vee_\Delta$.

Corollary: $\quad \mathrm{rank} G - \mathrm{rank} L = \mathrm{rank} G^\vee_\Delta - \mathrm{rank} L^\vee_{\Delta_{red}}$.

Approach 2 - Distinguished polarization

The paper [BSV24] gives an algorithm when Δ is *distinguished polarized*:

1. ι is even, i.e., $\mathfrak{g}(1) = 0$.
2. There is a Lagrangian H-stable decomposition $S = S^+ \oplus S^-$.

The dual data is based on the spherical G-variety:

$$X = (S^+ \times G)/(HU)^{diag}.$$

Table 3 in [KS17] lists the dual group of $\Delta = (G, H, 1, 0)$, where $X = H \backslash G$.

Approach 3 - Strongly tempered data

[MWZ24] proposed the dual of "strongly tempered" hyperspherical data

$$\Delta^\vee = (G^\vee, G^\vee, \iota^\vee_\Delta, S^\vee_\Delta).$$

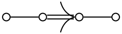
1. H, ι : inferred from [Kno06].
2. S: conjectured in an ad-hoc way.

Finding hyperspherical data

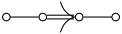
Determine Δ by ι, H, S successively.

1. ι: 16 nilpotent orbits of F_4, see [CM93].
2. H: spherical subgroup of L, see [BP17].
3. $S \oplus \mathfrak{g}(1)$: coisotropic repn. of H, see [Kno06].

Hyperspherical data of F_4

ι		$[\mathfrak{l},\mathfrak{l}]$	$\mathfrak{h}$	$\mathfrak{g}(1) \oplus S$	$\Delta^\vee$
0	0 0 0 0	$\mathfrak{f}_4$	$\mathfrak{f}_4$	0	Trivial.
			$\mathfrak{so}_9$	0	
			$\mathfrak{sp}_6 \oplus \mathfrak{sl}_2$	0	Disconnected.
A_1	1 0 0 0	$\mathfrak{sp}_6$	$\mathfrak{sp}_6$	$\bigwedge_0^3$	Not anomaly-free.
			$\mathfrak{sp}_6$	$\bigwedge_0^3 \oplus \mathrm{St}$	
$\widetilde{A_1}$	0 0 0 1	$\mathfrak{so}_7$	$\mathfrak{so}_6$	$\mathfrak{spin}_6 \oplus \mathfrak{spin}_6^\vee$	
$\widetilde{A_2}$	0 0 0 2	$\mathfrak{so}_7$	$\mathfrak{g}_2$	0	
B_2	2 0 0 1	$\mathfrak{sp}_4$	$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$	St	Not anomaly-free.
B_3	2 2 0 0	$\mathfrak{sl}_3$	$\mathfrak{so}_3$	0	Disconnected.
C_3	1 0 1 2	$\mathfrak{sl}_2$	$\mathfrak{sl}_2$	$\mathrm{St} \oplus \mathrm{St}^\vee$	
F_4	2 2 2 2	0	0	0	Regular.

Hyperspherical data of F_4

ι		$[\mathfrak{l},\mathfrak{l}]$	$\mathfrak{h}$	$\mathfrak{g}(1)\oplus S$	$\Delta^\vee$
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Hyperspherical data of F_4

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			$\mathfrak{sp}_6 \oplus \mathfrak{sl}_2$	0	Disconnected.
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$\widetilde{A_2}$	0 0 0 2	$\mathfrak{so}_7$	$\mathfrak{g}_2$	0	Self dual.
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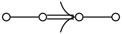
B_3	2 2 0 0	$\mathfrak{sl}_3$	$\mathfrak{so}_3$	0	Disconnected.
C_3	1 0 1 2	$\mathfrak{sl}_2$	$\mathfrak{sl}_2$	$\mathrm{St} \oplus \mathrm{St}^\vee$	$(\mathfrak{so}_9, 0, 0)$
F_4	2 2 2 2	0	0	0	Regular.

Hyperspherical data of F_4

ι		$[\mathfrak{l},\mathfrak{l}]$	$\mathfrak{h}$	$\mathfrak{g}(1) \oplus S$	$\Delta^\vee$
0	$0\ 0\ 0\ 0$	$\mathfrak{f}_4$	$\mathfrak{f}_4$	0	Trivial.
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$\widetilde{A_2}$	$0\ 0\ 0\ 2$	$\mathfrak{so}_7$	$\mathfrak{g}_2$	0	Self dual.
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F_4	$2\ 2\ 2\ 2$	0	0	0	Regular.

Hyperspherical data of F_4

Expectations of hyperspherical duality of F_4:

Trivial $(\mathfrak{f}_4, \mathfrak{f}_4, 0, 0)$	$\longleftrightarrow$	Regular $(\mathfrak{f}_4, 0, F_4, 0)$
$T^*(SO_9\backslash F_4)$ $(\mathfrak{f}_4, \mathfrak{so}_9, 0, 0)$	$\longleftrightarrow$	$T^*(St_2)$ $(\mathfrak{f}_4, \mathfrak{sl}_2, C_3, \mathrm{St} \oplus \mathrm{St}^\vee)$

The rest are self-dual:

1. $(\mathfrak{f}_4, \mathfrak{sp}_6, A_1, \bigwedge_0^3 \oplus \mathrm{St})$,
2. $(\mathfrak{f}_4, \mathfrak{so}_6, \widetilde{A_1}, \mathfrak{spin}_6 \oplus \mathfrak{spin}_6^\vee)$,
3. $(\mathfrak{f}_4, \mathfrak{g}_2, \widetilde{A_2}, 0)$.

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