Workshop: Analysing the Role of Signs in Mathematical Inventions
Aarhus University, Ny Munkegade 118, 8000 Aarhus C | 1–2 June 2026

Program

Monday, June 1- in 1531-113 D1

10:00 Welcome and introduction

10:30 – 11:40 Atsushi Shimojima (Doshisha University, Japan): *Three (or So) Pervasive Properties of Diagrammatic Systems*

12:00-12:45 David Waszek (ENS, France): *Signs and the individuation of mathematical claims*

Lunch

14:00 – 15:10 Véronique Izard (CNRS & U. Paris Cité, France): *Cognitive Foundations of Geometry*

15:15- 16:00 John Mumma (CSU San Bernadino, US): *Essentially Diagrammatic Mathematical subjects*

16:30 -17:40 Valeria Giardino (Institut Jean Nicod, France): *What does the Lascaux's rectangle really tell us?*

18:00 – 20:00 Dinner

Tuesday, June 2 – in 1531-113 D1

9:30-10:40 Jemma Lorenat (Pitzer College, US): *Curves and their readers at the turn of the twentieth century*

10:40-11:25 Danielle Macbeth (Haverford College; US): *Expressive Notation in Mathematical Practice*

12:00-12:45 Sandra Visokolskis (University of Cordoba, Argentina): *New Insights in Mathematics through a Change of Soft Machinery*

Lunch

14:00 – 15:10 Dirk Schlimm (McGill, Canada): *Notational abstractness and its effects*

15:50-16:35 Madeline Muntersbjorn (University of Toledo, US): *From APOS Theory to the Algebra Project: the changing roles of notation in the cultivation of mathematical understanding*

16:40-17:50 Marcus Giaquinto (UCL, UK): A philosophical problem about matrices (with some remarks on definitions).

Abstracts:

Marcus Giaquinto (University College London, UK)

A philosophical problem about matrices (*with some remarks on definitions*).

This talk is about the matrices of linear algebra: What, strictly speaking, are they? This is not a practical problem. You can recognise matrices and manipulate them to solve problems. But there are conflicting views among mathematicians about what constitutes a matrix, as revealed by a survey of their definitions. The problem is philosophical. I will use Alexander Paseau's valuable presentation of the matter and his proposed resolution of the conflict in his 2019 paper "Philosophy of the Matrix" (*Philosophia Mathematica* (III) 25(2) 246-267). Though initially in agreement with Paseau's proposed answer, I became increasingly dissatisfied over time. I will give my reasons for dissatisfaction, argue for an alternative answer, and propose a wider view of mathematical entities than is currently standard.

Valeria Giardino (Institut Jean Nicod, Paris)

What does the Lascaux's rectangle really tell us?

In a very recent book entitled "Le Rectangle de Lascaux. Et Homo sapiens inventa la géometrie" (Lascaux's rectangle. And Homo sapiens invented geometry) (Dehaene, 2026), Stanislas Dehaene defends a view according to which the small rectangle that appears on the walls of the cave at Lascaux, next to the well-known impressive depictions of animals, is an illustration of a fundamental capacity of human cognition, that is, the capacity for formulating symbolic thoughts and on this base for building an infinite pyramid of concepts. I will take Dehaene's view as a starting point to discuss the extent to which we can really talk in terms of a "rectangle" when it comes to the shape that we see in Lascaux; along Wittgensteinian lines, I will argue that to give an account of humans' symbolic powers, it is necessary to consider cognitive capacities, which are surely central, but also specific symbolic practices - in some cases, mathematical practices. For this reason, I will introduce the notion of "diagramming" as the capacity of putting together several cognitive systems in view of some inferential aim and distinguish between different levels at which diagramming recruits spatial cognition and action for higher level tasks. Diagramming is indeed related to a fundamental characteristics of human cognition, that is, representational thought; however, such form of thought presents itself differently depending on the practice at stake and on the cognitive tools involved by such a practice.

Véronique Izard (CNRS & Université Paris Cité, France)

Cognitive foundations of geometry

The research conducted in my lab aims at identifying *cognitive foundations* for mathematics, *i.e.* systems of representations that possess mathematical content and are present in humans prior to any formal education in mathematics. The existence of such cognitive foundations has been well established for one branch of mathematics, arithmetic. Indeed, from the first months of life, young children can perceive numeric quantities and perform additive or multiplicative operations on quantities. These abilities support the acquisition of number concepts later in life, and have been proposed to enable humans' arithmetic cognition. In this talk, I will present two recent studies looking for cognitive foundations for another major branch of mathematics: geometry. The first study focused on Euclidean geometry, and found that children and adults encode a rich repertoire of geometric properties, at several levels of abstraction. The second study probed intuitions for non-Euclidean geometry and revealed the existence of a pervasive Euclidean bias in adults, setting non-Euclidean geometry outside the limits of human spontaneous intuitions. These two studies shed light on the scope and limits of the geometric content of form perception systems.

Jemma Lorenat (Pitzer College, US)

Curves and their readers at the turn of the twentieth century

By the end of the nineteenth century, figures of curves did not speak for themselves. Footnotes and captions stood alongside graphic forms to convey clarifications and justifications. In the study of singularities, geometers employed displacement, enlargement, distortion, and other visual manipulations to direct attention to certain features of a curve, while rendering other at best accidental, at worst misleading. By privileging "distinctness" over illustrative qualities, these drawing practices motivated creating more abstract and enumerative extratextual tools. At the same time, in mathematical statistics purely geometric curves approximated discrete frequency distributions. Here, too, not everything that could be seen was meaningful as statisticians cautioned their readers (a heterogeneous mix). Despite the labor and expense, such figures were deemed necessary in both settings — to provide concrete examples in geometry and to suggest more general hypotheses in statistics. The two disciplines also demanded symmetrical training in curve tracing from equations and curve fitting from data. These simultaneous, contrasting use cases raise questions about the different potentials of geometric figures in pure and applied mathematics.

Danielle Macbeth (Haverford College, US)

Expressive Notation in Mathematical Practice

As one might wonder about the role of signs in mathematical invention so one might wonder about the role of invention in mathematical signs, wonder, that is, what sort of invention, if any, is involved in developing signs for mathematical use. Jourdain suggests an answer in his little monograph *The Nature of Mathematics* (1912). He writes: "the long and strenuous work of the most gifted minds was necessary to provide us with simple and expressive notation which, in nearly all parts of mathematics, enables even the less gifted of us to reproduce theorems which needed the greatest genius to discover" (p. 16). How should we understand this? How, for example, is an expressive notation different from one that is not expressive? And what is it to *reproduce* "theorems which needed the greatest genius to discover"? Are there productions of such theorems that are not equally reproductions? My attempt to answer such questions begins with a distinction that Wittgenstein (early and late) highlights between seeing and seeing-as (or aspect-seeing), a distinction that we can apply not only to mathematical signs and mathematical claims as expressed in such signs but to whole notational systems. So applying the distinction is complicated, however, by the fact that the distinction between seeing and seeing-as can itself be thought about in different ways, under different aspects. We need explicitly to recognize and understand what the possibilities are here. Once we do, our understanding can then be applied to a disagreement that has arisen in relation to numerical signs and their mathematical uses: between those who think that one calculates in, say, the Roman numeration system in just the same sense as one calculates in a system such as that of Arabic numeration, on the one hand, and those who see the two systems of signs as quite different in their expressive power and so in their utility in calculating, on the other. What we find—with some assistance from linguists' reflections on the nature and structure of language and ways we tend to misunderstand it—is that the two parties to the debate are seeing both the signs of the two systems and calculations in such signs in essentially different ways. They are picking up on different aspects of those signs. The hope is that by clarifying the nature and source of the disagreement in this way it might be resolved. And having achieved thereby a better understanding of the nature and possibility of expressive notation in mathematics, we will be better placed to ask the original question: what, if any, is the role of signs in mathematical invention?

Madeline Muntersbjorn (University of Toledo, US)

From APOS Theory to the Algebra Project: the changing roles of notation in the cultivation of mathematical understanding

Ed Dubinsky developed APOS theory as a model of how students cultivate understanding as they learn mathematics. Dubinsky developed his model after experiencing the familiar frustration of teaching students who could execute algorithms correctly but did not really understand what they are doing at a conceptual level. Following Piaget, his model is divided into four stages that take place over time: action, process, object, and schema. Each stage can be subdivided further, and an individual student can be in many stages at once relative to their mathematical proficiency and conceptual understanding of specific content areas. Notation plays an important role at each stage in the learning progression and, significantly, the same notations play different roles for students at different stages. At the action stage, an equation is a thing in the world to be transformed by direct manipulation of its component parts while, after interiorization, the equation becomes the mental representation of a repeatable activity. After encapsulation, the equation can symbolize an object and, ultimately, actions, processes, and objects can be related to one another in a conceptual framework, or schema. As Dubinsky notes, everyone reaches plateaus in the cultivation of their mathematical understanding over time requiring reflection, if not intervention, to move beyond. For this reason, "teaching should consist of helping students use the mental structures they have to develop an understanding of as much mathematics as those available structures can handle. For students to move further, teaching should help them build new, more powerful structures to handle more and more advanced mathematics" (402). What does it mean to say that a structure "can handle" mathematics? Bob Moses' Algebra Project is both a philosophy of mathematics education and a plan of action—a movement and a message. The goal of the movement is to raise the floor of math literacy. The Accessible Calculus Project is an offshoot of the Algebra Project developed by Bill Crombie that argues that treating calculus as the capstone course of a four-year trajectory overlooks opportunities to deepen student understanding of calculus concepts that are implicit in prior algebra and geometry classes: In Dubinsky's words, these courses can handle more mathematics than what is typically taught. Making key concepts, such as rate of change and area under a curve, explicit foci in preparatory classes, means students are better able to make sense of derivatives and integrals as abstract objects with origins in the actions and processes of algebra and geometry. Critically, a system of formal notation is going to be more successful than its rivals if it supports mathematical reasoning at different stages in the cultivation process, from the novice just learning to manipulate extant symbols as things in the world to the expert who innovates new ways of representing their activities, processes, and objects as part of new schemas at the edges of mathematical discovery.

Dubinsky, Ed & Robert P. Moses. (March 2011) "Philosophy, Math Research, Math Ed Research, K–16 Education, and the Civil Rights Movement: A Synthesis." *Notices of the AMS*: 401-409.

Crombie, Bill. (January 2026) "Five Misconceptions about Calculus Access" posted at the Algebra Project website: https://algebra.org/wp-content/uploads/2026/01/Five-Misconceptions-about-Calculus-Access-Combine.pdf

John Mumma (California State University, San Bernardino, US)

Essentially diagrammatic mathematical subjects

In my talk I explore the prospects of identifying the content of a mathematical subject via the means by which new concepts are defined and new theorems proven by practitioners of the subject. I explore, in particular, whether this approach to mathematical content can be fruitfully used to characterize elementary geometry and category theory as *essentially diagrammatic*. The main obstacle to such an understanding of the two subjects is the familiar and well-established fact that any mathematical concept or proof can be presented in propositional form via a logical analysis. That the concepts and proofs of elementary geometry and category theory are subject to logical analysis is standardly taken to show that diagrams are inessential to their content. I confront this view with the equally familiar and well-established fact that logical analysis renders the proofs of elementary geometry and category theory unreadable. It becomes impossible, in particular, to discern the keys steps of the proofs when they are presented in purely propositional form. This I argue gives us reason to locate (part of) the content of the proofs in the diagrams with which they are originally presented.

Dirk Schlimm (McGill, Montreal, Canada)

Notational abstractness and its effects

To represent a subject matter symbolically, we need to decide which aspects of the subject matter are represented by which elements of the notation. This relation can be more or less explicit or implicit. The latter requires more cognitive effort by the user to extract the relevant information, giving rise to a notation that is often considered to be "more abstract." In this talk, I introduce a distinction between notations in terms of their level of abstractness, and, using various examples, discuss the effect on algorithms for symbolic manipulations and on mathematical inventions.

Atsushi Shimojima (Doshisha University, Kyoto, Japan)

Three (or So) Pervasive Properties of Diagrammatic Systems

Some diagrammatic systems are known to have the following properties, which deeply affect their functions as communicative and inferential tools:

1. Expressing a set of information in diagrams can result in the expression of its consequences,
2. Expressing a set of information in diagrams can serve as a basic check of its consistency, and
3. Expressing a set of information in diagrams can mandate the selective expression of other, non-consequential information.

In this talk, I will demonstrate how pervasive these properties are, by considering primitive systems of tabular diagrams, connection charts, quantity charts, and area diagrams. I will also sketch our current attempt to explain the cause of this pervasiveness.

Sandra Visokolskis (National University of Cordoba, Cordoba, Argentina)

New Insights in Mathematics through a Change of Soft Machinery

When dealing with a mathematical problem that lacks an immediately deductible solution using conventional methods, it becomes necessary to construct some kind of new soft machinery (various notations, iconic diagrams, and all kinds of representations of the ideas involved) that allows access to the problem from a novel perspective. Thus, creative originality in these cases stems from a change of format. But how can a change in form affect the underlying matter or content?

In this sense, and seeking to answer this controversial question, this paper aims to demonstrate a technique implicit in an ancient style of mathematical problem-solving that appears to be common in both ancient Greek mathematics and the mathematics produced in Mesopotamia during the Old Babylonian era. While in the Greek case it takes the name "application of areas", in the Babylonian case it results in a kind of latent geometric algebra in rhetorical writings. The focus of this supposed technique, which we will call the "double view procedure", is to highlight the central role of a change in semiotic format when trying to solve a problem that, from the outset, due to its original representation, does not show immediately deducible results. This would lead to appealing to a change in notation or semiotic anchoring, which would ultimately configure a powerful and creative heuristic technique. At the same time, by solving the given problem in terms of a format different from the starting one, it allows for the introduction of original novelty, contributing new insights in mathematics.

The work will proceed with the following methodology: (1) The "double view procedure" will be presented as a method for extracting creative solutions when faced with a problem representation format that leads to aporia, forcing the search for alternative configurations that creatively resolve the problem. (2) This dual mechanism will be described as a transformation of semiotic forms: surrogative, analytical (understanding "analysis" here in the ancient Greek sense as a decompositional process of discovery), abductive (according to the various versions that the American philosopher Charles S. Peirce conceived, the only reasoning that introduces novelty), and theorematic (a term applied by Peirce to a type of innovative deductive reasoning based on ad hoc constructions, thus contemplating the mathematical deductive vein and, at the same time, in some sense, an ampliative one). (3) two paradigmatic case studies of this double-view procedure will be offered: (3.i) concerning the strategy of applying areas that Euclid addresses in *Elements* (Book I, Props. 42-45); and (3.ii) relating to the reformulation of surfaces from one format to another, allowing for immediate and evident resolution of geometric problems applied to everyday life, in Paleo-Babylonian mathematics. In particular, the following Mesopotamian texts will be analyzed: BM 13901 #1, AO 8862 #1, and YBC 7289. Finally (4) both cases will be compared in search of a common link between them, which explains how the applied dual technique is generalizable to multiple mathematical problems, not only from the past but also current ones that are even more sophisticated in their formulations.

David Waszek (École Normale Supérieure, France)

Signs and the individuation of mathematical claims

There is an intuitive sense in which the mathematical signs one uses impact how one individuates mathematical claims. For instance, Manders (1996) argued that the expressive means of traditional Euclid-style geometry and those of 19th-century projective geometry lead to different ways of carving out what should properly count—or what can be handled—as an individual geometrical theorem or proof step. Drawing upon examples of notational differences as well as diagram use, my first task will be to refine and assess the prospects of this idea, in particular by distinguishing matters of expressive power from pragmatic concerns of accessibility and tractability. I will then argue that this raises a difficulty for recent work by Hunt (2021, forthcoming) that aims, in wide generality, at demarcating those cases in which a difference in 'formulation' (including differences in signs or notations) is genuinely 'epistemic', as opposed to merely 'psychological' or 'practical'. Hunt's claim is that differences in formulation are epistemic where they underpin differences in problem-solving plans. The difficulty is that it is not obvious that the relevant problem-solving plans, and the 'inferential relations' that constitute them, can be individuated in a way that does not ultimately rest on the pragmatic concerns that ground language choices in the first place. I conclude by discussing how this difficulty could be solved, and drawing conclusions about possible accounts of mathematical content.

References.

Manders K (1996). Diagram Contents and Representational Granularity. In Seligman J & Westerståhl D (eds), *Logic, Language and Computation*, vol. 1, Stanford: CSLI Publications, pp. 389–404.

Hunt J (2021). Understanding and Equivalent Reformulations. *Philosophy of Science* 88(5), pp. 810–823.

— (forthcoming). On the Value of Reformulating. *Journal of Philosophy*.