Continuous and fractal parabolic Anderson models

From flat space to geometric contexts

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July 5, 2026

Contents

CHAPTER 1

Introduction

These notes are an invitation to the *parabolic Anderson model* (PAM), a stochastic partial differential equation which sits at the intersection between mathematical physics, probability theory and analysis. Our two guiding goals are the following. First, on the Euclidean space $\mathbb{R}^d$, we want to explain what the model is, why physicists and probabilists care about it, and which mathematical phenomena (chiefly *intermittency* and its polymer counterpart, *localization*) it is designed to capture. Second, and this is the more original thread of the course, we want to explain how the model, and the exponents attached to it, are affected when the flat space $\mathbb{R}^d$ is replaced by a genuinely *geometric* background: a sub-Riemannian manifold such as a Heisenberg group, or a fractal such as the Sierpiński gasket.

This introductory chapter has no proofs. Its purpose is to fix the equation once and for all (Section 1.1), to describe the two phenomena that will motivate every estimate proved later (Sections 1.2 and 1.3), to list the analytic tools we shall need (Section 1.4), and finally to explain why the underlying geometry deserves a course of its own (Section 1.5).

Notation 1.0.1

Throughout the course $(\Omega, \mathcal{F}, \mathbb{P})$ denotes an underlying probability space, and $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. We write $a \lesssim b$ when there is an unspecified constant $C > 0$ such that $a \leq C\, b$, and $a \asymp b$ when both $a \lesssim b$ and $b \lesssim a$ hold.

1.1 The parabolic Anderson model

The name of the model goes back to P. W. Anderson's work on the quantum mechanics of disordered media, in which an electron evolves in a random potential and may become spatially *localized* rather than delocalized. The stationary object behind this picture is the random Schrödinger operator, the *Anderson Hamiltonian*

$$\mathcal{H} = \tfrac{1}{2}\Delta + V, \tag{1.1}$$

where Δ is the Laplacian on $\mathbb{R}^d$ and $V = V(x)$ is a random potential. The *parabolic* model is the heat flow generated by (1.1): one studies the large-time behavior of $\exp(t\,\mathcal{H})$, equivalently of the solution to the Cauchy problem $\partial_t u = \mathcal{H}u$. When the static potential V is replaced by a *space-time* noise one is led to the equation we will focus on for the entire course. Writing $\dot{W}$ for the noise, the parabolic Anderson model on $\mathbb{R}_+ \times \mathbb{R}^d$ reads

$$\partial_t u(t,x) = \tfrac{1}{2}\Delta\, u(t,x) + \beta\, u(t,x)\, \dot{W}(t,x), \qquad u(0,x) = u_0(x), \tag{1.2}$$

where $\beta > 0$ is a parameter tuning the intensity of the noise (the inverse temperature, in the language of statistical mechanics).

Three features of equation (1.2) should be emphasized from the outset.

(i) *The noise is multiplicative.* The unknown u multiplies the noise $\dot{W}$ in the right-hand side of (1.2). This is what makes the model genuinely "Anderson": the medium acts on the solution proportionally to the solution itself, which is exactly the mechanism that will produce the tall, rare peaks discussed in Section 1.2. Replacing $u\,\dot{W}$ by $\dot{W}$ alone yields the much tamer *additive* stochastic heat equation, which we shall nonetheless meet in Chapter 3 as an intermediate object.

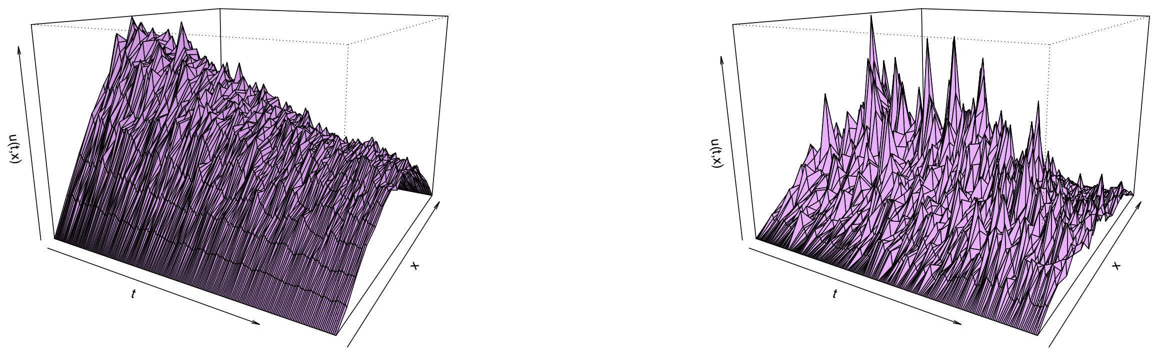
(ii) *The noise is rough.* The typical driving noise $\dot{W}$ is white in time and either white or colored in space. Such a $\dot{W}$ is a random distribution, not a function, so the product $u\,\dot{W}$ in (1.2) has no classical meaning. Giving (1.2) a rigorous sense requires a stochastic integration theory; we shall adopt the Itô–Walsh–Dalang point of view, and solutions will be understood in the Itô (or Itô–Skorohod) sense unless stated otherwise. The competing Stratonovich interpretation, which is the natural one for pathwise and polymer questions, will be commented upon when polymers enter the picture. When the noise is rougher still (for instance the space-time white noise in spatial dimension $d \geq 2$) even this stochastic-integration approach breaks down: the product $u\,\dot{W}$ becomes genuinely ill-posed and the equation must be *renormalized*. Making sense of the resulting renormalized model is one of the achievements of the theory of regularity structures [32]; we refer to [33] for the construction of the continuum parabolic Anderson model on $\mathbb{R}^2$, and to [15] for moment estimates in this renormalized setting. These pathwise techniques fall outside the scope of these notes, in which we shall stay within the range of regularity for which the Itô–Walsh–Dalang theory applies.

(iii) *The regularity of the noise is a parameter.* A recurring theme is that the spatial regularity of $\dot{W}$ (encoded later by a parameter α) decides both *whether* equation (1.2) can be solved at all and *how* its solution behaves in large time. The interplay between this regularity and the geometry of the underlying space is, in a nutshell, the subject of these notes.

1.2 Intermittency

The single most striking feature of the parabolic Anderson model is *intermittency*. Loosely speaking, an intermittent field is one whose mass, instead of spreading out as it would under the plain heat equation, concentrates as time grows on a few small, widely separated islands where it reaches huge heights. This is precisely the behavior observed for magnetic fields in turbulent astrophysical plasmas, the context in which intermittency in random media was popularized by Zel'dovich and his co-authors [54]. The mathematical theory of the parabolic Anderson model was then founded by Carmona and Molchanov [12], and is surveyed in [26, 39].

Figures 1.1 and 1.2 illustrate the phenomenon on simulations of the solution $u(t,x)$ of (1.2) for increasing values of the noise intensity λ. For a small intensity the surface is only a mild random perturbation of the smooth heat-equation profile (Figure 1.1). As λ grows, the profile is progressively reshaped, until for $\lambda = 2$ almost all of the mass has collapsed onto a handful of sharp, isolated spikes against an essentially flat background. This is the clearest manifestation of intermittency (Figure 1.2).



$\lambda = 0.1$ $\lambda = 0.5$

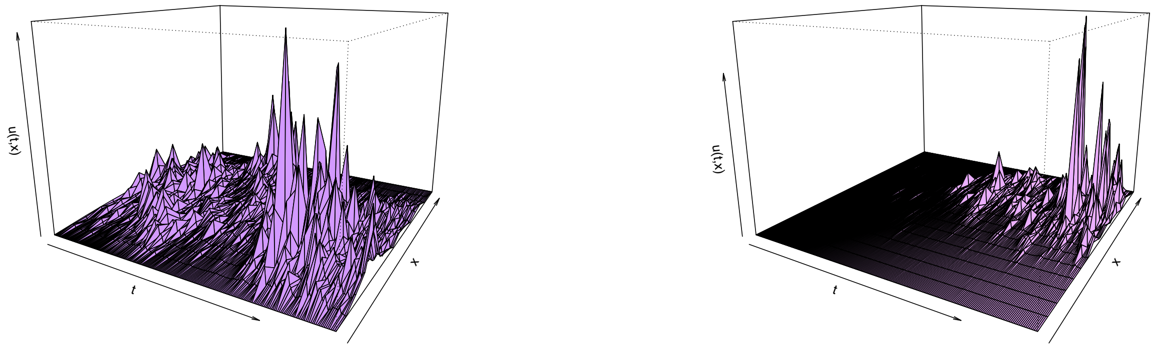
Figure 1.1 – The solution $u(t,x)$ of (1.2) for small noise intensity λ: only a mild random perturbation of the smooth heat-equation profile.

The quantitative counterpart of this picture is phrased in terms of the moments of the solution. For $p \geq 1$ one defines the p-th *moment Lyapunov exponent* (or upper Lyapunov exponent) by

$$\gamma(p) := \limsup_{t\to+\infty} \frac{1}{t} \ln \mathbb{E}[\,u(t,x)^p\,]\,. \tag{1.3}$$

Whenever the relevant quantities exist and are finite, the model is said to be (fully) *intermittent* if the normalized exponents are strictly increasing, that is

$$\frac{\gamma(1)}{1} < \frac{\gamma(2)}{2} < \cdots < \frac{\gamma(p)}{p} < \cdots\,. \tag{1.4}$$



$\lambda = 1$ $\lambda = 2$

Figure 1.2 – The same solution for larger noise intensity λ: the mass concentrates on a few tall, well-separated peaks (the onset of intermittency).

The strict inequalities in (1.4) are the analytic signature of the spikes seen in Figure 1.2: higher moments are dominated by the rare, very high peaks, so they grow strictly faster, on the exponential scale, than any power of the lower moments would predict. Establishing two-sided bounds on the exponents (1.3) is one of the central goals of the course, and is the subject of Chapter 4.

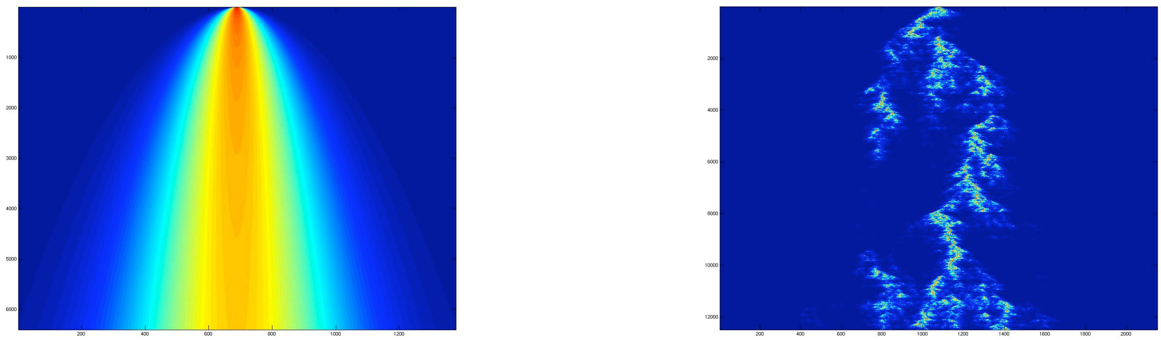
1.3 Directed polymers and localization

There is a second, equivalent way of reading equation (1.2), which comes from statistical mechanics and which gives intermittency a vivid geometric meaning. By the Feynman–Kac formula, the solution of (1.2) can be represented as

$$u(t,x) \;=\; \mathbb{E}_B\Big[\, u_0(B_t)\, \exp\Big(\textstyle\int_0^t \beta\, \dot W(t-s, B_s)\, ds \Big) \Big], \tag{1.5}$$

where B is a Brownian motion started at x and $\mathbb{E}_B$ is the expectation over B only, the noise $\dot W$ being frozen. The right-hand side of (1.5) is the *partition function* of a *directed polymer*: a Brownian path is reweighted by the exponential of the energy it collects along the random environment $\dot W$. Normalizing this weight defines the *polymer measure*, a random probability measure on paths under which favorable regions of the environment are visited more often. The parameter β in (1.2) is then literally an inverse temperature.

This dictionary turns the analytic question of intermittency into a pathwise question about the polymer, organized around the classical dichotomy between *weak* and *strong* disorder. In the weak-disorder regime the energy collected along the path is too weak to overcome the entropy of Brownian motion: the polymer measure stays close to the Brownian one and the path remains *diffusive*, spreading like $\sqrt{t}$. In the strong-disorder regime energy wins over entropy: the path *localizes*, concentrating on a thin corridor of exceptionally favorable sites. The two regimes are displayed in Figure 1.3, which shows the weight of the endpoint of the polymer as a function of space (horizontal) and time (vertical).



(a) diffusive behavior (b) localized behavior

Figure 1.3 – The weight of the polymer endpoint in space (horizontal) and time (vertical). *(a)* In the weak-disorder regime the weight spreads parabolically: diffusive behavior. *(b)* In the strong-disorder regime the weight concentrates on a thin wandering filament: localization.

Directed polymers have a long and rich history, both in the discrete setting of random walks on $\mathbb{Z}^d$ and in the continuous setting underlying equation (1.2); we refer to the Saint-Flour lecture notes of Comets [16] for a panorama, and to [17, 40, 46, 51] for results of the type that will guide us. The polymer viewpoint, and the localization it makes visible, is one of the main reasons the parabolic Anderson model is worth studying beyond the flat case: the localization corridor of Figure 1.3 (b) is a path in the underlying space, and its behavior must therefore depend on the geometry of that space.

1.4 Overview of the toolkit

The analysis of equation (1.2) draws on a small but sophisticated toolbox, which we now list. Each item will be developed in its own right later in the course; the purpose here is only to advertise the methods and to indicate where they will be used.

(i) *Stochastic calculus in infinite dimensions.* Giving a meaning to the product $u\dot{W}$ in (1.2) requires a theory of stochastic integration with respect to a space-time noise. We shall use the random-field approach of Walsh and Dalang [18, 53], together with the Malliavin calculus [48] when the Itô–Skorohod interpretation is in force. This is the content of Chapters 2 and 3.

(ii) *The Feynman–Kac formula.* Representation (1.5) is the bridge between the equation and the polymer, and is the main vehicle for moment estimates. See Chapter 4.

(iii) *Intersection local times.* Computing the moments $\mathbb{E}[u(t,x)^p]$ through (1.5) leads to expressions involving the mutual intersection of p independent copies of the underlying Brownian motion. The corresponding intersection local times control the exponents (1.3).

(iv) *Wiener chaos expansions.* Because the noise is Gaussian, the solution of (1.2) can be expanded in a series of multiple stochastic integrals. This chaos expansion yields both existence-uniqueness results and a route to moment formulas; see Chapter 3.

(v) *Heat semigroup estimates.* Every quantity above is ultimately governed by the heat kernel $p_t(x,y)$ of the underlying Laplacian. Sharp upper and lower bounds on p_t are therefore the technical backbone of the whole theory (and, as we explain next, the precise point at which the geometry of the space enters).

1.5 From Euclidean space to geometric structures

On the flat space $\mathbb{R}^d$ the model (1.2) is by now fairly well understood, and the foundational analytic results we shall rely on go back to [11, 12, 25] and, in the stochastic-PDE formulation, to [34, 35]. The central question of this course is what survives, and what changes, when $\mathbb{R}^d$ is replaced by a space whose geometry is non-trivial.

The mechanism by which the geometry enters has already been named: it is the heat kernel. On $\mathbb{R}^d$ the kernel is Gaussian, $p_t(x,y) \asymp t^{-d/2}\exp(-|x-y|^2/2t)$, and the single number d controls everything. On a geometric space the Gaussian bound is replaced by a sub-Gaussian one,

$$p_t(x,y) \asymp t^{-d_h/d_w} \exp\left(-C\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right), \tag{1.6}$$

in which *two* exponents now appear: the Hausdorff dimension d_h, governing the growth of the volume of balls, and the walk dimension $d_w \geq 2$, governing how fast the Brownian motion escapes them. The Euclidean case is recovered for $d_h = d$ and $d_w = 2$, whereas on the Sierpiński gasket, for instance, one has $d_h = \ln 3/\ln 2$ and $d_w = \ln 5/\ln 2$. The combination that turns out to govern the parabolic Anderson model is the *spectral dimension*

$$d_s := \frac{2\,d_h}{d_w}, \tag{1.7}$$

which measures the growth of the eigenvalues of the Laplacian.

The effect of replacing d by the pair (d_h, d_w) is not cosmetic. Two concrete instances will recur throughout the course.

(i) *Sub-Riemannian spaces.* On the Heisenberg group $\mathbb{H}^n$ the sub-Riemannian structure is degenerate, and the relevant notion of dimension is no longer the topological dimension $d = 2n+1$ but the Hausdorff dimension $Q = 2n+2$. The threshold of noise regularity for which equation (1.2) can be solved shifts accordingly: written in terms of Q it coincides with the Euclidean condition, but written in terms of d it does not, an effect that can be traced back entirely to the degeneracy of the geometry [8, 9].

(ii) *Fractals.* On post-critically finite fractals the exponents d_h and d_w take a whole range of non-classical values, and with them the Lyapunov exponents (1.3) acquire a correspondingly rich dependence on the noise regularity and on the boundary conditions [10]. In particular the condition $d_s < 2$, i.e. $d_h < d_w$, makes even the space-time white noise admissible in (1.2), a phenomenon with no Euclidean analogue in dimension $d \geq 2$.

These two examples are the destination of the course. The route to them is as follows. Chapter 2 sets up the Gaussian noises and measures their regularity in Besov scales. Chapter 3 builds the stochastic integral and proves existence and uniqueness for the additive equation and then, by a contraction argument, for the parabolic Anderson model itself, together with its chaos expansion. Chapter 4 is devoted to the Feynman–Kac representation of moments and to the resulting estimates on the Lyapunov exponents (1.3). Finally Chapters 5 and 6 carry the whole programme over from $\mathbb{R}^d$ to the hypoelliptic and fractal settings outlined above. The former sets up the geometric framework, while the latter develops the corresponding PAM theory.

CHAPTER 2

Gaussian noises and their regularity

2.1 Wiener space and Malliavin calculus

In this course, the noise $\dot{W}$ driving the PAM equation (1.2) will be a random distribution. We will thus see it as a random linear functional on suitable test functions. The Gaussian Hilbert space structure of this noise, together with the Malliavin calculus built on top of it, constitute the basis of all subsequent analysis. This section sets up that framework. The canonical references are Nualart [48] for the Malliavin calculus, and Hu, Huang, Nualart and Tindel [35] for the specific Gaussian noises we work with.

2.1.1 The driving noise

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of Notation 1.0.1, the space-time noise W is encoded by a centered Gaussian family $\{W(\varphi);\, \varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^d)\}$. Throughout this course the noise is *white in time* and correlated in space only, with spatial covariance function $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$:

$$\mathbb{E}\Big[W(\varphi)\, W(\psi)\Big] = \int_{\mathbb{R}_+ \times \mathbb{R}^{2d}} \varphi(t,x)\, \psi(t,y)\, \Lambda(x-y)\, dx\, dy\, dt. \tag{2.1}$$

We assume throughout that Λ is non-negative definite, so that its Fourier transform $\mu := \mathcal{F}\Lambda$ is a non-negative tempered measure on $\mathbb{R}^d$, called the *spectral measure* of the noise. For the stochastic integrals against W to be well-defined in $L^2(\Omega)$, the spectral measure must not charge high frequencies too heavily: the heat kernel p_t, whose Fourier transform is $e^{-t|\xi|^2/2}$, must belong to $\mathcal{H}$ for each $t > 0$, which requires $\int_{\mathbb{R}^d} e^{-t|\xi|^2} \mu(d\xi) < \infty$. A uniform version of this requirement, sufficient for all $t \geq 0$, is Dalang's condition [18].

Assumption 2.1.1

The spectral measure μ satisfies Dalang's condition:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{2.2}$$

This integrability condition is exactly what is needed for the stochastic integral with respect to W to be well-defined (Chapter 3).

The natural Hilbert space in which W acts as an isometry is built by completing the space of test functions under the inner product induced by the covariance (2.1). Let thus $\mathcal{H}$ be the completion of $\mathcal{D}([0,\infty)\times\mathbb{R}^d)$ with respect to the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}_+\times\mathbb{R}^{2d}} \varphi(t,x)\,\psi(t,y)\,\Lambda(x-y)\,dx\,dy\,dt. \tag{2.3}$$

Equivalently, writing $\mathcal{F}_x$ for the Fourier transform in the space variable, one can write (2.3) as

$$\langle \varphi, \psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}_+\times\mathbb{R}^{d}} \mathcal{F}_x\varphi(t,\xi)\,\overline{\mathcal{F}_x\psi(t,\xi)}\,\mu(d\xi)\,dt. \tag{2.4}$$

The covariance identity $\mathbb{E}[W(\varphi)W(\psi)] = \langle\varphi,\psi\rangle_{\mathcal{H}}$ means that $\varphi \mapsto W(\varphi)$ extends to a linear isometry

$$W : \mathcal{H} \longrightarrow L^2(\Omega), \qquad \mathbb{E}[W(h)\,W(k)] = \langle h, k\rangle_{\mathcal{H}}, \tag{2.5}$$

and throughout the course we will work with Wiener integrals written as

$$W(h) = \int_0^\infty \int_{\mathbb{R}^d} h(s,x)\,W(ds,dx), \quad \text{for} \quad h\in\mathcal{H}.$$

Example 2.1.2: Riesz kernel

The primary example throughout the course is the η-*Riesz spatial covariance*, defined for $0<\eta<d$ by

$$\Lambda(x) = |x|^{-\eta}, \qquad \mu(d\xi) = c_{\eta,d}\,|\xi|^{-(d-\eta)}\,d\xi, \tag{2.6}$$

where $c_{\eta,d} > 0$ is a normalizing constant. Assumption 2.1.1 holds precisely when $0<\eta<2$ (Lemma 2.3.3 in Section 2.3). The limiting case $\eta\to 0$ corresponds informally to white noise in space.

2.1.2 Isonormal Gaussian processes and Wiener chaos

The isometry (2.5) is the defining property of an abstract class of Gaussian processes that unifies the examples above.

Definition 2.1.3: Isonormal Gaussian process

Let $\mathcal{H}$ be a separable real Hilbert space. A centered Gaussian family $W = \{W(h);\ h \in \mathcal{H}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called an *isonormal Gaussian process on* $\mathcal{H}$ if

$$\mathbb{E}[W(h_1)\,W(h_2)] = \langle h_1, h_2\rangle_{\mathcal{H}} \qquad \text{for all } h_1, h_2 \in \mathcal{H}. \tag{2.7}$$

The noise of Section 2.1.1 is an isonormal Gaussian process on $\mathcal{H}$. The Hilbert space structure of $\mathcal{H}$ supports a canonical orthogonal decomposition of $L^2(\Omega)$, whose building blocks are defined via Hermite polynomials. Recall that the n-th *probabilists' Hermite polynomial* H_n is characterized by

$$\exp\Big(tz - \frac{t^2}{2}\Big) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, H_n(z), \qquad t, z \in \mathbb{R}, \tag{2.8}$$

giving the first values

$$H_0(z) = 1, \qquad H_1(z) = z, \qquad H_2(z) = z^2 - 1, \qquad H_3(z) = z^3 - 3z. \tag{2.9}$$

Since $W(h)$ is a standard Gaussian random variable for every unit vector $h \in \mathcal{H}$, the composition $H_n(W(h))$ is a well-defined element of $L^2(\Omega)$ for each $n \geq 0$. Varying h over all unit vectors and closing in $L^2(\Omega)$ produces, for each degree n, a distinguished closed subspace that captures the "degree-n part" of the Gaussian structure.

Definition 2.1.4: Wiener chaos

For $n \geq 1$, the *n-th homogeneous Wiener chaos* $\mathbf{H}_n$ is the closed subspace of $L^2(\Omega)$ generated by

$$\Big\{H_n(W(h));\ h \in \mathcal{H},\ \|h\|_{\mathcal{H}} = 1\Big\}. \tag{2.10}$$

We set $\mathbf{H}_0 := \mathbb{R}$ (the constants). The subspaces $\{\mathbf{H}_n\}_{n\geq 0}$ are pairwise orthogonal in $L^2(\Omega)$.

Each chaos subspace $\mathbf{H}_n$ admits a natural parametrization by symmetric kernels. The *n-th multiple stochastic integral* $I_n(f)$ of a kernel $f \in \mathcal{H}^{\odot n}$ (the n-th symmetric tensor product of $\mathcal{H}$) is the unique element of $\mathbf{H}_n$ that extends, by linearity and L^2-continuity, the assignment

$$I_n(h^{\otimes n}) := H_n(W(h)), \qquad h \in \mathcal{H},\ \|h\|_{\mathcal{H}} = 1. \tag{2.11}$$

The resulting maps $I_n : \mathcal{H}^{\odot n} \to \mathbf{H}_n$ are isometries: for all $f \in \mathcal{H}^{\odot n}$ and $g \in \mathcal{H}^{\odot m}$,

$$\mathbb{E}\Big[I_n(f)\, I_m(g)\Big] = \mathbf{1}_{n=m} \cdot n!\, \langle f, g\rangle_{\mathcal{H}^{\otimes n}}. \tag{2.12}$$

The factor $n!$ reflects the symmetrization: it arises because each symmetric kernel f is built from $n!$ permutations of its arguments. The pairwise orthogonality stated in Definition 2.1.4 is an immediate consequence of (2.12) with $n \neq m$.

With the maps I_n in place, the question of whether the orthogonal direct sum $\bigoplus_{n\geq 0} \mathbf{H}_n$ exhausts $L^2(\Omega, \sigma(W), \mathbb{P})$ has a clean and complete answer.

Theorem 2.1.5: Wiener chaos decomposition; [48, Theorem 1.1.1]

Every $F \in L^2(\Omega, \sigma(W), \mathbb{P})$ admits a unique orthogonal decomposition

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{2.13}$$

where the series converges in $L^2(\Omega)$ and $f_n \in \mathcal{H}^{\odot n}$ for each $n \geq 1$.

Decomposition (2.13) will be called the *Wiener chaos expansion* of F.

2.1.3 Malliavin derivative and Skorohod integral

The Wiener chaos structure supports two canonical operators: a derivative D and its L^2 adjoint δ. To begin with, the derivative D is best understood as a gradient with respect to the underlying Gaussian randomness. Given a direction $h \in \mathcal{H}$, one can perturb the noise W by the deterministic shift εh: for every $k \in \mathcal{H}$, replace $W(k)$ by $W(k) + \varepsilon\langle k, h\rangle_{\mathcal{H}}$. For a smooth cylindrical random variable $F = f(W(h_1), \ldots, W(h_n))$, this transforms F into

$$f(W(h_1) + \varepsilon\langle h_1, h\rangle_{\mathcal{H}}, \ldots, W(h_n) + \varepsilon\langle h_n, h\rangle_{\mathcal{H}}).$$

Differentiating this expressions at $\varepsilon = 0$ gives the *directional Malliavin derivative of F in the direction h*:

$$\begin{aligned} D_h F &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f\Big(W(h_1) + \varepsilon\langle h_1, h\rangle_{\mathcal{H}}, \ldots, W(h_n) + \varepsilon\langle h_n, h\rangle_{\mathcal{H}}\Big) \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n))\, \langle h_j, h\rangle_{\mathcal{H}}. \end{aligned} \tag{2.14}$$

The map $h \mapsto D_h F$ is a bounded linear functional on $\mathcal{H}$, so by the Riesz representation theorem there exists a unique $\mathcal{H}$-valued random variable DF such that

$$D_h F = \langle DF, h\rangle_{\mathcal{H}} \qquad \text{for all } h \in \mathcal{H}. \tag{2.15}$$

This is the Malliavin derivative DF. Reading off the $\mathcal{H}$-representative from (2.14) yields the following explicit formula.

Definition 2.1.6: Malliavin derivative

A random variable is *smooth cylindrical* if it has the form $F = f(W(h_1), \dots, W(h_n))$ for some $h_1, \dots, h_n \in \mathcal{H}$ and $f \in C_p^\infty(\mathbb{R}^n)$ (smooth with all derivatives of polynomial growth). The *Malliavin derivative* DF is the $\mathcal{H}$-valued random variable

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n))\, h_j. \tag{2.16}$$

Formula (2.16) defines D only on smooth cylindrical variables. To extend it to a larger class of random variables, one needs to know that limits of smooth cylindrical approximations have a well-defined derivative, i.e. that D is closable. Its proof can be found in [48, Proposition 1.2.1].

Proposition 2.1.7: Closability of D

The Malliavin derivative D, defined on smooth cylindrical variables by (2.16), is closable as an operator from $L^2(\Omega)$ to $L^2(\Omega; \mathcal{H})$: if (F_n) is a sequence of smooth cylindrical variables satisfying $F_n \to 0$ in $L^2(\Omega)$ and $DF_n \to G$ in $L^2(\Omega; \mathcal{H})$, then $G = 0$.

Proposition 2.1.7 guarantees that the norm below is well-defined: if two approximating sequences of smooth cylindricals converge to the same limit in $L^2(\Omega)$, their derivatives converge to the same limit in $L^2(\Omega; \mathcal{H})$, so the extension of D is unambiguous. The *Malliavin–Sobolev space* $\mathbb{D}^{1,2}$ is thus defined as the closure of the smooth cylindrical variables under

$$\|F\|_{1,2}^2 = \mathbb{E}[F^2] + \mathbb{E}\Big[\|DF\|_{\mathcal{H}}^2\Big]. \tag{2.17}$$

On chaos expansions, the derivative acts as a raising/lowering operator: if $F = \sum_{n\geq 0} I_n(f_n)$ belongs to $\mathbb{D}^{1,2}$, then

$$DF = \sum_{n=1}^{\infty} n\, I_{n-1}\Big(f_n(\,\cdot\,, \star)\Big), \tag{2.18}$$

where $\star$ stands for the free variable in $\mathcal{H}$ and $f_n(\,\cdot\,, \star)$ denotes f_n with one argument singled out.

Since D is a closed operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$, it has a well-defined L^2-adjoint. This adjoint is an operator from $\mathcal{H}$-valued square-integrable processes back to scalar random variables, and it plays the role of a stochastic integral for non-adapted integrands.

Definition 2.1.8: Skorohod integral

The *Skorohod integral* δ, also called the *divergence operator*, is the $L^2(\Omega;\mathcal{H})$-adjoint of D. An element $u \in L^2(\Omega;\mathcal{H})$ belongs to $\mathrm{Dom}(\delta)$ if the map $F \mapsto \mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]$ extends to a bounded linear functional on $L^2(\Omega)$. In that case, $\delta(u)$ is the unique element of $L^2(\Omega)$ satisfying the duality relation

$$\mathbb{E}\Big[\delta(u)\,F\Big] = \mathbb{E}\Big[\langle DF, u\rangle_{\mathcal{H}}\Big] \qquad \text{for all } F \in \mathbb{D}^{1,2}. \tag{2.19}$$

Unpacking the inner product in (2.19) via (2.3) gives an explicit space-time formula: for $u \in \mathrm{Dom}(\delta)$ and $F \in \mathbb{D}^{1,2}$,

$$\mathbb{E}\Big[\delta(u)\,F\Big] = \mathbb{E}\int_{\mathbb{R}_+\times\mathbb{R}^{2d}} D_{t,x}F\; u_{t,y}\;\Lambda(x-y)\,dx\,dy\,dt, \tag{2.20}$$

where $D_{t,x}F$ denotes the Malliavin derivative of F evaluated at $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. When u is adapted to the filtration generated by W, the Skorohod integral $\delta(u)$ coincides with the Itô–Walsh integral $\int u\,W(dt,dx)$; in general δ extends the Itô integral to non-adapted integrands.

We close this section with three identities that will be invoked repeatedly.

(i) For any $F \in \mathbb{D}^{1,2}$ and any $h \in \mathcal{H}$,

$$F\,W(h) = \delta(F\,h) + \langle DF, h\rangle_{\mathcal{H}}. \tag{2.21}$$

(ii) For any $F \in \mathbb{D}^{2,2}$ and $h, g \in \mathcal{H}$,

$$\mathbb{E}\Big[F\,W(h)\,W(g)\Big] = \mathbb{E}\Big[\langle D^2F, h\otimes g\rangle_{\mathcal{H}^{\otimes 2}}\Big] + \mathbb{E}[F]\,\langle h, g\rangle_{\mathcal{H}}. \tag{2.22}$$

(iii) For any $v \in \mathrm{Dom}(\delta)$ with chaos decomposition $v = \sum_{n=0}^{\infty} I_n(g_n(\,\cdot\,,\star))$, where $\star$ denotes the free variable in $\mathcal{H}$ (as in (2.18)) and each $g_n(\,\cdot\,,\star) \in \mathcal{H}^{\odot n}$ is symmetric in its first n variables, one can express $\delta(v)$ as

$$\delta(v) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{g}_n)\,, \tag{2.23}$$

where $\widetilde{g}_n$ denotes the symmetrization of $g_n(\,\cdot\,,\star)$ in all $n+1$ variables of $\mathcal{H}$.

Identities (2.21) and (2.22) follow from the duality formula (2.19); identity (2.23) follows from the action (2.18) of D on chaos. See [48, Chapter 1] for proofs.

We close the section with a key norm comparison for Wiener chaos elements: the L^k norm can be bounded by the L^2 norm at the cost of an explicit factor in n and k. This will be one of the main analytic tool in the moment estimates of Section 4.3.

Theorem 2.1.9: Nelson hypercontractivity

Let $k \geq 2$ be an integer. For any $n \geq 0$ and any $f_n \in \mathcal{H}^{\odot n}$, the following holds true:

$$\left\| I_n(f_n) \right\|_{L^k(\Omega)} \leq (k-1)^{n/2} \left\| I_n(f_n) \right\|_{L^2(\Omega)}. \tag{2.24}$$

Consequently, for any F admitting the chaos decoposition $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$ with $f_n \in \mathcal{H}^{\odot n}$, we have

$$\left\| F \right\|_{L^k(\Omega)} \leq \sum_{n=0}^{\infty} (k-1)^{n/2} \sqrt{n!}\, \|f_n\|_{\mathcal{H}^{\otimes n}}. \tag{2.25}$$

Proof. Inequality (2.25) follows immediately from (2.24) by the triangle inequality in $L^k(\Omega)$ and the isometry $\|I_n(f_n)\|_{L^2} = \sqrt{n!}\, \|f_n\|_{\mathcal{H}^{\otimes n}}$ from (2.12). It therefore suffices to prove (2.24).

In order to show (2.24), we introduce the *Ornstein-Uhlenbeck semigroup* $(P_t^{\mathrm{OU}})_{t \geq 0}$ on $L^2(\Omega)$, which acts on each Wiener chaos level by

$$P_t^{\mathrm{OU}} I_n(f_n) = e^{-nt} I_n(f_n), \tag{2.26}$$

so that $I_n(f_n)$ is an eigenfunction of the generator with eigenvalue n. Nelson's theorem asserts that $P_t^{\mathrm{OU}} : L^2(\Omega) \to L^q(\Omega)$ with operator norm at most 1 whenever $q \leq 1 + e^{2t}$; see [48, Theorem 1.4.1]. Setting $t := \frac{1}{2} \log(k-1)$, so that $1 + e^{2t} = k$, and using (2.26), we obtain

$$\left\| I_n(f_n) \right\|_{L^k(\Omega)} = e^{nt} \left\| P_t^{\mathrm{OU}} I_n(f_n) \right\|_{L^k(\Omega)} \leq e^{nt} \left\| I_n(f_n) \right\|_{L^2(\Omega)} = (k-1)^{n/2} \left\| I_n(f_n) \right\|_{L^2(\Omega)},$$

which is (2.24). ∎

2.2 Besov spaces

Besov spaces provide a flexible scale of function spaces that interpolates between classical Sobolev and Hölder spaces, and is well adapted to measuring the regularity of distributions such as the noise $\dot{W}$. Their definition relies on the *Littlewood–Paley decomposition*, which splits a tempered distribution into spectrally localized blocks and reads off regularity from the dyadic decay of those blocks.

2.2.1 Littlewood–Paley decomposition

The decomposition is built from a smooth, dyadic partition of unity in Fourier space.

Notation 2.2.1: Littlewood–Paley cutoffs

Fix two smooth, radial, nonnegative functions $\chi, \varphi : \mathbb{R}^d \to [0, 1]$ with compact support satisfying:

(i) the support of χ is contained in a ball $\{|\xi| \le b\}$ and the support of φ is contained in an annulus $C = \{a \le |\xi| \le b\}$ with $a = 3/4$ and $b = 8/3$;

(ii) $\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d$;

(iii) $\mathrm{supp}(\chi) \cap \mathrm{supp}(\varphi(2^{-i}\cdot)) = \varnothing$ for $i \ge 1$, and $\mathrm{supp}(\varphi(2^{-i}\cdot)) \cap \mathrm{supp}(\varphi(2^{-j}\cdot)) = \varnothing$ whenever $|i - j| > 1$.

The existence of such χ, φ is classical [2, Proposition 2.10]. We write $\varphi_j := \varphi(2^{-j}\cdot)$ for $j \ge 0$.

The interpretation of those functions is as follows: the function χ concentrates near zero frequency, while φ_j selects frequencies of order 2^j. Together they tile the frequency axis in dyadic annuli and allow one to localize any distribution spectrally. The resulting building blocks are the Littlewood–Paley projections.

Notation 2.2.2: Schwartz space and tempered distributions

We denote by $\mathcal{S}(\mathbb{R}^d)$ the *Schwartz space* of rapidly decreasing smooth functions on $\mathbb{R}^d$, equipped with the usual Fréchet topology generated by the seminorms

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|.$$

Its topological dual $\mathcal{S}'(\mathbb{R}^d)$ is the space of *tempered distributions*. The Fourier transform $\mathcal{F}$ extends continuously from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, so that the operations in Definition 2.2.3 below are well-defined for every tempered distribution.

Definition 2.2.3: Littlewood–Paley blocks

Let $u \in \mathcal{S}'(\mathbb{R}^d)$. The *Littlewood–Paley blocks* are

$$\Delta_{-1}u := \mathcal{F}^{-1}(\chi\,\mathcal{F}u), \qquad \Delta_j u := \mathcal{F}^{-1}(\varphi_j\,\mathcal{F}u) \quad \text{for } j \ge 0. \tag{2.27}$$

The *low-frequency projection* at level $k \ge 0$ is $S_k u := \sum_{j=-1}^{k-1} \Delta_j u$.

Oberve that each block $\Delta_j u$ is smooth: it coincides with the convolution $K_j * u$, where

$$K_j(z) := 2^{jd}(\mathcal{F}^{-1}\varphi)(2^j z)$$

is a Schwartz-class kernel. The spectral disjointness condition in Notation 2.2.1-(iii) implies that $\Delta_i u$ and $\Delta_j u$ have disjoint Fourier supports whenever $|i-j|>1$. In addition, owing to Notation 2.2.1-(ii), the blocks sum to u in $\mathcal{S}'(\mathbb{R}^d)$:

$$u = \sum_{j\geq -1} \Delta_j u\,, \qquad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{2.28}$$

2.2.2 Besov–Hölder spaces

The Littlewood–Paley decomposition $u=\sum_{j\geq -1}\Delta_j u$ (equation (2.28)) provides a canonical way to measure the regularity of a distribution: a function is smooth if its high-frequency blocks $\Delta_j u$ are small, and rough if they grow. The natural quantification of this idea leads to the Besov–Hölder scale. This section introduces three nested levels of Besov spaces, each adapted to a different aspect of the problem.

(i) *Unweighted Besov–Hölder spaces* $\mathcal{B}^\kappa(\mathbb{R}^d)$ (Definition 2.2.4): the baseline scale, built from the L^∞ norms of the LP blocks. The exponent $\kappa\in\mathbb{R}$ is the regularity index; for $\kappa\in(0,1)$ these spaces coincide with Hölder spaces, while negative κ captures rough distributions such as the noise $\dot W$.

(ii) *Weighted Besov–Hölder spaces* $\mathcal{B}^\kappa_w(\mathbb{R}^d)$ (Definition 2.2.6): the L^∞ norm of each block is replaced by $\|w\,\Delta_j f\|_{L^\infty}$, where w is a weight function. The polynomial weight ρ_σ (Notation 2.2.5) is the only instance needed here; it controls a moderate spatial growth of f at infinity.

(iii) *L^p-weighted Besov spaces* $\mathcal{B}^\kappa_{p,w}(\mathbb{R}^d)$ (Definition 2.2.7): the L^∞ norm of each block is replaced by the weighted L^p_w norm, and the sup over j is replaced by a ℓ^p-type sum. The main instance is the space $\mathcal{A}_q=\mathcal{B}^{-\kappa'}_{2q,\rho_\sigma}(\mathbb{R}^d)$, which appears as an intermediate space in the proof that the noise lies in $\mathcal{B}^{-\kappa}_{\rho_\sigma}$.

We start with the unweighted baseline.

Definition 2.2.4: Besov–Hölder space

Let $\kappa\in\mathbb{R}$. The *Besov–Hölder space* $\mathcal{B}^\kappa(\mathbb{R}^d):=\mathcal{B}^\kappa_{\infty,\infty}(\mathbb{R}^d)$ consists of all tempered distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|_{\mathcal{B}^\kappa} := \sup_{j\geq -1} 2^{j\kappa}\,\|\Delta_j f\|_{L^\infty(\mathbb{R}^d)} < \infty. \tag{2.29}$$

Recall that the exponent κ in Definition 2.2.4 represents the regularity index: a large positive κ forces the high-frequency blocks to be small (smooth functions), while a negative κ permits polynomial growth of the blocks (rough distributions). The connection to classical function spaces is transparent for $\kappa\in(0,1)$: in that range, $\mathcal{B}^\kappa(\mathbb{R}^d)$ coincides with the space of κ-Hölder continuous functions $\mathcal{C}^\kappa(\mathbb{R}^d)$ [2, Theorem 2.36]. The noise $\dot W$ will be shown in Section 2.4 to belong almost surely to $\mathcal{B}^\kappa(\mathbb{R}^d)$ for appropriate negative κ, quantifying its distributional roughness.

When the solution u to the PAM equation is not spatially bounded, one needs to track its growth at infinity. This is done by inserting a weight that penalizes large values of $|x|$; the relevant weight in this course is the polynomial decay function ρ_σ below.

Notation 2.2.5: Polynomial weight

For $\sigma > 0$, the *polynomial weight* $\rho_\sigma : \mathbb{R}^d \to (0, \infty)$ is defined by

$$\rho_\sigma(x) \;:=\; \Big(1 + |x|^2\Big)^{-\sigma/2}. \tag{2.30}$$

Thus $\rho_\sigma(x) \sim |x|^{-\sigma}$ as $|x| \to \infty$. In particular, $\rho_\sigma \in L^p(\mathbb{R}^d)$ whenever $p\sigma > d$, so by choosing σ large enough one can ensure any desired integrability at infinity.

With the weight ρ_σ at hand, one obtains a weighted analogue of Definition 2.2.4 simply by replacing the L^∞ norm of each LP block by a weighted L^∞ norm.

Definition 2.2.6: Weighted Besov–Hölder space

Let $w : \mathbb{R}^d \to (0, \infty)$ be a measurable weight and $\kappa \in \mathbb{R}$. The *weighted Besov–Hölder space* $\mathcal{B}^\kappa_w(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ (Notation 2.2.2) for which

$$\|f\|_{\mathcal{B}^\kappa_w} \;:=\; \sup_{j \geq -1} 2^{j\kappa} \Big\| w\, \Delta_j f \Big\|_{L^\infty(\mathbb{R}^d)} \;<\; \infty. \tag{2.31}$$

When $w \equiv 1$ this reduces to Definition 2.2.4: $\mathcal{B}^\kappa_1(\mathbb{R}^d) = \mathcal{B}^\kappa(\mathbb{R}^d)$. The primary instance in this course is $w = \rho_\sigma$ (Notation 2.2.5), giving the space $\mathcal{B}^\kappa_{\rho_\sigma}(\mathbb{R}^d)$ of distributions whose LP blocks satisfy $\sup_j 2^{j\kappa}\|\rho_\sigma \Delta_j f\|_{L^\infty} < \infty$.

The space $\mathcal{B}^\kappa_{\rho_\sigma}$ uses L^∞ to measure each block; replacing it by an $L^p_{\rho_\sigma}$ norm and summing over j yields a finer scale that is better suited to moment estimates. The specific instance needed below is $\mathcal{A}_q$, which serves as an intermediate space in the proof that the noise lies in $\mathcal{B}^{-\kappa}_{\rho_\sigma}$.

Definition 2.2.7: L^p-weighted Besov space and the space $\mathcal{A}_q$

Let $w : \mathbb{R}^d \to (0, \infty)$, $p \geq 1$, and $\kappa \in \mathbb{R}$. For $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ set

$$\|g\|^p_{L^p_w} := \int_{\mathbb{R}^d} |g(x)|^p \, w(x) \, dx. \tag{2.32}$$

The *L^p-weighted Besov space* $\mathcal{B}^\kappa_{p,w}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ for which

$$\|f\|^p_{\mathcal{B}^\kappa_{p,w}} := \sum_{j \geq -1} 2^{jp\kappa} \, \|\Delta_j f\|^p_{L^p_w(\mathbb{R}^d)} < \infty. \tag{2.33}$$

Given $\kappa' > 0$, $\sigma > 0$, and $q \geq 1$, we define

$$\mathcal{A}_q := \mathcal{B}^{-\kappa'}_{2q,\,\rho_\sigma}(\mathbb{R}^d), \tag{2.34}$$

so that $\|f\|^{2q}_{\mathcal{A}_q} = \sum_{j \geq -1} 2^{-2qj\kappa'} \|\Delta_j f\|^{2q}_{L^{2q}_{\rho_\sigma}}$. By Besov embedding, $\mathcal{A}_q \hookrightarrow \mathcal{B}^{-\kappa}_{\rho_\sigma}$ for any $\kappa > \kappa'$, provided q is large enough (see [52, Chapter 6]).

Two properties of Besov–Hölder spaces will be used repeatedly. The first controls the regularity gain when the heat semigroup $p_t * f$ is applied to a distribution of regularity κ; the second is a product estimate.

Theorem 2.2.8: Heat semigroup on Besov spaces; [35, Lemma 2.4]

Let $\kappa \in \mathbb{R}$ and $f \in \mathcal{B}^\kappa(\mathbb{R}^d)$. Then for all $t > 0$ and $\hat{\kappa} > \kappa$,

$$\|p_t * f\|_{\mathcal{B}^{\hat{\kappa}}} \leq c \, t^{-\frac{\hat{\kappa} - \kappa}{2}} \, \|f\|_{\mathcal{B}^\kappa}, \tag{2.35}$$

where $c > 0$ depends only on d, κ, and $\hat{\kappa}$.

Proposition 2.2.9: Product in Besov spaces; [2, Theorem 2.52]

Let $\kappa_1, \kappa_2 \in \mathbb{R}$ with $\kappa_2 < 0 < \kappa_1$ and $\kappa_1 > |\kappa_2|$. Then the pointwise product extends to a continuous bilinear map

$$\mathcal{B}^{\kappa_1}(\mathbb{R}^d) \times \mathcal{B}^{\kappa_2}(\mathbb{R}^d) \longrightarrow \mathcal{B}^{\kappa_2}(\mathbb{R}^d), \tag{2.36}$$

with bound $\|f_1 f_2\|_{\mathcal{B}^{\kappa_2}} \leq c \, \|f_1\|_{\mathcal{B}^{\kappa_1}} \|f_2\|_{\mathcal{B}^{\kappa_2}}$.

Proposition 2.2.8 will underpin the existence and uniqueness argument in Chapter 3: convolution with the heat kernel gains $(\hat{\kappa} - \kappa)/2$ derivatives at the cost of a power of t. Proposition 2.2.9 will be invoked to make sense of the product $u \dot{W}$ in the PAM equation when u is smooth and $\dot{W}$ is a rough distribution.

2.3 Definition of our classes of noises

The framework for the noise (the covariance structure (2.1), the spectral measure $\mu = \mathcal{F}\Lambda$, the Hilbert space $\mathcal{H}$ with inner product (2.4), the isometry (2.5), and Dalang's condition (2.2)) are all set up in Subsection 2.1.1. The present section collects concrete examples of spatial correlations Λ. For each one we specify $\Lambda(x)$ and the spectral measure $\mu(d\xi)$, and we verify Assumption 2.1.1 in a short lemma.

Example 2.3.1: Riesz noise

For $\eta \in (0, d)$, the *Riesz correlation* is

$$\Lambda_\eta(x) := |x|^{-\eta}, \qquad x \in \mathbb{R}^d \setminus \{0\}. \tag{2.37}$$

It is nonneg-definite and locally integrable for $\eta \in (0, d)$. Its Fourier transform is

$$\mu_\eta(d\xi) := c_{\eta,d}\, |\xi|^{-(d-\eta)}\, d\xi, \qquad c_{\eta,d} = \frac{2^{d-\eta}\pi^{d/2}\,\Gamma\left(\frac{d-\eta}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)}. \tag{2.38}$$

Remark 2.3.2. *The spectral density $|\xi|^{-(d-\eta)}$ is homogeneous of degree $\eta - d$. Also note that this spectral density is singular at the origin and non-integrable at infinity for $\eta \geq 0$.*

The threshold $\eta < 2$ is dictated by the high-frequency behavior of μ_η: the singularity at the origin is always integrable, but the power-law decay at infinity fails to be square-summable against $(1 + |\xi|^2)^{-1}$ once $\eta \geq 2$.

Lemma 2.3.3: Dalang's condition for Riesz noise

Assumption 2.1.1 holds for the noise of Example 2.3.1 if and only if $\eta < 2$.

Proof. Inserting (2.38) into (2.2), the condition to verify is

$$\int_{\mathbb{R}^d} \frac{c_{\eta,d}\, |\xi|^{-(d-\eta)}}{1 + |\xi|^2}\, d\xi = c_{\eta,d} \left(\int_{|\xi| \leq 1} \frac{|\xi|^{-(d-\eta)}}{1 + |\xi|^2}\, d\xi + \int_{|\xi| > 1} \frac{|\xi|^{-(d-\eta)}}{1 + |\xi|^2}\, d\xi \right) < \infty. \tag{2.39}$$

Each piece in the right hand side of (2.39) is handled separately.

Low frequencies $|\xi| \leq 1$. Since $1 + |\xi|^2 \geq 1$, one has $\int_{|\xi| \leq 1} |\xi|^{-(d-\eta)}\, d\xi \sim \int_0^1 r^{\eta-1}\, dr < \infty$ for all $\eta > 0$.

High frequencies $|\xi| > 1$. Here $(1 + |\xi|^2)^{-1} \sim |\xi|^{-2}$, so

$$\int_{|\xi|>1} |\xi|^{-(d-\eta)-2}\, d\xi \sim \int_1^\infty r^{\eta-3}\, dr,$$

which converges if and only if $\eta < 2$. Combining both pieces, the integral in (2.39) is finite if and only if $\eta < 2$, which completes the proof. ∎

Before stating our second example, we recall how the operator $(I-\Delta)^{-\eta/2}$ can be written as an integral of the heat semigroup. We denote by $P_t = e^{t\frac{1}{2}\Delta}$ the heat semigroup generated by $\frac{1}{2}\Delta$ on $\mathbb{R}^d$ (the operator driving the PAM (1.2)), whose integral kernel is the Gaussian heat kernel p_t introduced in (2.2)'s discussion, namely

$$p_t(x) \;=\; (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right), \qquad t>0,\; x\in\mathbb{R}^d, \tag{2.40}$$

with spatial Fourier transform $\widehat{p}_t(\xi) = e^{-t|\xi|^2/2}$. The full Laplacian entering the Bessel operator generates $e^{t\Delta} = P_{2t}$, whose Gaussian kernel is $p_{2t}(x) = (4\pi t)^{-d/2}\, e^{-|x|^2/(4t)}$. The starting point is the elementary Laplace-transform identity for the Gamma function: for every $\lambda>0$ and $\eta>0$,

$$\lambda^{-\eta/2} \;=\; \frac{1}{\Gamma(\eta/2)} \int_0^\infty t^{\eta/2-1}\, e^{-t\lambda}\, dt. \tag{2.41}$$

Applying (2.41) through the spectral calculus with λ replaced by the (positive, self-adjoint) operator $I-\Delta$, and using $e^{-t(I-\Delta)} = e^{-t}\, P_{2t}$, we obtain the subordination formula

$$(I-\Delta)^{-\eta/2} \;=\; \frac{1}{\Gamma(\eta/2)} \int_0^\infty t^{\eta/2-1}\, e^{-t}\, P_{2t}\, dt. \tag{2.42}$$

Reading off the integral kernel of both sides of (2.42) via the kernel p_{2t} of P_{2t} obtained above yields the kernel $\Lambda^{\mathrm{B}}_\eta$ recorded in the example below.

Example 2.3.4: Bessel noise.

For $\eta>0$, the *Bessel correlation* $\Lambda^{\mathrm{B}}_\eta$ is the integral kernel of the operator $(I-\Delta)^{-\eta/2}$:

$$\Lambda^{\mathrm{B}}_\eta(x) = c\int_0^\infty w^{(\eta-d)/2-1}\, e^{-w}\, e^{-\frac{|x|^2}{4w}}\, dw, \qquad x\in\mathbb{R}^d, \tag{2.43}$$

with normalizing constant $c = (4\pi)^{-d/2}\Gamma(\eta/2)^{-1}$, as follows from the subordination formula (2.42). Its Fourier transform is

$$\mu^{\mathrm{B}}_\eta(d\xi) \;:=\; (1+|\xi|^2)^{-\eta/2}\, d\xi. \tag{2.44}$$

For $0<\eta<d$, the kernel $\Lambda^{\mathrm{B}}_\eta$ has the same Riesz-type singularity $|x|^{\eta-d}$ at the origin, but (unlike the Riesz correlation) it decays exponentially at infinity (see [27, Section 1.2.1]):

$$\Lambda^{\mathrm{B}}_\eta(x) \asymp \begin{cases} |x|^{\eta-d} & \text{as } |x|\to 0,\\ |x|^{(\eta-d-1)/2}\, e^{-|x|} & \text{as } |x|\to\infty. \end{cases} \tag{2.45}$$

For $\eta\ge d$ the kernel is bounded near the origin, the singularity in (2.45) being replaced by a finite value. The spectral density $(1+|\xi|^2)^{-\eta/2}$ has polynomial decay for large $|\xi|$, and no singularity at $\xi=0$.

Remark 2.3.5. *Here is an interesting duality: the Bessel kernel $\Lambda^{\mathrm{B}}_\eta$ may be used either as the correlation Λ (with spectral density (2.44)) or, conversely, as the spectral density $\hat{\Lambda}(\xi)\,d\xi$ (in which case the correlation function is $(1+|x|^2)^{-\eta/2}$, a polynomial-decay function). Both choices lead to well-defined noises; the conditions on η differ.*

Lemma 2.3.6: Dalang's condition for Bessel noise

Assumption 2.1.1 holds for the noise of Example 2.3.4 if and only if $\eta > d-2$.

Proof. Substituting (2.44) into (2.2), we have to check the finiteness of the following integral:

$$\int_{\mathbb{R}^d} (1+|\xi|^2)^{-(\eta+2)/2}\,d\xi \sim \int_0^\infty r^{d-1}(1+r^2)^{-(\eta+2)/2}\,dr.$$

The integrand near $r=0$ is $O(1)$, hence integrable. For $r\to\infty$ it behaves like $r^{d-\eta-3}$, and the integral converges if and only if $d-\eta-3<-1$, i.e., $\eta>d-2$. ∎

The two previous examples isolate the two features of a correlation function that matter for the PAM, namely its singularity at the origin and its decay at infinity. Each example controls these features by a single parameter. On the one hand, the Bessel kernel of Example 2.3.4 produces, through (2.45), a prescribed power singularity $|x|^{\eta-d}$ at the origin (for $0<\eta<d$), together with a rapid (exponential) decay at infinity. On the other hand, the polynomial-decay function $(1+|x|^2)^{-s/2}$ is bounded near the origin and behaves like $|x|^{-s}$ as $|x|\to\infty$. Adding one of each type therefore lets us tune the two exponents independently: the Bessel term fixes the order of the singularity at 0, while the polynomial term fixes the power of the decay at ∞. This is the content of our next example, in which any pair of power behaviors at 0 and at ∞ can be realized.

Example 2.3.7: Riesz-type noise

For $s_1,s_2\in(0,d)$, the *Riesz-type correlation with parameters* (s_1,s_2) is defined by

$$\Lambda_{s_1,s_2}(x) := \Lambda^{\mathrm{B}}_{s_1}(x) + (1+|x|^2)^{-s_2/2}, \tag{2.46}$$

combining the Bessel kernel of Example 2.3.4 and a polynomial-decay tail. Its spectral density is

$$\hat{\Lambda}_{s_1,s_2}(\xi) := (1+|\xi|^2)^{-s_1/2} + \Lambda^{\mathrm{B}}_{s_2}(\xi), \tag{2.47}$$

where $\Lambda^{\mathrm{B}}_{s_2}(\xi)$ denotes the Bessel kernel of Example 2.3.4 evaluated at ξ. The kernel Λ_{s_1,s_2} interpolates between Riesz-type behavior near the origin and polynomial decay at infinity:

$$\Lambda_{s_1,s_2}(x) \asymp \begin{cases} |x|^{s_1-d} & \text{as } |x|\to 0,\\ |x|^{-s_2} & \text{as } |x|\to\infty. \end{cases} \tag{2.48}$$

The parameter s_1 controls the singularity at the origin, and s_2 controls the spatial decay at infinity.

Lemma 2.3.8: Dalang's condition for Riesz-type noise

For any $s_2 \in (0, d)$, Assumption 2.1.1 holds for the noise of Example 2.3.7 if and only if $s_1 > d - 2$.

Proof. The spectral measure μ_{s_1,s_2} of Λ_{s_1,s_2} satisfies

$$\mu_{s_1,s_2} = \mu^{\mathrm{B}}_{s_1} + \nu, \quad \text{where} \quad \nu(d\xi) = \Lambda^{\mathrm{B}}_{s_2}(\xi)\, d\xi$$

The contribution of $\mu^{\mathrm{B}}_{s_1}$ is handled by Lemma 2.3.6: it satisfies (2.2) iff $s_1 > d-2$. For ν, we split at $|\xi| = 1$ in the following way:

Low frequencies. Near $\xi = 0$ the Bessel kernel satisfies $\Lambda^{\mathrm{B}}_{s_2}(\xi) \asymp |\xi|^{s_2-d}$ according to (2.45), so that $\int_{|\xi|\leq 1} \Lambda^{\mathrm{B}}_{s_2}(\xi)\, d\xi \sim \int_0^1 r^{s_2-1}\, dr < \infty$ for $s_2 > 0$.

High frequencies. The Bessel kernel decays exponentially: $\Lambda^{\mathrm{B}}_{s_2}(\xi) \lesssim e^{-c|\xi|}$ for $|\xi| > 1$, so $\int_{|\xi|>1} \Lambda^{\mathrm{B}}_{s_2}(\xi)/(1+|\xi|^2)\, d\xi < \infty$.

Hence $\int_{\mathbb{R}^d} \nu(d\xi)/(1+|\xi|^2) < \infty$ for all $s_2 \in (0,d)$, and the condition reduces entirely to $s_1 > d-2$. ∎

All the correlations considered so far (the Riesz kernel of Example 2.3.1, the Bessel kernel of Example 2.3.4, and the Riesz-type kernel of Example 2.3.7) are isotropic: they depend on x only through the Euclidean norm $|x|$, and hence treat all directions of $\mathbb{R}^d$ in the same way. Our last example breaks this symmetry. It is built as a tensor product over the d coordinates, with a separate Hurst parameter H_i attached to each direction, so that the roughness of the noise is allowed to differ from one coordinate axis to the next.

Example 2.3.9: Fractional spatial noise

Let $H_i \in \left(\frac{1}{2}, 1\right)$ for $i = 1, \dots, d$. The *fractional spatial correlation* is

$$\Lambda_H(x) := \prod_{i=1}^{d} H_i(2H_i - 1)\, |x_i|^{2H_i-2}, \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d \setminus \{0\}. \tag{2.49}$$

Each factor is (up to a constant) the covariance kernel of a one-dimensional fractional Brownian motion with Hurst parameter H_i. The spectral measure is

$$\mu_H(d\xi) := C_H \prod_{i=1}^{d} |\xi_i|^{1-2H_i}\, d\xi, \tag{2.50}$$

where $C_H > 0$ is a constant depending on the H_i. The spectral density is singular at each coordinate hyperplane $\{\xi_i = 0\}$; the condition $H_i > 1/2$ ensures that this singularity is integrable.

Lemma 2.3.10: Dalang's condition for fractional spatial noise

Assumption 2.1.1 holds for the noise of Example 2.3.9 if and only if $\sum_{i=1}^{d} H_i > d - 1$.

Proof. Set $\bar{H} := \sum_{i=1}^{d} H_i$. We divide again the verification of (2.2) in two steps.

Near the origin. Since $1 - 2H_i \in (-1, 0)$, each factor $|\xi_i|^{1-2H_i}$ is integrable near $\xi_i = 0$. Evaluating separately in each coordinate, we get

$$\int_{[-1,1]^d} \mu_H(d\xi) = \int_{[-1,1]^d} \prod_i |\xi_i|^{1-2H_i}\, d\xi = \prod_i \int_{-1}^{1} |\xi_i|^{1-2H_i}\, d\xi_i < \infty.$$

Near infinity. The spectral density $\prod_i |\xi_i|^{1-2H_i}$ is homogeneous of degree $\sum_i (1 - 2H_i) = d - 2\bar{H}$. Passing to polar coordinates $\xi = r\omega$ with $r = |\xi|$ and $\omega \in S^{d-1}$, so that $d\xi = r^{d-1}\, dr\, d\sigma(\omega)$, we obtain

$$\prod_i |\xi_i|^{1-2H_i} = r^{d-2\bar{H}} \prod_i |\omega_i|^{1-2H_i}.$$

The angular integral is finite, that is

$$\int_{S^{d-1}} \prod_i |\omega_i|^{1-2H_i}\, d\sigma(\omega) < \infty,$$

because each exponent satisfies $1 - 2H_i > -1$. This makes the singularities on the coordinate hyperplanes $\{\omega_i = 0\}$ integrable, exactly as in the previous step. Hence

$$\int_{|\xi|>1} \frac{\prod_i |\xi_i|^{1-2H_i}}{1 + |\xi|^2}\, d\xi \sim \int_1^\infty \frac{r^{d-2\bar{H}}}{1 + r^2}\, r^{d-1}\, dr \sim \int_1^\infty r^{2d-2\bar{H}-3}\, dr,$$

which converges if and only if $2d - 2\bar{H} - 3 < -1$. We thus obtain the desired condition $\bar{H} > d - 1$. ■

Remark 2.3.11 (Further examples)**.** *The examples below are widely considered in the literature, although they will play a more modest role in our notes.*

(i) White noise in space. *Setting $\Lambda = \delta_0$ gives $\mu =$ Lebesgue, and (2.2) reduces to $\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-1}\, d\xi < \infty$, which holds if and only if $d = 1$. Hence space-time white noise is only admissible in one spatial dimension.*

(ii) Fractional noise in time. *The framework extends to include time correlations by replacing the white-noise factor dt in (2.1) with a convolution kernel $\gamma(s - t)\, ds\, dt$, where $\gamma(t) = H_0(2H_0 - 1)|t|^{2H_0-2}$, $H_0 \in (\frac{1}{2}, 1)$, is the covariance of a one-dimensional fractional Brownian motion with Hurst parameter H_0. The full covariance is then*

$$\mathbb{E}\Big[W(\varphi)\, W(\psi)\Big] = \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{2d}} \varphi(s, x)\, \psi(t, y)\, \gamma(s - t)\, \Lambda(x - y)\, dx\, dy\, ds\, dt, \tag{2.51}$$

and Dalang's condition acquires an additional integrability factor from $\hat{\gamma}$; we refer to [35, Section 2.1] for details. The four spatial examples above all extend naturally to this setting.

2.4 Inclusion in Besov spaces

We are now in a position to make precise the statement, announced after Definition 2.2.4, that the driving noise $\dot{W}$ belongs almost surely to a Besov space of negative regularity. Since W is white in time (see (2.1)), the natural object is not a single distribution but the whole trajectory $t \mapsto W_t$, where W_t is the spatial distribution obtained by integrating W over the time window $[0, t]$. The main result of this section, Theorem 2.4.4 below, states that this trajectory is Hölder continuous in time with values in a weighted Besov space $\mathcal{B}^{-\kappa}_{\rho_\sigma}$ (Definition 2.2.6). This type of regularity is exploited when solving stochastic PDEs in the Stratonovich sense, but will not be directly featured in the Itô setting adopted for the current notes. We still find it a valuable information, since it gives a precise idea of the noise singularity. The result is taken from [35, Proposition 5.5], specialized here to the white-in-time setting (2.1). We first record the distribution-valued process associated with W.

Notation 2.4.1: The noise as a distribution-valued process

For $t \geq 0$, let W_t denote the random tempered distribution acting on test functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$ by

$$W_t(\varphi) := W\Big(\mathbf{1}_{[0,t]} \otimes \varphi\Big), \tag{2.52}$$

with W the Gaussian family of Subsection 2.1.1. For $0 \leq s \leq t$ we write the increment

$$\delta W_{st} := W_t - W_s, \qquad \text{so that} \qquad \delta W_{st}(\varphi) = W\Big(\mathbf{1}_{[s,t]} \otimes \varphi\Big). \tag{2.53}$$

To measure the joint space-time regularity of $t \mapsto W_t$ we combine the weighted Besov norm of Definition 2.2.6 with a Hölder norm in time.

Definition 2.4.2: Time-Hölder, Besov-valued space

Fix $T > 0$, $\theta \in (0,1)$, $\kappa \in \mathbb{R}$, and $\sigma > 0$, and let ρ_σ be the polynomial weight of Notation 2.2.5. The space $\mathcal{C}^{\theta,-\kappa}_{\rho_\sigma}([0,T])$ consists of all maps $f : [0,T] \to \mathcal{B}^{-\kappa}_{\rho_\sigma}(\mathbb{R}^d)$ for which

$$\|f\|_{\mathcal{C}^{\theta,-\kappa}_{\rho_\sigma}} := \sup_{0\leq t\leq T} \|f_t\|_{\mathcal{B}^{-\kappa}_{\rho_\sigma}} + \sup_{0\leq s<t\leq T} \frac{\|f_t - f_s\|_{\mathcal{B}^{-\kappa}_{\rho_\sigma}}}{(t-s)^\theta} < \infty. \tag{2.54}$$

The regularity of W_t in space is dictated by the integrability of the spectral measure μ at high frequencies. Recall that Assumption 2.1.1 (Dalang's condition) only guarantees $\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-1} \mu(d\xi) < \infty$. Obtaining a genuine Hölder regularity in the Besov scale requires a small amount of extra integrability, quantified by a parameter α.

Assumption 2.4.3

There exists $\alpha \in (0,1)$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}} < \infty. \tag{2.55}$$

Since $2(1-\alpha) < 2$, Assumption 2.4.3 is stronger than Assumption 2.1.1, to which it reduces in the limit $\alpha \to 0$. The gap between the two is the price to pay for pathwise (rather than merely L^2) regularity of the noise. For the running examples of Section 2.3 the condition (2.55) is easy to read off.

(i) For the Riesz noise of Example 2.3.1, with $\mu(d\xi) \asymp |\xi|^{-(d-\eta)}\,d\xi$ by (2.38), condition (2.55) holds for every $\alpha < 1 - \frac{\eta}{2}$, hence (for some $\alpha > 0$) under the same restriction $\eta < 2$ as Dalang's condition.

(ii) For the Bessel noise of Example 2.3.4, with $\mu_\eta^{\mathrm{B}}(d\xi) = (1+|\xi|^2)^{-\eta/2}\,d\xi$ by (2.44), condition (2.55) holds for every $\alpha < 1 - \frac{d-\eta}{2}$, i.e. whenever $\eta > d-2$.

We can now state the main result of this section.

Theorem 2.4.4: Pathwise Besov regularity of the white-in-time noise.

Let W be the white-in-time Gaussian noise with covariance (2.1) and spectral measure μ, and suppose that Assumption 2.4.3 holds for some $\alpha \in (0,1)$. Fix a horizon $T > 0$ and a weight exponent $\sigma > d$. Then for every $\theta \in \left(0, \frac{1}{2}\right)$ and every $\kappa > 1-\alpha$, the process $t \mapsto W_t$ of Notation 2.4.1 admits a modification belonging almost surely to $\mathcal{C}^{\theta,-\kappa}_{\rho_\sigma}([0,T])$. Moreover the random variable $\|W\|_{\mathcal{C}^{\theta,-\kappa}_{\rho_\sigma}}$ has finite moments of all orders.

The proof rests on a single second-moment computation: the white-in-time structure turns the time average into a factor $(t-s)$, while the spatial integrability (2.55) controls the growth of the Littlewood–Paley blocks. The intermediate space $\mathcal{A}_q$ of Definition 2.2.7 converts these moment bounds into pathwise Hölder regularity through the Garsia–Rodemich–Rumsey lemma [24].

Proof. Fix $\kappa > \kappa' > 1-\alpha$ and recall from (2.34) the space $\mathcal{A}_q = \mathcal{B}^{-\kappa'}_{2q,\rho_\sigma}(\mathbb{R}^d)$, whose norm is

$$\|f\|_{\mathcal{A}_q}^{2q} = \sum_{j\geq -1} 2^{-2qj\kappa'}\,\|\Delta_j f\|^{2q}_{L^{2q}_{\rho_\sigma}} = \sum_{j\geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}^d} |\Delta_j f(x)|^{2q}\,\rho_\sigma(x)\,dx, \tag{2.56}$$

the last equality being the definition (2.32) of the weighted L^{2q} norm. We will prove that, for q large enough,

$$\mathbb{E}\Big[\|\delta W_{st}\|^{2q}_{\mathcal{A}_q}\Big] \leq c_q\,(t-s)^q, \qquad 0 \leq s \leq t \leq T. \tag{2.57}$$

Starting from (2.57), the proof is concluded as follows. The Besov embedding recorded after (2.34) gives $\mathcal{A}_q \hookrightarrow \mathcal{B}^{-\kappa}_{\rho_\sigma}$ for q large, so (2.57) yields $\mathbb{E}[\|\delta W_{st}\|^{2q}_{\mathcal{B}^{-\kappa}_{\rho_\sigma}}] \le c_q(t-s)^q$. Hence by the Garsia–Rodemich–Rumsey lemma [24], a moment bound of the form $\mathbb{E}[\|\delta W_{st}\|^{2q}] \le c_q(t-s)^q$ implies that $t \mapsto W_t$ has a modification which is Hölder continuous of any order $\theta < \frac{1}{2} - \frac{1}{2q}$ with values in $\mathcal{B}^{-\kappa}_{\rho_\sigma}$, and that the corresponding Hölder norm has moments of all orders. Letting $q \to \infty$ covers every $\theta < \frac{1}{2}$ and every $\kappa > 1-\alpha$, which is the assertion of the theorem. It remains to prove (2.57), which we do in three steps.

Step 1: Reduction to a second moment. By Notation 2.4.1, $\Delta_j(\delta W_{st})$ is the random field obtained by applying the Littlewood–Paley block Δ_j (Definition 2.2.3) to the spatial distribution δW_{st}. Writing $\Delta_j f = K_j * f$ and setting $K_{j,x}(y) := K_j(x-y)$, we have

$$\Delta_j(\delta W_{st})(x) = \delta W_{st}(K_{j,x}) = W(\mathbf{1}_{[s,t]} \otimes K_{j,x}),$$

by (2.53). Each such quantity is a centered Gaussian random variable, so its even moments are controlled by its variance,

$$\mathbb{E}\Big[\Big|W(\mathbf{1}_{[s,t]} \otimes K_{j,x})\Big|^{2q}\Big] \;=\; (2q-1)!!\; \mathbb{E}\Big[\Big|W(\mathbf{1}_{[s,t]} \otimes K_{j,x})\Big|^{2}\Big]^{q}. \tag{2.58}$$

Taking expectations in (2.56) applied to $f = \delta W_{st}$ and inserting (2.58), we obtain

$$\mathbb{E}\Big[\|\delta W_{st}\|^{2q}_{\mathcal{A}_q}\Big] \;=\; c_q \sum_{j\ge -1} 2^{-2qj\kappa'} \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big|W(\mathbf{1}_{[s,t]} \otimes K_{j,x})\Big|^{2}\Big]^{q} \rho_\sigma(x)\,dx. \tag{2.59}$$

Step 2: The second moment. Using the spectral form (2.4) of the covariance and the white-in-time structure (2.1), the time integration reduces to the diagonal and produces a factor $(t-s)$:

$$\mathbb{E}\Big[\Big|W(\mathbf{1}_{[s,t]} \otimes K_{j,x})\Big|^{2}\Big] = \int_s^t \int_{\mathbb{R}^d} \Big|\mathcal{F}K_{j,x}(\xi)\Big|^2 \mu(d\xi)\,dr = (t-s)\int_{\mathbb{R}^d} \Big|\varphi_j(\xi)\Big|^2 \mu(d\xi), \tag{2.60}$$

where we used $|\mathcal{F}K_{j,x}(\xi)| = |\varphi_j(\xi)|$ (the translation x only contributes a unit-modulus phase), with $\varphi_j = \varphi(2^{-j}\cdot)$ from Notation 2.2.1 for $j \ge 0$. In (2.60), we also use $\varphi_{-1} = \chi$. In particular the right-hand side of (2.60) does not depend on x. To estimate it, introduce the finite measure

$$\nu(d\xi) \;:=\; \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}}, \tag{2.61}$$

which is finite by Assumption 2.4.3. For $j \ge 0$ the function φ_j is supported in the annulus $\{2^j a \le |\xi| \le 2^j b\}$ (Notation 2.2.1), where $1+|\xi|^{2(1-\alpha)} \le c\,2^{2(1-\alpha)j}$; hence

$$\int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 \mu(d\xi) \;\le\; \int_{\{|\xi|\le 2^j b\}} \Big[1+|\xi|^{2(1-\alpha)}\Big]\,\nu(d\xi) \;\le\; c_\mu\, 2^{2(1-\alpha)j}. \tag{2.62}$$

The low-frequency block $j = -1$ contributes the finite constant $\int |\chi(\xi)|^2 \mu(d\xi)$ and is consistent with (2.62).

Step 3: Summation. Insert (2.60) and (2.62) into (2.59). Since the integrand no longer depends on x, the spatial integral contributes the finite factor $\int_{\mathbb{R}^d} \rho_\sigma(x)\, dx < \infty$, which is where the assumption $\sigma > d$ is used (Notation 2.2.5). We are left with

$$\mathbb{E}\Big[\|\delta W_{st}\|_{\mathcal{A}_q}^{2q}\Big] \le c_q\,(t-s)^q \sum_{j\ge -1} 2^{-2qj\kappa'}\, 2^{2q(1-\alpha)j} = c_q\,(t-s)^q \sum_{j\ge -1} 2^{2qj(1-\alpha-\kappa')}.$$

Because $\kappa' > 1-\alpha$, the geometric series converges, and we obtain (2.57). This completes the proof. ■

Theorem 2.4.4 is the precise sense in which the noise is “rough”: for any $\alpha > 0$ allowed by (2.55), the spatial regularity index $-\kappa$ can be taken just below $\alpha - 1 < 0$, confirming that $\dot{W}$ is a genuine distribution and not a function. This negative index is the obstruction that makes the product $u\,\dot{W}$ in (1.2) delicate.

CHAPTER 3

Existence and uniqueness for the stochastic heat equation

3.1 Integration theory with respect to the Gaussian noise

The mild (Duhamel) formulation of the PAM equation (1.2) reads

$$u(t,x) = P_t u_0(x) + \beta \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, u(s,y)\, W(ds,dy), \tag{3.1}$$

where $P_t = e^{t\frac{1}{2}\Delta}$ is the heat semigroup introduced at the beginning of the Bessel-noise example (see (2.40)), whose action on u_0 is the ordinary convolution $P_t u_0 = p_t * u_0$ with the heat kernel p_t (2.40). Two features of the stochastic integral on the right make it non-trivial. First, the deterministic factor $s \mapsto p_{t-s}(x-\cdot)$ is not an element of $\mathcal{H}$ in the most direct sense: the issue is not that it is ill-defined, but rather that checking it belongs to $\mathcal{H}$ requires a Fourier computation against the spectral measure μ. Second, the random factor $u(s,y)$ is adapted but not deterministic, so the Hilbert-space isometry (2.5) does not apply to the product $p_{t-s}(x-y)\, u(s,y)$ directly. The goal of this section is to set up an integration theory that handles both features in a unified way. We follow Dalang [18], who extended Walsh's martingale measure integral [53] to allow distributional integrands. One of Dalang's main ideas is to use a Fourier-based norm that is weaker than Walsh's.

3.1.1 Martingale measure

Although we have introduced W as a centered Gaussian family indexed by test functions (Subsection 2.1.1), the white-in-time covariance (2.1) shows that $\varphi \mapsto W(\varphi)$ extends σ-additively to a σ-finite $L^2(\Omega)$-valued measure on bounded Borel subsets of $[0,\infty) \times \mathbb{R}^d$. This object is usually called a *martingale measure*.

Notation 3.1.1: Worthy martingale measure associated with W

For $B \in \mathcal{B}_b(\mathbb{R}^d)$ (the bounded Borel subsets of $\mathbb{R}^d$), define

$$M_t(B) \;:=\; W\Big(\mathbf{1}_{[0,t]\times B}\Big)\,, \qquad t \ge 0, \tag{3.2}$$

where the right-hand side is an isometric extension of the map $\varphi \mapsto W(\varphi)$ of Subsection 2.1.1 to indicator functions. The covariance (2.1) gives

$$\mathbb{E}\Big[M_t(A)\, M_t(B)\Big] \;=\; t \int_A\int_B \Lambda(x-y)\, dx\, dy, \qquad A, B \in \mathcal{B}_b(\mathbb{R}^d). \tag{3.3}$$

Under Assumption 2.1.1, the process $t \mapsto M_t(B)$ is a continuous, square-integrable $(\mathcal{F}_t)$-martingale, where

$$\mathcal{F}_t \;:=\; \sigma\Big\{M_s(B);\; s \le t,\; B \in \mathcal{B}_b(\mathbb{R}^d)\Big\} \vee \mathcal{N}, \tag{3.4}$$

with $\mathcal{N}$ the σ-field of $\mathbb{P}$-null events. In the language of Walsh [53], the triple

$$(M_t(B), \mathcal{F}_t, Q)$$

is a *worthy martingale measure*, where the covariation measure Q is given by (3.3). We retain the differential notation $W(ds, dx)$ for the stochastic increment of M, writing

$$M_t(B) \;=\; \int_0^t \int_B W(ds, dx). \tag{3.5}$$

Remark 3.1.2. *The filtration $(\mathcal{F}_t)_{t\ge 0}$ defined in (3.4) is the natural filtration of the noise W: it is the smallest filtration making the process $t \mapsto M_t(B)$ adapted for every $B \in \mathcal{B}_b(\mathbb{R}^d)$. It coincides with the filtration alluded to on line (2.20) of Subsection 2.1.3, where we stated that the Skorohod integral $\delta(u)$ reduces to the Itô–Walsh integral $\int u\, W(dt, dx)$ when u is adapted to "the filtration generated by W": that filtration is precisely $(\mathcal{F}_t)_{t\ge 0}$ as defined in (3.4).*

Remark 3.1.3 (The covariation measure Q)**.** *Let us make precise the third entry of the triple $(M_t(B), \mathcal{F}_t, Q)$ of Notation 3.1.1. In Walsh's terminology [53], a martingale measure M is* worthy *when there exists a (random) signed measure $Q(dx, dy, ds)$ on $\mathbb{R}^d \times \mathbb{R}^d \times [0, \infty)$, symmetric in (x, y), positive definite, and σ-finite, that represents the covariation of the two martingales $t \mapsto M_t(A)$ and $t \mapsto M_t(B)$ in the sense*

$$\langle M(A), M(B)\rangle_t \;=\; Q\Big(A \times B \times (0, t]\Big), \qquad A, B \in \mathcal{B}_b(\mathbb{R}^d), \tag{3.6}$$

and that is controlled by a (positive) dominating measure *K in the sense that $|Q(\Gamma)| \le K(\Gamma)$ for every $\Gamma \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d \times [0, \infty))$. Here $\langle M(A), M(B)\rangle$ denotes the predictable* *covariation of the two square-integrable martingales of Notation 3.1.1, i.e. the predictable process making $M_t(A)\, M_t(B) - \langle M(A), M(B)\rangle_t$ a martingale.*

For the noise W considered here, the covariance computation (3.3) identifies Q explicitly: it is the deterministic *measure*

$$Q(dx, dy, ds) \;=\; \Lambda(x-y)\, dx\, dy\, ds, \tag{3.7}$$

with Λ the spatial covariance function of (2.1). Indeed, evaluating (3.7) on $A \times B \times (0,t]$ gives

$$Q\big(A \times B \times (0,t]\big) \;=\; t \int_A \int_B \Lambda(x-y)\, dx\, dy,$$

which is exactly the right-hand side of (3.3). Two comments are in order.

(i) Worthiness is automatic here. *Recall from the covariance structure (2.1) that the spatial covariance $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ is non-negative, and that it is non-negative definite (the sentence following (2.1)). The first property gives $Q \geq 0$ and the second that Q is positive definite, so that the measure Q of (3.7) dominates itself: one may take the dominating measure $K = Q$, which is what makes M worthy. Note also that the required symmetry of Q in (x,y) is automatic, since a real non-negative definite Λ is even, $\Lambda(x-y) = \Lambda(y-x)$.*

(ii) The bracket is deterministic. *Because Q does not depend on ω, the covariation (3.6) is deterministic as well, reflecting the Gaussian, spatially homogeneous and white-in-time nature of W. This is the structural feature that will let us reduce, in Proposition 3.1.5, the second-moment of the stochastic integral to the explicit bilinear form (3.11).*

3.1.2 Walsh's stochastic integral

In his landmak lecture notes [53], Walsh built the stochastic integral $\int g\, W(ds, dx)$ for predictable processes g via the following L^2 completion.

Definition 3.1.4: Elementary processes and Walsh's integral

A process $(g(s,x;\omega))$ is *elementary* if

$$g(s,x;\omega) \;=\; X(\omega)\, \mathbf{1}_{(a,b]}(s)\, \mathbf{1}_A(x), \tag{3.8}$$

where $0 \leq a < b$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and X is a bounded, $\mathcal{F}_a$-measurable random variable. Denote by $\mathcal{E}$ the vector space of all finite linear combinations of elementary processes. For $g \in \mathcal{E}$, the *Walsh stochastic integral* is defined by linearity from

$$\int_0^T \int_{\mathbb{R}^d} X\, \mathbf{1}_{(a,b]}(s)\, \mathbf{1}_A(x)\, W(ds, dx) \;:=\; X\, \Big(M_{T\wedge b}(A) - M_{T\wedge a}(A)\Big). \tag{3.9}$$

So far the integral is only defined on the dense class $\mathcal{E}$ of elementary processes (3.8). The next result is what turns this into a genuine stochastic integral: an L^2-isometry that controls the second moment of (3.9) by a deterministic bilinear form and thereby extends the integral, by closure, to a large space of predictable integrands.

Proposition 3.1.5: Walsh's isometry and extension

For a predictable process g, set

$$\|g\|_+^2 := \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(s,x)|\, \Lambda(x-y)\, |g(s,y)|\, dx\, dy\, ds\right], \tag{3.10}$$

and let P_+ be the completion of $(\mathcal{E}, \|\cdot\|_+)$. For every $g \in \mathcal{E}$, the Walsh integral (3.9) satisfies

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}^d} g\, W(ds,dx)\right|^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s,x)\, \Lambda(x-y)\, g(s,y)\, dx\, dy\, ds\right] \le \|g\|_+^2. \tag{3.11}$$

Consequently $g \mapsto \int_0^T \int_{\mathbb{R}^d} g\, W(ds,dx)$ extends uniquely to an $L^2(\Omega)$-bounded linear map on all of P_+, the *Walsh stochastic integral.*

Proof. We prove the equality in (3.11) on the elementary integrands (3.8) and extend it by bilinearity; the inequality and the passage to P_+ are then immediate.

Step 1: The identity on a single elementary process. Fix one elementary process

$$g(s,x;\omega) = X(\omega)\, \mathbf{1}_{(a,b]}(s)\, \mathbf{1}_A(x),$$

with X bounded and $\mathcal{F}_a$-measurable, as in (3.8). By the definition (3.9) of the integral,

$$\int_0^T \int_{\mathbb{R}^d} g\, W(ds,dx) = X\Big(M_{T\wedge b}(A) - M_{T\wedge a}(A)\Big). \tag{3.12}$$

The process $t \mapsto M_t(A)$ is a square-integrable $(\mathcal{F}_t)$-martingale (Notation 3.1.1), and its increments after time a are independent of $\mathcal{F}_a$; hence the covariation formula (3.3) gives the conditional second moment

$$\mathbb{E}\Big[\Big(M_{T\wedge b}(A) - M_{T\wedge a}(A)\Big)^2 \,\Big|\, \mathcal{F}_a\Big] = (T\wedge b - T\wedge a) \int_A \int_A \Lambda(x-y)\, dx\, dy. \tag{3.13}$$

Since X is $\mathcal{F}_a$-measurable, multiplying (3.13) by X^2, taking expectations, and using (3.12) yields

$$\mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}^d} g\, W(ds,dx)\right|^2\right] = \mathbb{E}[X^2]\, (T\wedge b - T\wedge a) \int_A \int_A \Lambda(x-y)\, dx\, dy. \tag{3.14}$$

Feeding the same g into the bilinear form on the right-hand side of (3.11) and using $\int_0^T \mathbf{1}_{(a,b]}(s)\, ds = T\wedge b - T\wedge a$ produces exactly the right-hand side of (3.14). This proves equality (3.11) for a single elementary process.

Step 2: Polarization to finite sums. For a general $g = \sum_i g_i \in \mathcal{E}$, define I_i as

$$I_i = \int_0^T \int_{\mathbb{R}^d} g_i \, W(ds, dx).$$

Expanding the square on the left of (3.11) and the bilinear form on its right gives the two double sums

$$\mathbb{E}\left[\left|\sum_i I_i\right|^2\right] = \sum_{i,j} \mathbb{E}\big[I_i I_j\big], \qquad \sum_{i,j} \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_i(s,x)\, \Lambda(x-y)\, g_j(s,y)\, dx\, dy\, ds\right]. \tag{3.15}$$

Each cross term $\mathbb{E}[I_i I_j]$ is evaluated exactly as in (3.13)-(3.14), conditioning on $\mathcal{F}_a$ for the later of the two starting times: martingale increments over disjoint time intervals are orthogonal, while on overlapping intervals the covariation (3.3) again produces the spatial factor $\int_{A_i} \int_{A_j} \Lambda(x-y)\, dx\, dy$. Term by term the two sums in (3.15) agree, which is the equality in (3.11) for every $g \in \mathcal{E}$.

Step 3: Inequality and extension. The subsequent inequality in (3.11) is the pointwise bound $g(s,x)\, g(s,y) \leq |g(s,x)|\, |g(s,y)|$ together with $\Lambda \geq 0$. The integral then extends to all of P_+ by $L^2(\Omega)$-closure. See Walsh [53, Chapter 2] for the worthy martingale-measure framework on which (3.13) relies. ■

3.1.3 Dalang's extension

The space P_+ is insufficient for our purposes because the norm $\|\cdot\|_+$ involves the absolute values $|g(s,x)|$, which are meaningless when $g(s,\cdot)$ is a distribution. The decisive observation of Dalang [18] is that the isometry in (3.11) uses the inner product $\int g(s,x)\Lambda(x-y)g(s,y)\, dx\, dy$ *without* absolute values: the left-hand side is thus controlled by a strictly weaker norm, whose completion $\mathcal{H}_\omega$ is much larger than P_+ and does accommodate distributional integrands.

Definition 3.1.6: Extended norm and Hilbert space $\mathcal{H}_\omega$

For predictable integrands g, h, average the Hilbert-space inner product (2.3) over the randomness:

$$\langle g, h\rangle_{\mathcal{H}_\omega} := \mathbb{E}\Big[\langle g, h\rangle_{\mathcal{H}}\Big] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,x)\,\Lambda(x-y)\,h(s,y)\,dx\,dy\,ds\right], \tag{3.16}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is the inner product (2.3) applied to the sample paths $(s,x)\mapsto g(s,x;\omega)$ (supported in $[0,T]$). In other words, $\langle\cdot,\cdot\rangle_{\mathcal{H}_\omega}$ is the inner product of $L^2(\Omega;\mathcal{H})$, the random counterpart of $\mathcal{H}$. The associated norm

$$\|g\|_{\mathcal{H}_\omega} := \langle g, g\rangle_{\mathcal{H}_\omega}^{1/2} = \mathbb{E}\Big[\|g\|_{\mathcal{H}}^2\Big]^{1/2} \tag{3.17}$$

is the norm under which Dalang [18] performs the extension, where it is denoted $\|\cdot\|_0$. We write $\mathcal{H}_\omega$ for the completion of $(\mathcal{E}, \|\cdot\|_{\mathcal{H}_\omega})$; it is a Hilbert space, canonically identified with $L^2(\Omega;\mathcal{H})$, and the subscript $\mathcal{H}_\omega$ in (3.16) and (3.17) records both the inner product and the space at once. The same construction will be performed for the modified martingale measure M^Z in Notation 3.1.14, yielding the norm $\|\cdot\|_{\mathcal{H}_Z}$ of (3.29) and its associated Hilbert space $\mathcal{H}_Z$.

Remark 3.1.7 (Fourier form of the extended inner product)**.** *Applying the Fourier identity (2.4) pathwise to the sample paths $(s,x)\mapsto g(s,x;\omega)$ and $(s,x)\mapsto h(s,x;\omega)$ and then averaging, the right-hand side of (3.16) can be written in Fourier mode as*

$$\langle g, h\rangle_{\mathcal{H}_\omega} = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \mathcal{F}_x g(s,\cdot)(\xi)\,\overline{\mathcal{F}_x h(s,\cdot)(\xi)}\,\mu(d\xi)\,ds\right], \tag{3.18}$$

where $\mathcal{F}_x$ is the spatial Fourier transform and $\mu = \mathcal{F}\Lambda$ is the spectral measure of the noise (2.1). Dropping the expectation in (3.18) recovers the deterministic identity (3.21) (Remark 3.1.9).

The point of this weaker norm is that the Walsh integral remains an isometry for it, so that the construction of Proposition 3.1.5 carries over verbatim to the larger space $\mathcal{H}_\omega$.

Theorem 3.1.8: Extension of the Walsh integral to $\mathcal{H}_\omega$

One has $P_+ \subset \mathcal{H}_\omega$, and the Walsh integral of Proposition 3.1.5 extends by continuity to an isometry

$$\begin{aligned} \mathcal{H}_\omega &\longrightarrow \Big\{\text{continuous square-integrable } (\mathcal{F}_t)\text{-martingales}\Big\} \\ g &\longmapsto \int_0^\cdot \int_{\mathbb{R}^d} g\, W(ds, dx). \end{aligned} \tag{3.19}$$

In particular, for every $g \in \mathcal{H}_\omega$ the second moment of the integral is given by the isometry identity

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^T \int_{\mathbb{R}^d} g\, W(ds, dx)\right|^2\right] &= \|g\|^2_{\mathcal{H}_\omega} \\ &= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(s,x)\, \Lambda(x-y)\, g(s,y)\, dx\, dy\, ds\right], \end{aligned} \tag{3.20}$$

the norm $\|\cdot\|_{\mathcal{H}_\omega}$ being the one of (3.17).

Proof of Theorem 3.1.8. Since $\Lambda \geq 0$, the integrand in (3.16) is dominated, for $g = h$, by the one in (3.10), so that $\|g\|_{\mathcal{H}_\omega} \leq \|g\|_+$ for every $g \in \mathcal{E}$. Hence the identity map on $\mathcal{E}$ is $\|\cdot\|_+$-to-$\|\cdot\|_{\mathcal{H}_\omega}$ continuous and extends to the inclusion $P_+ \subset \mathcal{H}_\omega$. By the isometry (3.11) of Proposition 3.1.5, the middle term of which equals $\|g\|^2_{\mathcal{H}_\omega}$, the map $g \mapsto \int g\, W(ds, dx)$ is an isometry from $(\mathcal{E}, \|\cdot\|_{\mathcal{H}_\omega})$ into $L^2(\Omega)$; since $\mathcal{E}$ is dense in $\mathcal{H}_\omega$ by construction, it extends to all of $\mathcal{H}_\omega$, yielding both the isometry (3.19) and the identity (3.20). ■

Remark 3.1.9 (The deterministic case)**.** *For deterministic g, h the expectation in the Fourier identity* (3.18) *(Remark 3.1.7) is vacuous, so $\langle g, h\rangle_{\mathcal{H}_\omega} = \langle g, h\rangle_{\mathcal{H}}$ and it reduces to*

$$\langle g, h\rangle_{\mathcal{H}} = \int_0^T \int_{\mathbb{R}^d} \mathcal{F}_x g(s,\cdot)(\xi)\, \overline{\mathcal{F}_x h(s,\cdot)(\xi)}\, \mu(d\xi)\, ds. \tag{3.21}$$

This Fourier representation extends to distributional integrands: it is well-defined as soon as $\mathcal{F}_x S(t,\cdot)$ is a function, even when $S(t,\cdot) \in \mathcal{S}'(\mathbb{R}^d)$ (Notation 2.2.2).

The following proposition makes the claim of Remark 3.1.2 rigorous for processes in $\mathcal{H}_\omega$: once the Walsh integral has been extended to $\mathcal{H}_\omega$ by Theorem 3.1.8, every adapted process in $\mathcal{H}_\omega$ belongs to $\mathrm{Dom}(\delta)$ and the Skorohod integral reduces to the Itô–Walsh integral. We adapt this result from [48, Proposition 1.3.3], which handled the case of integrals with respect to a classical Brownian motion.

Proposition 3.1.10: Itô–Walsh = Skorohod for adapted integrands

Let $v \in \mathcal{H}_\omega$ be adapted to the filtration $(\mathcal{F}_s)_{s\geq 0}$ of (3.4). Then $v \in \mathrm{Dom}(\delta)$ and

$$\delta(v) \;=\; \int_0^T \int_{\mathbb{R}^d} v(s,y)\, W(ds,dy), \tag{3.22}$$

where the right-hand side is the Itô–Walsh integral of Theorem 3.1.8.

Proof. Step 1: Locality of D for adapted functionals. Let $0 \leq a \leq T$, let $F \in \mathbb{D}^{1,2}$ be $\mathcal{F}_a$-measurable, and let $g \in \mathcal{H}$ be a deterministic element satisfying $g(r,\cdot) = 0$ for $r \leq a$. We claim that

$$\langle DF, g\rangle_{\mathcal{H}} = 0 \qquad \text{a.s.} \tag{3.23}$$

Indeed, the typical example of for a smooth cylindrical functional being $\mathcal{F}_a$-measurable is of the form

$$F = f(W(h_1), \ldots, W(h_n))\,,$$

with each $h_j \in \mathcal{H}$ satisfying $h_j(r,\cdot) = 0$ for $r > a$. For this type of functional, the derivative formula (2.16) gives

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n))\, h_j.$$

Since $h_j(r,\cdot) = 0$ for $r > a$ and $g(r,\cdot) = 0$ for $r \leq a$, the inner product (2.3) yields $\langle h_j, g\rangle_{\mathcal{H}} = 0$ for every j, hence $\langle DF, g\rangle_{\mathcal{H}} = 0$. The general case $F \in \mathbb{D}^{1,2} \cap L^2(\mathcal{F}_a)$ follows by approximating F by $\mathcal{F}_a$-measurable smooth cylindricals (which are dense in $L^2(\mathcal{F}_a)$) and invoking the closability of D (Proposition 2.1.7).

Step 2: Elementary adapted integrands. Let $0 \leq a < b \leq T$, let $g \in \mathcal{H}$ be deterministic with $g(r,\cdot) = 0$ for $r \notin (a,b]$, and let $F \in \mathbb{D}^{1,2} \cap L^2(\mathcal{F}_a)$. The product rule (2.21) applied to F and the deterministic element $g \in \mathcal{H}$ gives

$$F\, W(g) = \delta(Fg) + \langle DF, g\rangle_{\mathcal{H}}.$$

By Step 1, $\langle DF, g\rangle_{\mathcal{H}} = 0$ a.s. On the other hand, the Itô–Walsh integral of the elementary adapted process Fg equals $\int_0^T \int_{\mathbb{R}^d} F\, g\, W(ds,dy) = F\, W(g)$ by the isometry of Theorem 3.1.8. Hence $Fg \in \mathrm{Dom}(\delta)$ and

$$\delta(Fg) = F\, W(g) = \int_0^T \int_{\mathbb{R}^d} Fg\, W(ds,dy).$$

Equation (3.22) therefore holds for every such elementary adapted integrand $v = Fg$.

Step 3: Extension by density. Finite linear combinations of elementary adapted processes Fg as in Step 2 are dense in the subspace of adapted processes in $\mathcal{H}_\omega$. To see this, note that for each $a \geq 0$ the smooth cylindricals $f(W(h_1), \ldots, W(h_n))$ with h_j supported in $[0, a] \times \mathbb{R}^d$ form a subspace of $\mathbb{D}^{1,2} \cap L^2(\mathcal{F}_a)$ that is dense in $L^2(\mathcal{F}_a)$; combining this with step-function approximations in the time variable yields the claimed density in $\mathcal{H}_\omega$. Let $v \in \mathcal{H}_\omega$ be adapted and let (v_n) be such an approximating sequence. By Step 2 and linearity, $\delta(v_n) = \int v_n \, W$ for each n. The Walsh isometry (3.20) gives

$$\left\| \int v_n \, W - \int v \, W \right\|_{L^2(\Omega)} = \|v_n - v\|_{\mathcal{H}_\omega} \to 0.$$

Since δ is a closed operator (it is the adjoint of D; Definition 2.1.8), the convergence $v_n \to v$ in $\mathcal{H}_\omega$ and $\delta(v_n) \to \int v \, W$ in $L^2(\Omega)$ together imply $v \in \mathrm{Dom}(\delta)$ and $\delta(v) = \int_0^T \int_{\mathbb{R}^d} v \, W(ds, dy)$. ■

3.1.4 Integration against a modified martingale measure

The second difficulty announced at the beginning of this section is that the integrand in (3.1) is the product of the (deterministic) heat kernel and the (random) solution u. In Picard's iteration scheme, this means one needs to integrate a deterministic function $S(s, \cdot)$ against the modified noise measure $u_n \cdot W(ds, dx)$, where u_n is the n-th iterate. Dalang [18] handles this by replacing the worthy martingale measure M of Notation 3.1.1 with a *modified* martingale measure M^Z, in which Z plays the role of the current iterate.

Definition 3.1.11: Modified martingale measure M^Z

Let $Z = (Z(s, x),\ 0 \leq s \leq T,\ x \in \mathbb{R}^d)$ be an $(\mathcal{F}_s)$-adapted process satisfying

$$\sup_{0 \leq s \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E}\Big[Z(s, x)^2\Big] < +\infty. \tag{3.24}$$

The *modified martingale measure* M^Z is defined by

$$M_t^Z(B) := \int_0^t \int_B Z(s, y) \, W(ds, dy), \qquad B \in \mathcal{B}_b(\mathbb{R}^d), \tag{3.25}$$

where the integral is a Walsh stochastic integral in the sense of Definition 3.1.4 and Proposition 3.1.5.

To extend integration against M^Z to distributional integrands, the natural analogue of (3.21) involves the effective spectral measure of the product $Z \cdot W$, which depends on the two-point statistics of Z. The following spatial homogeneity condition (see [18, Section 2]) ensures that these statistics are well structured.

Assumption 3.1.12: Spatial homogeneity of Z

The process Z satisfying (3.24) is *spatially homogeneous in mean square*: for all $s \geq 0$ and $x, y \in \mathbb{R}^d$,

$$\mathbb{E}\Big[Z(s,x)\,Z(s,y)\Big] \;=\; \mathbb{E}\Big[Z(s,0)\,Z(s,x-y)\Big]. \tag{3.26}$$

Under this homogeneity the two-point statistics of Z depend only on the spatial increment, which lets us attach to M^Z a spectral measure exactly as the noise itself carries the spectral measure $\mu = \mathcal{F}\Lambda$ of (2.1).

Proposition 3.1.13: Effective spectral measure of M^Z

Under Assumption 3.1.12, the function

$$g_s(z) \;:=\; \mathbb{E}\Big[Z(s,0)\,Z(s,z)\Big], \qquad z \in \mathbb{R}^d, \tag{3.27}$$

is well-defined, non-negative definite, and independent of the reference point. Consequently the pointwise product $\Lambda(\cdot)\,g_s(\cdot)$ is non-negative definite, and its Fourier transform is a non-negative tempered measure

$$\mu_s^Z(d\xi) \;:=\; \mathcal{F}\Big[\Lambda(\cdot)\,g_s(\cdot)\Big](d\xi), \tag{3.28}$$

called the *effective spectral measure* of M^Z at time s.

Proof. By (3.26) the right-hand side of (3.27) depends only on the increment $z = x-y$, so g_s is well-defined and independent of the reference point; it is non-negative definite, being the covariance function of the mean-square stationary field $x \mapsto Z(s,x)$. Since Λ is non-negative definite as well (the standing hypothesis following the covariance structure (2.1)), the pointwise product $\Lambda\,g_s$ is again non-negative definite. Indeed, fix points $x_1, \ldots, x_n \in \mathbb{R}^d$; saying that a function ϕ is non-negative definite means precisely that the matrix $[\phi(x_i - x_j)]_{1 \leq i,j \leq n}$ is positive semidefinite. The two matrices

$$\Big[\Lambda(x_i - x_j)\Big]_{i,j} \quad \text{and} \quad \Big[g_s(x_i - x_j)\Big]_{i,j}$$

are thus positive semidefinite, and Schur's product theorem (the Hadamard, i.e. entrywise, product of two positive semidefinite matrices is positive semidefinite) shows that their entrywise product $[\Lambda(x_i - x_j)\,g_s(x_i - x_j)]_{i,j}$ is positive semidefinite as well. Since $x_1, \ldots, x_n$ were arbitrary, $\Lambda\,g_s$ is non-negative definite. By Bochner's theorem its Fourier transform (3.28) is therefore a non-negative measure. ∎

Notation 3.1.14: Norm $\|\cdot\|_{\mathcal{H}_Z}$ and space $\mathcal{H}_Z$

For a deterministic function $S : [0,T] \to \mathcal{S}'(\mathbb{R}^d)$ such that $\mathcal{F}_x S(t,\cdot)$ is a measurable function for each t, set

$$\|S\|^2_{\mathcal{H}_Z} := \int_0^T \int_{\mathbb{R}^d} \left|\mathcal{F}_x S(t,\cdot)(\xi)\right|^2 \mu_t^Z(d\xi)\, dt. \tag{3.29}$$

The completion of the space of deterministic Schwartz functions under $\|\cdot\|_{\mathcal{H}_Z}$ is denoted $\mathcal{H}_Z$. For $S \in \mathcal{H}_Z$, the stochastic integral $\int S\, dM^Z$ is defined via the isometry

$$\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} S(s,x)\, M^Z(ds,dx)\right|^2\right] = \left\|S\,\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}_Z} = \int_0^t \int_{\mathbb{R}^d} \left|\mathcal{F}_x S(s,\cdot)(\xi)\right|^2 \mu_s^Z(d\xi)\, ds. \tag{3.30}$$

The main result of this section identifies a sufficient condition for a deterministic integrand S to belong to $\mathcal{H}_Z$. It is stated in terms of the spectral measure μ of the noise alone (not the effective μ^Z), which makes it easy to verify in practice using only Assumption 2.1.1. The hypothesis that $S(t)$ is a *non-negative* distribution ensures that the absolute values can be removed from the Walsh norm, which is the key step in the proof.

Theorem 3.1.15: Dalang's stochastic integral

Let Z satisfy (3.24) and Assumption 3.1.12. Let $t \mapsto S(t)$ be a deterministic function with values in the space of *non-negative tempered distributions with rapid decrease*, i.e. $S(t) \in \mathcal{S}'(\mathbb{R}^d)$ (Notation 2.2.2), $\langle S(t), \varphi \rangle \geq 0$ for all $\varphi \geq 0$, and $\mathcal{F}_x S(t, \cdot)$ is a locally bounded function for each t. Suppose that

$$\int_0^T \int_{\mathbb{R}^d} \left| \mathcal{F}_x S(t, \cdot)(\xi) \right|^2 \mu(d\xi)\, dt \;<\; +\infty. \tag{3.31}$$

Then $S \in \mathcal{H}_Z$ (see Notation 3.1.14) and the stochastic integral $\int S\, dM^Z$ is well-defined. Moreover, for every $0 \leq t \leq T$,

$$\mathbb{E}\left[\left| \int_0^t \int_{\mathbb{R}^d} S(s,x)\, M^Z(ds, dx) \right|^2 \right] \;=\; \int_0^t \int_{\mathbb{R}^d} \left| \mathcal{F}_x S(s, \cdot)(\xi) \right|^2 \mu_s^Z(d\xi)\, ds, \tag{3.32}$$

and the following L^2 bound holds:

$$\begin{aligned} &\mathbb{E}\left[\left| \int_0^t \int_{\mathbb{R}^d} S(s,x)\, M^Z(ds, dx) \right|^2 \right] \\ &\qquad \leq \left(\sup_{0 \leq s \leq t} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left[Z(s,x)^2 \right] \right) \int_0^t \int_{\mathbb{R}^d} \left| \mathcal{F}_x S(s, \cdot)(\xi) \right|^2 \mu(d\xi)\, ds. \end{aligned} \tag{3.33}$$

It is convenient to abbreviate the prefactor appearing in (3.33). This quantity will play a central role in the contraction argument for the stochastic heat equation (Section 3.3), where the weight Z is taken to be the unknown solution itself and the factor $\mathcal{N}_{t,2}(Z)$ is precisely what must be controlled to close the fixed-point estimate.

Notation 3.1.16

For a weight Z satisfying (3.24) and $t \geq 0$, we set

$$\mathcal{N}_{t,2}(Z) \;:=\; \sup_{0 \leq s \leq t} \sup_{x \in \mathbb{R}^d} \mathbb{E}\left[Z(s,x)^2 \right]. \tag{3.34}$$

Remark 3.1.17. *With the notation (3.34), writing $\| \cdot \|_{\mathcal{H}}$ for the deterministic (unweighted) norm defined by (2.4) and $\| \cdot \|_{\mathcal{H}_Z}$ for its weighted counterpart in Notation 3.1.14, the L^2 bound (3.33) can be written compactly as*

$$\left\| S\, \mathbf{1}_{[0,t]} \right\|_{\mathcal{H}_Z}^2 \leq \mathcal{N}_{t,2}(Z) \left\| S\, \mathbf{1}_{[0,t]} \right\|_{\mathcal{H}}^2. \tag{3.35}$$

Indeed, the left-hand side of (3.35) is the $L^2(\Omega)$ norm of the stochastic integral by the isometry (3.30), while the integral against μ on the right-hand side of (3.33) is exactly $\| S\, \mathbf{1}_{[0,t]} \|_{\mathcal{H}}^2$ by (2.4).

Proof of Theorem 3.1.15. Fix a standard mollifier $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi \geq 0$, $\int_{\mathbb{R}^d} \psi(x)\,dx = 1$, $\operatorname{supp}(\psi) \subset \{|x| \leq 1\}$, and set $\psi_n(x) := n^d \psi(nx)$. Since $\psi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$, one has $\mathcal{F}\psi_n(\xi) \to 1$ pointwise and $|\mathcal{F}\psi_n(\xi)| \leq 1$ for all ξ. Define the mollified integrand

$$S_n(t) := \psi_n * S(t), \qquad n \geq 1. \tag{3.36}$$

Since $S(t)$ is a distribution with rapid decrease, $S_n(t) = \psi_n * S(t)$ belongs to $\mathcal{S}(\mathbb{R}^d)$ for each t. Moreover, $S_n(t) \geq 0$ because both $\psi_n \geq 0$ and $S(t) \geq 0$. We prove in three steps that $\|S_n - S\|_{\mathcal{H}_Z} \to 0$, which yields $S \in \mathcal{H}_Z$ and the claimed identities.

Step 1: $S_n \in P_{+,Z}$ with a uniform bound. Let $\|\cdot\|_{+,Z}$ be the Walsh norm adapted to the modified martingale measure M^Z of Definition 3.1.11, i.e. the analogue of (3.10) in which the integrand is weighted by Z:

$$\|S\|_{+,Z}^2 := \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |S(s,x)|\,|Z(s,x)|\,\Lambda(x-y)\,|Z(s,y)|\,|S(s,y)|\,dx\,dy\,ds\right], \tag{3.37}$$

and let $P_{+,Z}$ be the completion of the deterministic Schwartz functions under $\|\cdot\|_{+,Z}$. Since $S_n(t) \in \mathcal{S}(\mathbb{R}^d)$ and $S_n(t) \geq 0$, the absolute values in (3.37) can be removed. Applying the Cauchy–Schwarz inequality to bound $\mathbb{E}[|Z(s,x)||Z(s,y)|] \leq \sup_{x'} \mathbb{E}[Z(s,x')^2]$, and then the Fourier identity (3.21), we obtain

$$\|S_n\|_{+,Z}^2 \leq \left(\sup_{0\leq s\leq T}\sup_{x\in\mathbb{R}^d} \mathbb{E}\big[Z(s,x)^2\big]\right) \int_0^T \int_{\mathbb{R}^d} \big|\mathcal{F}S_n(t)(\xi)\big|^2 \mu(d\xi)\,dt. \tag{3.38}$$

Since $|\mathcal{F}S_n(t)(\xi)| = |\mathcal{F}\psi_n(\xi)|\,|\mathcal{F}S(t)(\xi)| \leq |\mathcal{F}S(t)(\xi)|$, the right-hand side of (3.38) is bounded uniformly in n by conditions (3.24) and (3.31). Hence $S_n \in P_{+,Z} \subset \mathcal{H}_Z$ for every $n \geq 1$.

Step 2: $\|S\|_{\mathcal{H}_Z} < \infty$. Using the definition (3.29) of $\|\cdot\|_{\mathcal{H}_Z}$ and $|\mathcal{F}S_n(t)(\xi)|^2 \nearrow |\mathcal{F}S(t)(\xi)|^2$ as $n \to \infty$, Fatou's lemma gives

$$\|S\|_{\mathcal{H}_Z}^2 \leq \liminf_{n\to\infty} \|S_n\|_{\mathcal{H}_Z}^2 \leq \liminf_{n\to\infty} \|S_n\|_{+,Z}^2 < \infty,$$

where the second inequality uses $\|\cdot\|_{\mathcal{H}_Z} \leq \|\cdot\|_{+,Z}$ (valid since $S_n \geq 0$) and the last one is the uniform bound from Step 1.

Step 3: $\|S_n - S\|_{\mathcal{H}_Z} \to 0$. From the definition (3.29) and $\mathcal{F}(S_n(t) - S(t))(\xi) = (\mathcal{F}\psi_n(\xi) - 1)\,\mathcal{F}S(t)(\xi)$,

$$\|S_n - S\|_{\mathcal{H}_Z}^2 = \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\psi_n(\xi) - 1|^2\,|\mathcal{F}S(t)(\xi)|^2\,\mu_t^Z(d\xi)\,dt. \tag{3.39}$$

The integrand in (3.39) converges pointwise to zero since $\mathcal{F}\psi_n(\xi) \to 1$, and is dominated by $4\,|\mathcal{F}S(t)(\xi)|^2$, whose integral is $4\|S\|_{\mathcal{H}_Z}^2 < \infty$ by Step 2. The dominated convergence theorem therefore gives $\|S_n - S\|_{\mathcal{H}_Z} \to 0$.

Since $S_n \in \mathcal{H}_Z$ and $\|S_n - S\|_{\mathcal{H}_Z} \to 0$, the completeness of $\mathcal{H}_Z$ implies $S \in \mathcal{H}_Z$, and the isometry (3.32) is (3.30) applied to $S\,\mathbf{1}_{[0,t]}$. Finally, the bound (3.33) is obtained by passing to the limit in the Step 1 estimate. Since $S_n \geq 0$ one has $\|\cdot\|_{\mathcal{H}_Z} \leq \|\cdot\|_{+,Z}$, so restricting (3.38) to the interval $[0,t]$ gives

$$\|S_n\,\mathbf{1}_{[0,t]}\|^2_{\mathcal{H}_Z} \leq \Big(\sup_{0\leq s\leq t}\sup_{x\in\mathbb{R}^d}\mathbb{E}\big[Z(s,x)^2\big]\Big)\int_0^t\int_{\mathbb{R}^d}\big|\mathcal{F}_x S_n(s,\cdot)(\xi)\big|^2\,\mu(d\xi)\,ds. \tag{3.40}$$

Here the supremum in the prefactor may be restricted to $[0,t]$ because, after restriction to $[0,t]$, only the times $s\in[0,t]$ enter the integral, and the Cauchy–Schwarz step bounding $\mathbb{E}[Z(s,x)Z(s,y)] \leq \sup_{x'}\mathbb{E}[Z(s,x')^2]$ is applied for such s only. As $n\to\infty$, the left-hand side of (3.40) converges to $\|S\,\mathbf{1}_{[0,t]}\|^2_{\mathcal{H}_Z} = \mathbb{E}[\,|\int_0^t\int_{\mathbb{R}^d} S\,dM^Z|^2\,]$ (by Step 3 and (3.30)), while $|\mathcal{F}_x S_n(s,\cdot)(\xi)|^2 \nearrow |\mathcal{F}_x S(s,\cdot)(\xi)|^2$ allows the monotone convergence theorem on the right-hand side. Passing to the limit yields exactly (3.33). ■

3.1.5 A Burkholder–Davis–Gundy inequality for L^p moments

The L^2 bound (3.33) of Theorem 3.1.15 controls the *second* moment of the modified stochastic integral $\int S\,dM^Z$ in terms of the deterministic spectral integral (3.31). For the fixed-point argument of Section 3.3, however, the contraction has to be set up in a space of processes with finite p-th moments for a fixed $p\geq 2$ (this is also the natural setting in which one later reads off the moment Lyapunov exponents responsible for intermittency). We therefore need the analogue of (3.33) for every $L^p(\Omega)$ norm. Such an estimate is the colored-noise counterpart of Khoshnevisan's Burkholder–Davis–Gundy (BDG) inequality ([36, Proposition 4.4]), which is stated there for space-time white noise; we adapt it to the spatially homogeneous noise of Subsection 2.1.1 and to the modified martingale measure M^Z of Definition 3.1.11.

To begin with, we introduce a new piece of notation. Namely the role played by the prefactor $\mathcal{N}_{t,2}(Z)$ of (3.34) in the L^2 bound is now played by its L^p upgrade.

Notation 3.1.18: L^p moment functional

For $p\geq 2$, an $(\mathcal{F}_s)$-adapted process Z with $Z(s,x)\in L^p(\Omega)$ for every (s,x), and $t\geq 0$, we set

$$\mathcal{N}_{t,p}(Z) := \sup_{0\leq s\leq t}\sup_{x\in\mathbb{R}^d}\big\|Z(s,x)\big\|^2_{L^p(\Omega)} = \sup_{0\leq s\leq t}\sup_{x\in\mathbb{R}^d}\Big(\mathbb{E}\Big[\,|Z(s,x)|^p\,\Big]\Big)^{2/p}. \tag{3.41}$$

The requirement that $\mathcal{N}_{t,p}(Z)$ be *finite* is the L^p analogue of the second-moment bound (3.24). It will be assumed in Theorem 3.1.22 below.

Remark 3.1.19. *We have kept the* square *of the $L^p(\Omega)$ norm in* (3.41) *on purpose: for $p=2$ the functional $\mathcal{N}_{t,p}(Z)$ reduces exactly to $\mathcal{N}_{t,2}(Z)=\sup_{s\leq t}\sup_x\mathbb{E}[Z(s,x)^2]$ of* (3.34), *so that* (3.41) *extends the earlier Notation 3.1.16, and the BDG bound* (3.43) *below has the same algebraic shape as the L^2 bound* (3.33). *Moreover $p\mapsto\mathcal{N}_{t,p}(Z)$ is non-decreasing in p by Jensen's inequality, and non-decreasing in t.*

The functional $\mathcal{N}_{t,p}$ singles out a natural space of admissible weights, the one in which the fixed-point argument of Section 3.3 will be carried out.

Definition 3.1.20: The space $\mathcal{A}_{t,p}$

For $p \geq 2$ and $t \geq 0$, let $\mathcal{A}_{t,p}$ be the set of all jointly measurable, $(\mathcal{F}_s)$-adapted processes $Z = (Z(s,x))_{0\leq s\leq t,\, x\in\mathbb{R}^d}$ for which $\mathcal{N}_{t,p}(Z) < +\infty$ (Notation 3.1.18). In this context, observe that two processes coinciding $ds\,dx\,d\mathbb{P}$-almost everywhere are being identified. Equipped with the norm

$$\|Z\|_{\mathcal{A}_{t,p}} := \mathcal{N}_{t,p}(Z)^{1/2} = \sup_{0\leq s\leq t} \sup_{x\in\mathbb{R}^d} \|Z(s,x)\|_{L^p(\Omega)}, \tag{3.42}$$

$\mathcal{A}_{t,p}$ is a Banach space. Indeed (3.42) is the norm of $L^\infty\Big([0,t]\times\mathbb{R}^d; L^p(\Omega)\Big)$, which is easily seen to be complete.

Remark 3.1.21. *By the monotonicity recorded in Remark 3.1.19, the norm (3.42) increases with p and with t, so that $\mathcal{A}_{t,q} \subseteq \mathcal{A}_{t,p}$ for $q \geq p \geq 2$ and the restriction of any $Z \in \mathcal{A}_{t',p}$ to $[0,t]$ lies in $\mathcal{A}_{t,p}$ for $t' \geq t$. In particular $\mathcal{A}_{t,p} \subseteq \mathcal{A}_{t,2}$, so that every $Z \in \mathcal{A}_{t,p}$ satisfies the second-moment bound (3.24) on $[0,t]$.*

Theorem 3.1.22: BDG inequality for M^Z

Let $p \geq 2$, $0 \leq t \leq T$, let $Z \in \mathcal{A}_{t,p}$ (Definition 3.1.20), and let $t \mapsto S(t)$ be a deterministic non-negative integrand as in Theorem 3.1.15, satisfying the integrability condition (3.31). Then the modified stochastic integral $\int S\,dM^Z$, which by (3.25) coincides with the Walsh integral $\int S\,Z\,dW$ of Definition 3.1.4, is well-defined, belongs to $L^p(\Omega)$, and satisfies

$$\left\| \int_0^t \int_{\mathbb{R}^d} S(s,y)\, M^Z(ds,dy) \right\|_{L^p(\Omega)}^2 \leq 4p\,\mathcal{N}_{t,p}(Z) \int_0^t \int_{\mathbb{R}^d} \Big|\mathcal{F}_x S(s,\cdot)(\xi)\Big|^2 \mu(d\xi)\, ds. \tag{3.43}$$

Equivalently, with the deterministic norm $\|\cdot\|_{\mathcal{H}}$ of (2.4), the bound (3.43) reads

$$\left\| \int_0^t \int_{\mathbb{R}^d} S\, dM^Z \right\|_{L^p(\Omega)}^2 \leq 4p\,\mathcal{N}_{t,p}(Z) \left\| S\,\mathbf{1}_{[0,t]} \right\|_{\mathcal{H}}^2. \tag{3.44}$$

Remark 3.1.23. *In contrast with Theorem 3.1.15, the moment bound (3.43) does* not *require the spatial homogeneity Assumption 3.1.12: it is read off directly from the quadratic variation of the Walsh integral $\int S\,Z\,dW$, bypassing the effective spectral measure μ^Z of (3.28). Homogeneity is needed only for the exact isometry (3.32), not for the inequality that drives the contraction. This is why $\mathcal{A}_{t,p}$, whose elements are merely measurable and adapted, is a legitimate domain for the stochastic integral.*

Remark 3.1.24. *For $p = 2$, the right-hand side of (3.43) coincides with that of the L^2 bound (3.33) up to the universal constant $4p = 8$; the sharp isometric constant 1* *of* (3.33) *is of course lost, since the BDG inequality replaces the exact second-moment identity* (3.32) *by the martingale inequality of Step 1 below. What matters for Section 3.3 is not the constant but the structure of* (3.44)*: the random weight Z enters only through the scalar $\mathcal{N}_{t,p}(Z)$, while the deterministic kernel S enters only through $\|S\,\mathbf{1}_{[0,t]}\|_{\mathcal{H}}$.*

Proof of Theorem 3.1.22. Well-definedness. Since $Z \in \mathcal{A}_{t,p} \subseteq \mathcal{A}_{t,2}$ (Remark 3.1.21) and $S \geq 0$ satisfies (3.31), the product $S\,Z$ is predictable and, by the Cauchy–Schwarz step bounding $\mathbb{E}[|Z(s,x)||Z(s,y)|] \leq \mathcal{N}_{t,2}(Z)$ in the Walsh norm (3.10),

$$\|S\,Z\|_+^2 \;\leq\; \mathcal{N}_{t,2}(Z)\,\Big\|S\,\mathbf{1}_{[0,t]}\Big\|_{\mathcal{H}}^2 \;<\; +\infty, \tag{3.45}$$

so $S\,Z \in P_+$ and $\int S\,Z\,dW$ is a well-defined Walsh integral by Proposition 3.1.5.

Step 0: the martingale and its bracket. Set

$$M_t \;:=\; \int_0^t\!\int_{\mathbb{R}^d} S(s,y)\,M^Z(ds,dy) \;=\; \int_0^t\!\int_{\mathbb{R}^d} S(s,y)\,Z(s,y)\,W(ds,dy), \tag{3.46}$$

the second equality being the definition (3.25) of the modified martingale measure. Then $t \mapsto M_t$ is a continuous, square-integrable $(\mathcal{F}_t)$-martingale. Thanks to the covariation formula (3.3) for the worthy martingale measure M (or, equivalently, by the Walsh isometry (3.11) read at the level of brackets), its quadratic variation is the random measure

$$\langle M\rangle_t \;=\; \int_0^t\!\int_{\mathbb{R}^d}\!\int_{\mathbb{R}^d} S(s,x)\,Z(s,x)\,\Lambda(x-y)\,Z(s,y)\,S(s,y)\,dx\,dy\,ds. \tag{3.47}$$

Step 1: BDG for the continuous martingale M. We apply the Burkholder–Davis–Gundy inequality for continuous L^2 martingales in the sharp form of [36, Theorem B.1], namely $\|M_t\|^2_{L^p(\Omega)} \leq 4p\,\|\langle M\rangle_t\|_{L^{p/2}(\Omega)}$. With the notation $\|Y\|_q := \|Y\|_{L^q(\Omega)}$, this is

$$\Big\|M_t\Big\|_p^2 \;\leq\; 4p\,\Big\|\langle M\rangle_t\Big\|_{p/2}. \tag{3.48}$$

Step 2: Minkowski's inequality on the bracket. Since $S(s,\cdot) \geq 0$ and the covariance $\Lambda \geq 0$ (recall that the noise model of Subsection 2.1.1 takes $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ non-negative in (2.1), the same property used for the worthiness of M in Remark 3.1.3), the deterministic weight $S(s,x)\,\Lambda(x-y)\,S(s,y)$ in (3.47) is non-negative. Minkowski's integral inequality in $L^{p/2}(\Omega)$ (valid since $p/2 \geq 1$) therefore lets us pull the norm inside the $dx\,dy\,ds$ integral,

$$\Big\|\langle M\rangle_t\Big\|_{p/2} \;\leq\; \int_0^t\!\int_{\mathbb{R}^d}\!\int_{\mathbb{R}^d} S(s,x)\,\Lambda(x-y)\,S(s,y)\,\Big\|Z(s,x)\,Z(s,y)\Big\|_{p/2}\,dx\,dy\,ds. \tag{3.49}$$

Moreover, owing to the Cauchy–Schwarz inequality on Ω, we have

$$\|Z(s,x)\,Z(s,y)\|_{p/2} \leq \|Z(s,x)\|_p\,\|Z(s,y)\|_p,$$

and by the definition (3.41) of $\mathcal{N}_{t,p}(Z)$ each factor is at most $\mathcal{N}_{t,p}(Z)^{1/2}$ for $s \leq t$, so that

$$\Big\|Z(s,x)\,Z(s,y)\Big\|_{p/2} \;\leq\; \sup_{x'\in\mathbb{R}^d}\|Z(s,x')\|_p^2 \;\leq\; \mathcal{N}_{t,p}(Z), \qquad 0 \leq s \leq t. \tag{3.50}$$

Step 3: Recognizing the deterministic norm. Substituting (3.50) into (3.49) pulls the scalar $\mathcal{N}_{t,p}(Z)$ out of the integral and leaves a purely deterministic double integral against Λ, which is exactly the $\mathcal{H}$-norm of $S\,\mathbf{1}_{[0,t]}$ in the form (2.3) (equivalently (2.4)):

$$\begin{aligned}\left\|\langle M\rangle_t\right\|_{p/2} &\le \mathcal{N}_{t,p}(Z)\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S(s,x)\,\Lambda(x-y)\,S(s,y)\,dx\,dy\,ds \\ &= \mathcal{N}_{t,p}(Z)\left\|S\,\mathbf{1}_{[0,t]}\right\|_{\mathcal{H}}^2 = \mathcal{N}_{t,p}(Z)\int_0^t\int_{\mathbb{R}^d}\left|\mathcal{F}_x S(s,\cdot)(\xi)\right|^2\mu(d\xi)\,ds. \end{aligned} \tag{3.51}$$

The last quantity is finite by the integrability condition (3.31) and $Z \in \mathcal{A}_{t,p}$, so $M_t \in L^p(\Omega)$. Inserting the chain (3.51) into (3.48) yields both the compact form (3.44) and, after expanding $\|S\,\mathbf{1}_{[0,t]}\|_{\mathcal{H}}^2$ through the Fourier identity (2.4), the bound (3.43). ■

In the contraction argument of Section 3.3 the integrand is not a single function S but the time-space shifted heat kernel $S^{(\tau,z)}(s,y) = p_{\tau-s}(z-y)$, one copy for each output point (τ,z), and the integral (3.43) is evaluated at that point. Applying Theorem 3.1.22 uniformly over (τ,z) turns the moment bound into a genuine *stability* statement: the map sending a weight Z to its stochastic convolution preserves the space $\mathcal{A}_{T,p}$ of Definition 3.1.20.

Theorem 3.1.25: Stability of $\mathcal{A}_{T,p}$

Let $p \ge 2$, $T > 0$, and let $S = \left(S^{(\tau,z)}\right)_{(\tau,z)\in[0,T]\times\mathbb{R}^d}$ be a family of deterministic non-negative integrands, each as in Theorem 3.1.15, jointly measurable in (τ,z,s,y), and such that

$$\|S\|_{*,T}^2 := \sup_{0\le\tau\le T}\sup_{z\in\mathbb{R}^d}\left\|S^{(\tau,z)}\,\mathbf{1}_{[0,\tau]}\right\|_{\mathcal{H}}^2 < +\infty. \tag{3.52}$$

Then for every $Z \in \mathcal{A}_{T,p}$ the random field

$$\mathcal{I}_S Z(\tau,z) := \int_0^\tau\int_{\mathbb{R}^d} S^{(\tau,z)}(s,y)\,Z(s,y)\,W(ds,dy), \qquad (\tau,z)\in[0,T]\times\mathbb{R}^d, \tag{3.53}$$

which by (3.25) is the modified stochastic integral $\int_0^\tau\int_{\mathbb{R}^d} S^{(\tau,z)}\,dM^Z$ to which Theorem 3.1.22 applies, admits a jointly measurable, adapted version belonging to $\mathcal{A}_{T,p}$, with

$$\mathcal{N}_{T,p}\left(\mathcal{I}_S Z\right) \le 4p\,\|S\|_{*,T}^2\,\mathcal{N}_{T,p}(Z). \tag{3.54}$$

In particular $Z \mapsto \mathcal{I}_S Z$ is a bounded linear operator on $\mathcal{A}_{T,p}$ with operator norm at most $(4p)^{1/2}\,\|S\|_{*,T}$.

Proof of Theorem 3.1.25. Fix $Z \in \mathcal{A}_{T,p}$. For each $(\tau,z) \in [0,T]\times\mathbb{R}^d$, the integrand $S^{(\tau,z)}$ satisfies the integrability condition (3.31) (since $\|S^{(\tau,z)}\mathbf{1}_{[0,\tau]}\|_{\mathcal{H}} \le \|S\|_{*,T} < \infty$), and $Z \in \mathcal{A}_{\tau,p}$ by restriction (Remark 3.1.21). So Theorem 3.1.22, applied on the interval $[0,\tau]$ in the compact form (3.44), gives

$$\left\|\mathcal{I}_S Z(\tau,z)\right\|_p^2 \le 4p\,\mathcal{N}_{\tau,p}(Z)\left\|S^{(\tau,z)}\,\mathbf{1}_{[0,\tau]}\right\|_{\mathcal{H}}^2 \le 4p\,\|S\|_{*,T}^2\,\mathcal{N}_{T,p}(Z), \tag{3.55}$$

where the second inequality uses $\mathcal{N}_{\tau,p}(Z) \le \mathcal{N}_{T,p}(Z)$ for $\tau \le T$ (Remark 3.1.19) and the definition (3.52) of $\|S\|_{*,T}$. The right-hand side of (3.55) is independent of (τ, z); taking the supremum over $(\tau, z) \in [0,T] \times \mathbb{R}^d$ on the left yields precisely (3.54). Since the kernel family is jointly measurable, the random field (3.53) admits a jointly measurable, adapted version (obtained, e.g., as the limit of the Walsh–Riemann sums defining the integral, each of which is jointly measurable and adapted); by (3.54) this version belongs to $\mathcal{A}_{T,p}$. Finally, $Z \mapsto \mathcal{I}_S Z$ is linear, since the integral (3.53) is linear in the integrand $S^{(\tau,z)}Z$ and hence in Z, and (3.54) is exactly the operator-norm bound $\|\mathcal{I}_S\|_{\mathcal{A}_{T,p}\to\mathcal{A}_{T,p}} \le (4p)^{1/2}\|S\|_{*,T}$. ■

Remark 3.1.26. *With the heat-kernel choice $S^{(\tau,z)}(s,y) = p_{\tau-s}(z-y)$, the operator $\mathcal{I}_S$ of (3.53) is exactly the stochastic convolution, and Theorem 3.1.25 says it maps the solution space $\mathcal{A}_{T,p}$ into itself with a norm governed by the deterministic quantity $\|S\|_{*,T}$ alone. Tuning the horizon T (or inserting an exponential weight in time) so that $4p\,\|S\|^2_{*,T} < 1$ is what will close the fixed point in Section 3.3.*

3.2 Additive noise case

We begin with the simplest instance of the stochastic heat equation, the one in which the noise enters *additively*. It is the tame model already announced in item **(i)** below the PAM (1.2). Although it lacks the multiplicative feedback responsible for intermittency, its definition requires at least to verify that the heat kernel is an admissible integrand for the noise, and to give the resulting stochastic convolution a sense in $L^2(\Omega)$). Isolating these difficulties from the fixed-point argument of Section 3.3 is the purpose of this section; the solution constructed here will moreover reappear there as an intermediate object.

3.2.1 The additive stochastic heat equation

Replacing the multiplicative term $\beta\, u\, \dot W$ in the PAM (1.2) by the forcing $\beta\, \dot W$ alone yields the *additive* stochastic heat equation

$$\partial_t u(t,x) \;=\; \tfrac{1}{2}\Delta\, u(t,x) \;+\; \beta\, \dot W(t,x), \qquad u(0,x) = u_0(x), \tag{3.56}$$

where $\dot W$ is the spatially homogeneous, white-in-time Gaussian noise of Subsection 2.1.1, with spatial covariance Λ and spectral measure $\mu = \mathcal{F}\Lambda$ determined by (2.1), and $u_0 : \mathbb{R}^d \to \mathbb{R}$ is a bounded measurable initial datum. Throughout this section the noise is assumed to satisfy Dalang's condition (2.2), that is Assumption 2.1.1.

3.2.2 Mild formulation

Because the product $u\,\dot W$ has disappeared, equation (3.56) is *linear* with a prescribed random forcing, and the variation-of-constants (Duhamel) principle applies: the heat semigroup with kernel p_t (2.40) propagates the initial datum, while the noise is accumulated and transported over $[0, t]$. Specializing the mild PAM (3.1) to the unit multiplier $u(s, y) \equiv 1$ inside the stochastic integral, we call u a *mild solution* of (3.56) if

$$u(t,x) \;=\; P_t u_0(x) \;+\; \beta \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, W(ds,dy), \tag{3.57}$$

the stochastic integral being understood in the Walsh–Dalang sense of Section 3.1. Here $P_t = e^{t\frac{1}{2}\Delta}$ is again the heat semigroup with Gaussian kernel p_t introduced in (2.40), so that $P_t u_0 = p_t * u_0$. The first term $P_t u_0(x)$ is an ordinary convolution, the classical solution of the homogeneous heat equation, and is bounded whenever u_0 is. All the probabilistic content of (3.57) is therefore carried by the second term, the *stochastic convolution*

$$v(t,x) \;:=\; \beta \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, W(ds,dy), \tag{3.58}$$

so that existence and uniqueness for (3.56) reduce to showing that v is a well-defined random field.

Remark 3.2.1. *We have phrased existence and uniqueness in the* mild *sense of* (3.57)*. This is no loss of generality: under the standing assumptions, the mild solution is also a* weak *(distributional) solution of* (3.56)*, and the two notions coincide. A proof of this equivalence can be found in [36, Section 5.3].*

3.2.3 The stochastic convolution is well defined

The integrand in (3.58) is the *deterministic* function $S(s, y) = p_{t-s}(x - y)$, so (3.58) is an integral against the martingale measure M associated with the noise W (Section 3.1), equivalently the modified measure M^Z of Definition 3.1.11 with the trivial weight $Z \equiv 1$. For this weight the correlation (3.27) is $g_s \equiv 1$, so that the effective spectral measure (3.28) reduces to $\mu_s^Z = \mu$ and the admissibility of S is governed by Dalang's condition for μ itself. We record this computation, in the form reused for general weights in Section 3.3, in the following proposition.

Proposition 3.2.2: The heat kernel as integrand

Fix $(t, x) \in (0, T] \times \mathbb{R}^d$. Under Dalang's condition (Assumption 2.1.1), the deterministic integrand $S(s, y) = p_{t-s}(x-y)$ is admissible for the trivial weight $Z \equiv 1$, so that the stochastic integral

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, W(ds,dy) \tag{3.59}$$

is well-defined.

Proof. Consider the deterministic integrand

$$S(s,y) \;:=\; p_{t-s}(x-y), \qquad 0 \le s < t, \tag{3.60}$$

where p_t is the heat kernel (2.40). Since S is deterministic, we may dispense with the weighted theory and work directly with the extended norm $\|\cdot\|_{\mathcal{H}_\omega}$ of Definition 3.1.6: by the deterministic Fourier identity (3.21) (Remark 3.1.9, with $g = h = S\,\mathbf{1}_{[0,t]}$),

$$\left\|S\,\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}_\omega} \;=\; \int_0^t \int_{\mathbb{R}^d} \left|\mathcal{F}_x S(s,\cdot)(\xi)\right|^2 \mu(d\xi)\,ds. \tag{3.61}$$

The spatial Fourier transform of the integrand is

$$\mathcal{F}_x S(s,\cdot)(\xi) \;=\; e^{-\imath\xi\cdot x}\, e^{-\frac{1}{2}(t-s)|\xi|^2}. \tag{3.62}$$

Inserting (3.62) into (3.61) and integrating in $s \in [0,t]$,

$$\left\|S\,\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}_\omega} \;=\; \int_{\mathbb{R}^d} \frac{1-e^{-t|\xi|^2}}{|\xi|^2}\,\mu(d\xi) \;\lesssim\; \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2}, \tag{3.63}$$

where the last inequality uses $\frac{1-e^{-t|\xi|^2}}{|\xi|^2} \le \frac{1}{|\xi|^2} \lesssim \frac{1}{1+|\xi|^2}$ for $|\xi| \ge 1$, and $\frac{1-e^{-t|\xi|^2}}{|\xi|^2} \le t$ for $|\xi| \le 1$. The right-hand side of (3.63) is finite by Assumption 2.1.1 (equation (2.2)). Hence $S\,\mathbf{1}_{[0,t]} \in \mathcal{H}_\omega$ under Dalang's condition alone, and Theorem 3.1.8 shows that the stochastic integral (3.59) is well-defined, with second moment given by the isometry (3.20). ■

Up to the factor β, the integral (3.59) is exactly the stochastic convolution (3.58) driving the additive equation (3.56). The general weighted case $Z = u$, needed when the noise is multiplied by the unknown solution, is treated in Section 3.3.

Proposition 3.2.3: Existence and uniqueness for the additive equation

Assume Dalang's condition (Assumption 2.1.1) and let u_0 be a bounded measurable initial datum. Then:

(i) The stochastic convolution (3.58) is, for every $(t,x) \in (0,T] \times \mathbb{R}^d$, a well-defined centered Gaussian random variable in $L^2(\Omega)$, with variance

$$\mathbb{E}\Big[v(t,x)^2\Big] \;=\; \beta^2 \int_{\mathbb{R}^d} \frac{1-e^{-t|\xi|^2}}{|\xi|^2}\,\mu(d\xi) \;<\; \infty, \tag{3.64}$$

finite and independent of x;

(ii) The additive equation (3.56) admits a unique mild solution, given explicitly by (3.57). It is a Gaussian field, bounded in $L^2(\Omega)$ uniformly on $[0,T] \times \mathbb{R}^d$.

Proof. By Proposition 3.2.2, the deterministic integrand $S(s,y) = p_{t-s}(x-y)$ satisfies $S\,\mathbf{1}_{[0,t]} \in \mathcal{H}_\omega$, so the integral (3.58) is well-defined by Theorem 3.1.8. Its second moment is the isometry (3.20) evaluated at S:

$$\mathbb{E}\Big[v(t,x)^2\Big] \;=\; \beta^2 \left\|S\,\mathbf{1}_{[0,t]}\right\|^2_{\mathcal{H}_\omega} \;=\; \beta^2 \int_{\mathbb{R}^d} \frac{1-e^{-t|\xi|^2}}{|\xi|^2}\,\mu(d\xi),$$

the last equality being the computation (3.63); it is finite by Dalang's condition (2.2) and, depending on t alone, independent of x. As an $L^2(\Omega)$-limit of Wiener integrals of deterministic integrands, $v(t,x)$ is centered Gaussian. This proves (i).

For (ii), define u by the right-hand side of (3.57), that is, $u(t,x) = P_t u_0(x) + v(t,x)$. The first term is the bounded deterministic function $P_t u_0$, and the second is the well-defined random field v just constructed, so u exists as a random field and solves (3.56) in the mild sense; this gives existence. Conversely, the forcing in (3.56) being prescribed (it does not involve the unknown), the mild formulation (3.57) determines $u(t,x)$ explicitly for every (t,x), so the mild solution is unique. Finally, u is Gaussian as the sum of the deterministic term $P_t u_0$ and the Gaussian field v, and the uniform L^2 bound follows from (3.64) together with the boundedness of $P_t u_0$. ■

3.2.4 Hölder continuity of the stochastic convolution

Proposition 3.2.3 constructs the stochastic convolution v of (3.58) as an $L^2(\Omega)$-valued field, but says nothing about the regularity of its trajectories $(t,x) \mapsto v(t,x)$. Under Dalang's condition (2.2) alone one can only show that v is continuous in $L^2(\Omega)$; genuine *pathwise* regularity requires the small amount of extra integrability already alluded to for the noise itself in Assumption 2.4.3. We recall that assumption here for convenience.

Strong Dalang condition. Throughout this subsection we assume that the spectral measure μ satisfies the strong Dalang condition of Assumption 2.4.3: there exists $\alpha \in (0,1)$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^{2(1-\alpha)}} < \infty, \tag{3.65}$$

which is (2.55). As noted after Assumption 2.4.3, relation (3.65) is stronger than Dalang's condition (2.2) (to which it reduces as $\alpha \to 0$), and the gap between the two is precisely the price for pathwise regularity. The parameter α will turn out to be the spatial Hölder exponent, the parabolic scaling of the heat equation halving it in the time variable.

In order to quantify pathwise Hölder regularity, we will rely on the following space-time version of Kolmogorov's continuity criterion, which we state in the parabolic form adapted to the heat equation (see [19, Chapter 3]).

Theorem 3.2.4: Kolmogorov's criterion, parabolic form

Equip $[0,T] \times \mathbb{R}^d$ with the parabolic distance

$$\rho\Big((t,x),(s,y)\Big) := \Big(|t-s| + |x-y|^2\Big)^{1/2}. \tag{3.66}$$

Let $\{X(t,x) : (t,x) \in [0,T] \times \mathbb{R}^d\}$ be a real-valued random field and suppose there exist $p \in [2,\infty)$, an exponent $\gamma \in (0,1]$ and a constant C such that

$$\mathbb{E}\Big[\,|X(t,x) - X(s,y)|^p\,\Big] \le C\,\rho\Big((t,x),(s,y)\Big)^{\gamma p}, \qquad (t,x),(s,y) \in [0,T] \times \mathbb{R}^d. \tag{3.67}$$

If $\gamma p > d+2$, then X admits a modification whose trajectories are, on every compact subset of $[0,T] \times \mathbb{R}^d$, almost surely Hölder continuous of every order $\theta < \gamma - \frac{d+2}{p}$ with respect to ρ; equivalently, of order θ in the space variable and $\theta/2$ in the time variable. The corresponding Hölder seminorms have finite moments of all orders.

Observe that the exponent $d+2$ in the threshold $\gamma p > d+2$ is the volume-growth dimension of the parabolic metric (3.66): a ball of radius r for ρ has Lebesgue measure of order $r^2 \cdot r^d = r^{d+2}$, the factor r^2 accounting for the time direction. We can now state the main result of this subsection.

Theorem 3.2.5: Hölder continuity of the stochastic convolution

Assume the strong Dalang condition (3.65) (Assumption 2.4.3) for some $\alpha \in (0,1)$, and let u_0 be a bounded measurable initial datum. Then the stochastic convolution v of (3.58) admits a modification whose trajectories are, on every compact subset of $[0,T] \times \mathbb{R}^d$, almost surely jointly Hölder continuous of every order $\theta_x < \alpha$ in the space variable and every order $\theta_t < \frac{\alpha}{2}$ in the time variable.

The proof follows a single principle: because the integrand in (3.58) is deterministic, every increment of v is a centered Gaussian variable, so its p-th moment is a power of its variance, and that variance is computed exactly through the Fourier identity (3.21) as in the proof of Proposition 3.2.2. The space and time increments are estimated separately, each through a decomposition of the heat-kernel integrand of the type used in the existence theory for the multiplicative equation.

Proof of Theorem 3.2.5. Since the integrand $\beta\, p_{t-s}(x-y)\, \mathbf{1}_{[0,t]}(s)$ in (3.58) is deterministic, for each (t,x) the random variable $v(t,x)$ is a centered Gaussian; so is every increment $v(t,x) - v(s,y)$, being a finite linear combination of Wiener integrals of deterministic integrands. Hence for every $p \ge 2$,

$$\mathbb{E}\Big[\,|v(t,x) - v(s,y)|^p\,\Big] = c_p\, \mathbb{E}\Big[\,|v(t,x) - v(s,y)|^2\,\Big]^{p/2}, \tag{3.68}$$

where the constant c_p can be expressed as $c_p = \mathbb{E}[|N|^p]$ for a standard Gaussian random variable $N \sim \mathcal{N}(0,1)$. By Kolmogorov's criterion (Theorem 3.2.4) it therefore suffices to control the second moment of the increments by the parabolic distance (3.66). Otherwise stated we have to prove

$$\mathbb{E}\Big[\,|v(t,x)-v(s,y)|^2\,\Big] \;\le\; C\,\Big(|t-s|+|x-y|^2\Big)^{\alpha}. \tag{3.69}$$

Indeed, inserting (3.69) into (3.68) yields (3.67) with $\gamma=\alpha$ for every p, and since the Gaussian field v has moments of all orders, letting $p\to\infty$ in Theorem 3.2.4 produces every space Hölder order $\theta_x<\alpha$ and every time Hölder order $\theta_t<\frac{\alpha}{2}$, which is the assertion of the theorem.

Kolmogorov's criterion is a local statement: only the bound (3.67) for points at parabolic distance at most 1 enters, so we may and do assume $|x-y|\le 1$ throughout (no restriction on $|t-s|\in[0,T]$ is needed). In addition, a simple application of the triangle inequality in $L^2(\Omega)$ yields

$$\mathbb{E}\Big[|v(t,x)-v(s,y)|^2\Big]^{1/2} \;\le\; \mathbb{E}\Big[|v(t,x)-v(t,y)|^2\Big]^{1/2}+\mathbb{E}\Big[|v(t,y)-v(s,y)|^2\Big]^{1/2}. \tag{3.70}$$

Therefore (3.69) follows once we establish the spatial bound

$$\mathbb{E}\Big[\,|v(t,x)-v(t,y)|^2\,\Big] \;\le\; C\,|x-y|^{2\alpha} \tag{3.71}$$

as well as the temporal bound

$$\mathbb{E}\Big[\,|v(t,x)-v(s,x)|^2\,\Big] \;\le\; C\,|t-s|^{\alpha}. \tag{3.72}$$

We will prove (3.71) and (3.72) in the two steps below.

Step 1: Spatial increment. The increment $v(t,x)-v(t,y)$ is the stochastic integral of the deterministic integrand $\beta[p_{t-\cdot}(x-\cdot)-p_{t-\cdot}(y-\cdot)]\mathbf{1}_{[0,t]}$. By the isometry (3.20) in its deterministic form (Remark 3.1.9) and the Fourier identity (3.21), exactly as in the proof of Proposition 3.2.2, we get

$$\mathbb{E}\Big[\,|v(t,x)-v(t,y)|^2\,\Big] \;=\; \beta^2\int_0^t\!\int_{\mathbb{R}^d}\Big|\mathcal{F}_x\Big[p_{t-s}(x-\cdot)-p_{t-s}(y-\cdot)\Big](\xi)\Big|^2\,\mu(d\xi)\,ds. \tag{3.73}$$

Owing to the form of the spatial Fourier transform (3.62) we have

$$\mathcal{F}_x[p_{t-s}(x-\cdot)](\xi)=e^{-\imath\xi\cdot x}\,e^{-\frac{1}{2}(t-s)|\xi|^2},$$

so the integrand in (3.73) equals

$$\Big|e^{-\imath\xi\cdot x}-e^{-\imath\xi\cdot y}\Big|^2\,e^{-(t-s)|\xi|^2} \;=\; \Big|1-e^{\imath\xi\cdot(x-y)}\Big|^2\,e^{-(t-s)|\xi|^2}. \tag{3.74}$$

Integrating in $s\in[0,t]$ as in (3.63), with $\int_0^t e^{-(t-s)|\xi|^2}\,ds=\frac{1-e^{-t|\xi|^2}}{|\xi|^2}\le\frac{1}{|\xi|^2}$, we obtain

$$\mathbb{E}\Big[\,|v(t,x)-v(t,y)|^2\,\Big] \;\le\; \beta^2\int_{\mathbb{R}^d}\frac{|1-e^{\imath\xi\cdot(x-y)}|^2}{|\xi|^2}\,\mu(d\xi). \tag{3.75}$$

We now invoke the elementary inequality: for all $\xi \in \mathbb{R}^d$ and all $h \in \mathbb{R}^d$ with $|h| \leq 1$,

$$\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2} \leq C_\alpha \frac{|h|^{2\alpha}}{1+|\xi|^{2(1-\alpha)}}. \tag{3.76}$$

To prove (3.76), we start from the two elementary bounds $|1-e^{\imath\xi\cdot h}| \leq |\xi|\,|h|$ and $|1-e^{\imath\xi\cdot h}| \leq 2$, and we split the ratio of interest into two factors,

$$\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2} = \left(\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2}\right)^{1-\alpha}\left(\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2}\right)^{\alpha}. \tag{3.77}$$

In the first factor of (3.77) we use $|1-e^{\imath\xi\cdot h}| \leq 2$, while in the second factor we use $|1-e^{\imath\xi\cdot h}| \leq |\xi|\,|h|$. This yields

$$\left(\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2}\right)^{1-\alpha} \leq \frac{4^{1-\alpha}}{|\xi|^{2(1-\alpha)}}, \qquad \left(\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2}\right)^{\alpha} \leq |h|^{2\alpha}. \tag{3.78}$$

Multiplying the two bounds in (3.78) gives

$$\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2} \leq C_\alpha \frac{|h|^{2\alpha}}{|\xi|^{2(1-\alpha)}}. \tag{3.79}$$

We now compare the right-hand side of (3.79) with that of (3.76) according to the size of $|\xi|$. For $|\xi| \geq 1$ we have $|\xi|^{2(1-\alpha)} \geq \frac{1}{2}\Big(1+|\xi|^{2(1-\alpha)}\Big)$, so (3.79) gives (3.76). For $|\xi| \leq 1$ we instead bound the ratio on the left hand side of (3.79) directly, using $|1-e^{\imath\xi\cdot h}| \leq |\xi|\,|h|$ and $|h| \leq 1$,

$$\frac{|1-e^{\imath\xi\cdot h}|^2}{|\xi|^2} \leq |h|^2 \leq |h|^{2\alpha}, \tag{3.80}$$

which together with $1+|\xi|^{2(1-\alpha)} \leq 2$ again yields (3.76). This proves the elementary inequality (3.76). Now inserting (3.76) with $h = x-y$ into (3.75), we end up with

$$\mathbb{E}\Big[\,|v(t,x)-v(t,y)|^2\,\Big] \leq C_\alpha\,\beta^2\,|x-y|^{2\alpha}\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}} = C\,|x-y|^{2\alpha},$$

finite by the strong Dalang condition (3.65). This proves (3.71).

Step 2: Temporal increment. Fix $0 \leq s \leq t \leq T$. We split the time increment into a near-diagonal and an off-diagonal part, $v(t,x)-v(s,x) = \beta\,(I_1+I_2)$, where

$$I_1 := \int_s^t\int_{\mathbb{R}^d} p_{t-r}(x-y)\,W(dr,dy), \tag{3.81}$$

$$I_2 := \int_0^s\int_{\mathbb{R}^d}\Big[p_{t-r}(x-y)-p_{s-r}(x-y)\Big]\,W(dr,dy). \tag{3.82}$$

Since I_1 and I_2 integrate the noise over the disjoint time slabs $[s,t]$ and $[0,s]$, and W has independent increments in time, they are independent centered Gaussian variables. Hence we obtain

$$\mathbb{E}\Big[\,|v(t,x)-v(s,x)|^2\,\Big] = \beta^2\Big(\mathbb{E}[I_1^2]+\mathbb{E}[I_2^2]\Big). \tag{3.83}$$

The near-diagonal term I_1. Similarly to what we did for (3.73), we apply the isometry (3.20) and the Fourier identity (3.21), using $|\mathcal{F}_x p_{t-r}(x-\cdot)(\xi)|^2 = e^{-(t-r)|\xi|^2}$ from (3.62). We discover that

$$\mathbb{E}[I_1^2] \;=\; \int_s^t \int_{\mathbb{R}^d} e^{-(t-r)|\xi|^2}\,\mu(d\xi)\,dr \;=\; \int_{\mathbb{R}^d} \frac{1-e^{-(t-s)|\xi|^2}}{|\xi|^2}\,\mu(d\xi), \tag{3.84}$$

the last equality being obtained by an elementary integration in the time variable. We will show how to bound the right hand side of (3.84) below.

The off-diagonal term I_2. As for I_1, invoke (3.20) and (3.21) to get

$$\mathbb{E}[I_2^2] \;=\; \int_0^s \int_{\mathbb{R}^d} \Big|\mathcal{F}_x\Big[p_{t-r}(x-\cdot) - p_{s-r}(x-\cdot)\Big](\xi)\Big|^2\,\mu(d\xi)\,dr. \tag{3.85}$$

Using $\mathcal{F}_x p_\tau(x-\cdot)(\xi) = e^{-\imath\xi\cdot x}e^{-\frac{1}{2}\tau|\xi|^2}$, the squared modulus of the bracket factors as

$$\Big(e^{-\frac{1}{2}(s-r)|\xi|^2} - e^{-\frac{1}{2}(t-r)|\xi|^2}\Big)^2 \;=\; e^{-(s-r)|\xi|^2}\Big(1-e^{-\frac{1}{2}(t-s)|\xi|^2}\Big)^2. \tag{3.86}$$

Integrating in $r\in[0,s]$, with $\int_0^s e^{-(s-r)|\xi|^2}\,dr = (1-e^{-s|\xi|^2})/|\xi|^2 \le 1/|\xi|^2$, and bounding the square by itself, $(1-e^{-\frac{1}{2}(t-s)|\xi|^2})^2 \le 1-e^{-\frac{1}{2}(t-s)|\xi|^2}$, we get

$$\mathbb{E}[I_2^2] \;\le\; \int_{\mathbb{R}^d} \frac{1-e^{-\frac{1}{2}(t-s)|\xi|^2}}{|\xi|^2}\,\mu(d\xi). \tag{3.87}$$

The right hand sides of (3.84) and (3.87) are of the same kind. Both are controlled by a second elementary inequality: for every $c>0$, every $0\le t-s\le T$, and every $\xi\in\mathbb{R}^d$,

$$\frac{1-e^{-c(t-s)|\xi|^2}}{|\xi|^2} \;\le\; C_{\alpha,c,T}\,\frac{(t-s)^\alpha}{1+|\xi|^{2(1-\alpha)}}. \tag{3.88}$$

To prove (3.88), we start from the elementary bound

$$1-e^{-a} \;\le\; \min(1,a) \;\le\; a^\alpha, \qquad a\ge 0,\;\; \alpha\in(0,1). \tag{3.89}$$

We now apply (3.89) according to the size of $|\xi|$, with $a = c(t-s)|\xi|^2$. For $|\xi|\ge 1$, the third bound in (3.89) gives $1-e^{-c(t-s)|\xi|^2} \le c^\alpha(t-s)^\alpha|\xi|^{2\alpha}$, hence

$$\frac{1-e^{-c(t-s)|\xi|^2}}{|\xi|^2} \;\le\; c^\alpha\,\frac{(t-s)^\alpha}{|\xi|^{2(1-\alpha)}} \;\le\; 2\,c^\alpha\,\frac{(t-s)^\alpha}{1+|\xi|^{2(1-\alpha)}}, \tag{3.90}$$

where in the last step we used $|\xi|^{2(1-\alpha)} \ge \frac{1}{2}(1+|\xi|^{2(1-\alpha)})$ for $|\xi|\ge 1$. For $|\xi|\le 1$, the first bound in (3.89) together with $t-s\le T$ gives instead

$$\frac{1-e^{-c(t-s)|\xi|^2}}{|\xi|^2} \;\le\; c\,(t-s) \;\le\; c\,T^{1-\alpha}\,(t-s)^\alpha \;\le\; 2\,c\,T^{1-\alpha}\,\frac{(t-s)^\alpha}{1+|\xi|^{2(1-\alpha)}}, \tag{3.91}$$

using $1 + |\xi|^{2(1-\alpha)} \leq 2$ in the last step. Bounds (3.90) and (3.91) together yield (3.88) with $C_{\alpha,c,T} = 2\max(c^\alpha, c\,T^{1-\alpha})$.

We can now conclude: plugging (3.88) (with $c = 1$ and $c = \frac{1}{2}$) into (3.84) and (3.87), then inserting into (3.83), we end up with

$$\mathbb{E}\Big[\,|v(t,x) - v(s,x)|^2\,\Big] \leq C\,(t-s)^\alpha \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^{2(1-\alpha)}} = C\,(t-s)^\alpha,$$

where the constant C is finite by (3.65). This proves (3.72) and completes the proof. ■

The Hölder orders α in space and $\frac{\alpha}{2}$ in time exhibit the parabolic scaling of the heat equation: time counts as two space derivatives. In the model case of space-time white noise in dimension $d = 1$, where $\mu(d\xi) = d\xi$, the strong Dalang condition (3.65) holds for every $\alpha < \frac{1}{2}$, and Theorem 3.2.5 recovers the classical regularity of the stochastic heat equation: Hölder continuous of every order below $\frac{1}{2}$ in space and below $\frac{1}{4}$ in time.

Remark 3.2.6. *The deterministic part $P_t u_0$ of the mild solution (3.57) is, for $t > 0$, of class $\mathcal{C}^\infty$ in (t,x) by the smoothing property of the heat semigroup. Consequently the mild solution $u = P_t u_0 + v$ of (3.56) inherits, on every compact subset of $(0,T] \times \mathbb{R}^d$, the Hölder regularity of v stated in Theorem 3.2.5: order $\theta_x < \alpha$ in space and $\theta_t < \frac{\alpha}{2}$ in time.*

3.3 Existence and uniqueness of PAM

Throughout this section we fix $p \geq 2$, a coupling constant $\beta \in \mathbb{R}$, and a bounded measurable initial datum $u_0 : \mathbb{R}^d \to \mathbb{R}$, and we assume Dalang's condition (Assumption 2.1.1). A *mild solution* of the PAM on $[0,T]$ is a process $u \in \mathcal{A}_{T,p}$ (Definition 3.1.20) satisfying the mild equation (3.1) for every $(t,x) \in [0,T] \times \mathbb{R}^d$.

Most existence-uniqueness results in this context (see e.g Dalang [18, Theorem 13]) constructs such a solution by a Picard iteration. We give here a slight variation: our construction hinges on a single application of the Banach fixed point theorem. The whole argument rests on the one quantitative input of Subsection 3.1.5, namely that $\mathcal{I}_S$ contracts $\mathcal{A}_{\tau,p}$ once the horizon τ is small.

3.3.1 The solution map and its contraction constant

Specialize the kernel family of Theorem 3.1.25 to the heat kernel: for $(t,x) \in [0,T] \times \mathbb{R}^d$ put

$$S^{(t,x)}(s,y) := p_{t-s}(x-y), \qquad 0 \leq s \leq t,\ y \in \mathbb{R}^d, \tag{3.92}$$

with p the heat kernel (2.40). Each $S^{(t,x)}$ is a non-negative integrand of the type required by Theorem 3.1.15, and $(t,x,s,y) \mapsto S^{(t,x)}(s,y)\,\mathbf{1}_{[0,t]}(s)$ is jointly measurable. For this family the operator $\mathcal{I}_S$ of (3.53) is the stochastic convolution

$$\mathcal{I}_S Z(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\,Z(s,y)\,W(ds,dy), \tag{3.93}$$

so that the mild equation (3.1) reads compactly as $u = P.u_0 + \beta\,\mathcal{I}_S u$.

The size of $\mathcal{I}_S$ over the horizon τ is measured by the constant $\|S\|_{*,\tau}^2$ of (3.52), which for the heat family is explicit.

Lemma 3.3.1

For $\tau \geq 0$ set

$$\Phi(\tau) \;:=\; \int_{\mathbb{R}^d} \frac{1 - e^{-\tau|\xi|^2}}{|\xi|^2}\,\mu(d\xi). \tag{3.94}$$

Under Assumption 2.1.1 the function Φ is finite, non-decreasing, and $\Phi(\tau) \to 0$ as $\tau \downarrow 0$. Moreover, for the heat family (3.92),

$$\|S\|_{*,\tau}^2 \;=\; \sup_{0\leq t\leq\tau}\,\sup_{x\in\mathbb{R}^d}\Big\|S^{(t,x)}\,\mathbf{1}_{[0,t]}\Big\|_{\mathcal{H}}^2 \;=\; \Phi(\tau). \tag{3.95}$$

Proof. By the Fourier identity (2.4) and the transform (3.62),

$$\Big\|S^{(t,x)}\,\mathbf{1}_{[0,t]}\Big\|_{\mathcal{H}}^2 \;=\; \int_0^t\int_{\mathbb{R}^d} e^{-(t-s)|\xi|^2}\,\mu(d\xi)\,ds \;=\; \Phi(t),$$

the last equality being the computation (3.63); it is independent of x and non-decreasing in t, which gives (3.95). Finiteness of $\Phi(\tau)$ is again (3.63): one has $\Phi(\tau) \lesssim \int_{\mathbb{R}^d}\mu(d\xi)/(1+|\xi|^2) < \infty$ by Dalang's condition (2.2). The integrand in (3.94) is non-negative and non-decreasing in τ, so Φ is non-decreasing; for $\tau \leq 1$ it is dominated by $\min(1,|\xi|^{-2}) \lesssim (1+|\xi|^2)^{-1} \in L^1(\mu)$ and tends to 0 pointwise as $\tau \downarrow 0$, so $\Phi(\tau) \to 0$ by dominated convergence. ■

We can now introduce the solution map.

Definition 3.3.2: Solution map

For $\tau \in (0,T]$ and $Z \in \mathcal{A}_{\tau,p}$, let $\Gamma_\tau Z$ be the process

$$\Gamma_\tau Z(t,x) \;:=\; P_t u_0(x) \;+\; \beta\,\mathcal{I}_S Z(t,x), \qquad (t,x) \in [0,\tau]\times\mathbb{R}^d. \tag{3.96}$$

Proposition 3.3.3: Γ_τ is an affine contraction

Let $\tau \in (0,T]$. Then Γ_τ maps $\mathcal{A}_{\tau,p}$ into itself, and for all $Z_1, Z_2 \in \mathcal{A}_{\tau,p}$,

$$\Big\|\Gamma_\tau Z_1 - \Gamma_\tau Z_2\Big\|_{\mathcal{A}_{\tau,p}} \;\leq\; 2\,|\beta|\,\sqrt{p\,\Phi(\tau)}\;\Big\|Z_1 - Z_2\Big\|_{\mathcal{A}_{\tau,p}}. \tag{3.97}$$

In particular Γ_τ is a contraction of $\mathcal{A}_{\tau,p}$ as soon as

$$4\,p\,\beta^2\,\Phi(\tau) \;<\; 1. \tag{3.98}$$

Proof. Stability. The free term $f(t,x) := P_t u_0(x)$ is deterministic, jointly measurable, and bounded by $\|u_0\|_\infty$ (the heat kernel being a probability density). Therefore we get that $f \in \mathcal{A}_{\tau,p}$ and

$$\mathcal{N}_{\tau,p}(f) = \sup_{t\leq\tau}\sup_x |P_t u_0(x)|^2 \leq \|u_0\|_\infty^2 < \infty.$$

In addition, owing to Lemma 3.3.1, $\|S\|_{*,\tau} = \Phi(\tau)^{1/2} < \infty$, so Theorem 3.1.25 gives $\mathcal{I}_S Z \in \mathcal{A}_{\tau,p}$. Since $\mathcal{A}_{\tau,p}$ is a vector space, $\Gamma_\tau Z = f + \beta\,\mathcal{I}_S Z \in \mathcal{A}_{\tau,p}$.

Contraction. The free term f cancels in the difference, and $\mathcal{I}_S$ is linear (Theorem 3.1.25), so $\Gamma_\tau Z_1 - \Gamma_\tau Z_2 = \beta\,\mathcal{I}_S(Z_1 - Z_2)$. By the operator bound (3.54) and the identity (3.95), we discover that

$$\begin{aligned}\left\|\Gamma_\tau Z_1 - \Gamma_\tau Z_2\right\|_{\mathcal{A}_{\tau,p}}^2 &= \beta^2\,\mathcal{N}_{\tau,p}\Big(\mathcal{I}_S(Z_1 - Z_2)\Big) \leq 4\,p\,\beta^2\,\|S\|_{*,\tau}^2\,\mathcal{N}_{\tau,p}(Z_1 - Z_2)\\ &= 4\,p\,\beta^2\,\Phi(\tau)\left\|Z_1 - Z_2\right\|_{\mathcal{A}_{\tau,p}}^2,\end{aligned}$$

which is (3.97). The Lipschitz constant is < 1 exactly under (3.98). ∎

3.3.2 Local existence and uniqueness

The contraction constant in (3.97) tends to 0 with τ, by Lemma 3.3.1. This is all the contraction argument needs.

Theorem 3.3.4: Local well-posedness

Assume Dalang's condition (Assumption 2.1.1) and let u_0 be bounded measurable. There is a horizon $\tau_* > 0$, depending only on p, β and the spectral measure μ (and *not* on u_0), such that for every $\tau \leq \tau_*$ the PAM (3.1) has a unique mild solution $u \in \mathcal{A}_{\tau,p}$ on $[0,\tau]$, where we recall that $\mathcal{A}_{\tau,p}$ is the space of Definition 3.1.20.

Proof. Since $\Phi(\tau) \to 0$ as $\tau \downarrow 0$ (Lemma 3.3.1), there is a largest $\tau_* > 0$ with $4p\beta^2\Phi(\tau_*) \leq \frac{1}{2}$, and τ_* depends only on p, β, μ. For every $\tau \leq \tau_*$ the smallness (3.98) holds, so by Proposition 3.3.3 the map Γ_τ is a contraction (of Lipschitz constant at most $1/\sqrt{2}$) of the Banach space $\mathcal{A}_{\tau,p}$ (Definition 3.1.20). By the Banach fixed point theorem it has a unique fixed point $u \in \mathcal{A}_{\tau,p}$, and the fixed point equation $u = \Gamma_\tau u$ is precisely the mild equation (3.1) on $[0,\tau]$. Uniqueness of the fixed point is uniqueness of the mild solution in $\mathcal{A}_{\tau,p}$. ∎

3.3.3 Globalization by patching

The horizon τ_* of Theorem 3.3.4 depends only on p, β, μ, not on the initial datum. This uniformity is what allows the local construction to be iterated over consecutive windows of equal length and the pieces patched into a global solution.

Theorem 3.3.5: Global existence and uniqueness for the PAM

Assume Dalang's condition (Assumption 2.1.1) and let u_0 be bounded measurable. Then for every $T > 0$ and every $p \geq 2$ the PAM (3.1) has a unique mild solution $u \in \mathcal{A}_{T,p}$.

Proof. Fix $\tau = \tau_*$ from Theorem 3.3.4 and let $N = \lceil T/\tau \rceil$, so that $[0, T] \subseteq \bigcup_{k=0}^{N-1}[k\tau, (k+1)\tau]$. We build the solution one window at a time.

Step 1: The restarted map. On a window $[a, a+\tau]$, given an $\mathcal{F}_a$-measurable random initial datum w with $\|w\|_{\infty,p} := \sup_x \|w(x)\|_p < \infty$, define for a process Z on that window

$$\Gamma^{a,w} Z(t, x) := P_{t-a}w(x) + \beta \int_a^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, Z(s, y)\, W(ds, dy), \quad t \in [a, a+\tau]. \tag{3.99}$$

Two facts make $\Gamma^{a,w}$ a contraction with the *same* constant as on $[0, \tau]$. First, the free term $P_{t-a}w$ is $\mathcal{F}_a$-measurable, hence adapted on $[a, a+\tau]$, and Minkowski's inequality gives $\sup_t \sup_x \|P_{t-a}w(x)\|_p \leq \|w\|_{\infty,p}$ (again because the heat kernel is a probability density). Second, the convolution over the window has the same size as on $[0, \tau]$,

$$\sup_{a \leq t \leq a+\tau} \sup_{x \in \mathbb{R}^d} \left\| p_{t-\cdot}(x - \cdot)\, \mathbf{1}_{[a,t]} \right\|_{\mathcal{H}}^2 = \sup_{a \leq t \leq a+\tau} \Phi(t-a) = \Phi(\tau),$$

by the very same computation (3.63), since it depends only on the elapsed time $t - a$ and the noise is homogeneous in time. Hence the proof of Proposition 3.3.3 applies verbatim (with $\mathcal{A}_{\tau,p}$ replaced by its time-translate to $[a, a+\tau]$), and for $\tau = \tau_*$ the map $\Gamma^{a,w}$ has a unique fixed point on $[a, a+\tau]$.

Step 2: Patching. Apply Step 1 successively. On $[0, \tau]$ with $w = u_0$ we recover the unique solution u of Theorem 3.3.4. Suppose u has been built up to time $k\tau$ with $\sup_{t \leq k\tau} \sup_x \|u(t, x)\|_p < \infty$. Set $w := u(k\tau, \cdot)$, which is $\mathcal{F}_{k\tau}$-measurable with $\|w\|_{\infty,p} \leq \mathcal{N}_{k\tau,p}(u)^{1/2} < \infty$. Then applying Step 1 on $[k\tau, (k+1)\tau]$, we produce the unique continuation. The concatenation solves the mild equation (3.1) on $[0, (k+1)\tau]$. Indeed, since u already solves (3.1) up to time $k\tau$, the restart datum is

$$u(k\tau, z) = P_{k\tau}u_0(z) + \beta \int_0^{k\tau} \int_{\mathbb{R}^d} p_{k\tau-s}(z-y)\, u(s, y)\, W(ds, dy). \tag{3.100}$$

Applying the operator $P_{t-k\tau}$, i.e. integrating (3.100) against $p_{t-k\tau}(x - \cdot)$, the restarted free term expands as

$$\begin{aligned}
&P_{t-k\tau}u(k\tau, \cdot)(x) \\
&= P_{t-k\tau}P_{k\tau}u_0(x) + \beta \int_0^{k\tau} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_{t-k\tau}(x-z)\, p_{k\tau-s}(z-y)\, dz \right) u(s, y)\, W(ds, dy) \\
&= P_t u_0(x) + \beta \int_0^{k\tau} \int_{\mathbb{R}^d} p_{t-s}(x-y)\, u(s, y)\, W(ds, dy),
\end{aligned}$$

where the first equality moves $P_{t-k\tau}$ through the stochastic integral by a stochastic Fubini theorem, and the second uses the semigroup property $P_{t-k\tau}P_{k\tau} = P_t$ on the free term together with the Chapman–Kolmogorov identity

$$\int_{\mathbb{R}^d} p_{t-k\tau}(x-z)\, p_{k\tau-s}(z-y)\, dz \;=\; p_{t-s}(x-y) \tag{3.101}$$

on the stochastic term. Hence the two stochastic integrals in (3.99) over $[0, k\tau]$ and $[k\tau, t]$ recombine into the single integral of (3.1) over $[0, t]$. After N steps we obtain $u \in \mathcal{A}_{T,p}$, its $\mathcal{A}_{T,p}$-norm being the maximum of the N finite window norms.

Step 3: Uniqueness. Let $u, \tilde{u} \in \mathcal{A}_{T,p}$ both solve (3.1). On $[0, \tau]$ both are fixed points of Γ_τ, hence equal by Theorem 3.3.4; in particular $u(\tau, \cdot) = \tilde{u}(\tau, \cdot)$. Assuming $u = \tilde{u}$ up to time $k\tau$, both restrict on $[k\tau, (k+1)\tau]$ to a fixed point of the same restarted map $\Gamma^{k\tau,\, u(k\tau,\cdot)}$, so they agree there too. After N steps, we get $u = \tilde{u}$ on $[0, T]$. ■

Remark 3.3.6. *The only place where the noise enters is through the scalar $\Phi(\tau)$ of (3.94), via the stability estimate (3.54). Compared with Dalang's Picard scheme [18, Theorem 13], the contraction argument trades the explicit iterates u_n for a single fixed-point statement, at the cost of working on a small horizon τ_* and patching; the gain is that convergence is immediate from completeness of $\mathcal{A}_{\tau,p}$, with no need to track the series $\sum_n \|u_{n+1} - u_n\|$. The same scheme will be reused, essentially unchanged, in the geometric settings of Chapter 6, where only the kernel p and the constant Φ change.*

3.3.4 The nonlinear case: multiplicative Lipschitz noise

The contraction scheme above never used the linearity of the map $r \mapsto \beta r$ that multiplies the noise: it used only that this map is globally Lipschitz and that $\mathcal{I}_S$ is a bounded operator on $\mathcal{A}_{\tau,p}$ (Theorem 3.1.25). Replacing $\beta\, u$ by a nonlinear coefficient $\sigma(u)$ therefore costs nothing beyond bookkeeping, as we now record.

Assumption 3.3.7

The function $\sigma\colon \mathbb{R} \to \mathbb{R}$ is globally Lipschitz: there is a constant $L_\sigma < \infty$ such that

$$|\sigma(r) - \sigma(r')| \;\le\; L_\sigma\, |r - r'|, \qquad r, r' \in \mathbb{R}. \tag{3.102}$$

In particular σ has linear growth, $|\sigma(r)| \le |\sigma(0)| + L_\sigma |r|$.

Corollary 3.3.8: Global well-posedness for multiplicative Lipschitz noise

Assume Dalang's condition (Assumption 2.1.1), let u_0 be bounded measurable, and let σ satisfy Assumption 3.3.7. Then for every $T > 0$ and every $p \geq 2$ the nonlinear stochastic heat equation

$$u(t,x) \;=\; P_t u_0(x) \;+\; \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, \sigma\Big(u(s,y)\Big)\, W(ds,dy) \tag{3.103}$$

has a unique mild solution $u \in \mathcal{A}_{T,p}$ (Definition 3.1.20). The linear PAM (3.1) is the special case $\sigma(r) = \beta r$, for which $L_\sigma = |\beta|$.

Proof. For a process Z write Z^σ for the process $(s,y) \mapsto \sigma(Z(s,y))$, and define the solution map

$$\Gamma^\sigma_\tau Z(t,x) \;:=\; P_t u_0(x) \;+\; \mathcal{I}_S Z^\sigma(t,x), \qquad (t,x) \in [0,\tau] \times \mathbb{R}^d, \tag{3.104}$$

with $\mathcal{I}_S$ the stochastic convolution of (3.93). The proofs of Proposition 3.3.3, Theorem 3.3.4 and Theorem 3.3.5 carry over verbatim once the following three points, the only places where σ enters, are checked.

(i) *Stability.* Since σ is continuous, Z^σ is jointly measurable and adapted whenever Z is. The linear growth in Assumption 3.3.7 gives, pointwise in ω, $|Z^\sigma(s,y)| \leq |\sigma(0)| + L_\sigma |Z(s,y)|$, so taking the $L^p(\Omega)$ norm and the supremum over $(s,y) \in [0,\tau] \times \mathbb{R}^d$,

$$\|Z^\sigma\|_{\mathcal{A}_{\tau,p}} \;\leq\; |\sigma(0)| \;+\; L_\sigma\, \|Z\|_{\mathcal{A}_{\tau,p}} \;<\; \infty, \qquad Z \in \mathcal{A}_{\tau,p}.$$

Hence $Z^\sigma \in \mathcal{A}_{\tau,p}$, and Theorem 3.1.25 gives $\mathcal{I}_S Z^\sigma \in \mathcal{A}_{\tau,p}$; with the deterministic free term, $\Gamma^\sigma_\tau Z \in \mathcal{A}_{\tau,p}$. (In the linear case $Z^\sigma = \beta Z$ and this step is immediate.)

(ii) *Contraction.* As in the proof of Proposition 3.3.3, the free term cancels in the difference and $\mathcal{I}_S$ is linear, so $\Gamma^\sigma_\tau Z_1 - \Gamma^\sigma_\tau Z_2 = \mathcal{I}_S\Big(Z_1^\sigma - Z_2^\sigma\Big)$. The operator bound (3.54) and the identity (3.95) give

$$\Big\|\Gamma^\sigma_\tau Z_1 - \Gamma^\sigma_\tau Z_2\Big\|^2_{\mathcal{A}_{\tau,p}} \;\leq\; 4\,p\,\Phi(\tau)\,\mathcal{N}_{\tau,p}\Big(Z_1^\sigma - Z_2^\sigma\Big).$$

The new ingredient is the pointwise Lipschitz bound (3.102), applied ω by ω, which after taking $L^p(\Omega)$ norms reads $\|Z_1^\sigma(s,y) - Z_2^\sigma(s,y)\|_p \;\leq\; L_\sigma\, \|Z_1(s,y) - Z_2(s,y)\|_p$, whence $\mathcal{N}_{\tau,p}(Z_1^\sigma - Z_2^\sigma) \leq L_\sigma^2\, \mathcal{N}_{\tau,p}(Z_1 - Z_2)$. Therefore

$$\Big\|\Gamma^\sigma_\tau Z_1 - \Gamma^\sigma_\tau Z_2\Big\|_{\mathcal{A}_{\tau,p}} \;\leq\; 2\,L_\sigma\,\sqrt{p\,\Phi(\tau)}\;\Big\|Z_1 - Z_2\Big\|_{\mathcal{A}_{\tau,p}}, \tag{3.105}$$

which is exactly (3.97) with $|\beta|$ replaced by L_σ. Thus Γ^σ_τ is a contraction as soon as $4\,p\,L_\sigma^2\,\Phi(\tau) \;<\; 1$, the analogue of (3.98); since $\Phi(\tau) \to 0$ (Lemma 3.3.1), this fixes a horizon $\tau_* > 0$ depending only on p, L_σ and μ.

(iii) *Patching.* Since u^σ is adapted whenever u is, the restarted map and the Chapman–Kolmogorov recombination (3.100)–(3.101) hold verbatim, with $u(s,y)$ replaced by $u^\sigma(s,y)$ in the stochastic integrals. The N-step patch then produces the global solution $u \in \mathcal{A}_{T,p}$, and uniqueness follows window by window exactly as in Theorem 3.3.5.

In short, the only change with respect to the linear PAM is that the role of $|\beta|$ is now played by the Lipschitz constant L_σ; the noise still enters solely through the scalar $\Phi(\tau)$. ■

CHAPTER 4

Moment estimates

4.1 Chaos expansions

Chapter 3 established existence and uniqueness of the mild solution (3.1) via a Banach fixed-point argument in the space $\mathcal{A}_{T,p}$. The present section revisits existence and uniqueness through a different lens: the *Wiener chaos expansion* of Section 2.1.2. Rather than iterating the solution map, one looks for an explicit, term-by-term formula for $u(t,x)$ as a series of multiple stochastic integrals. This approach has two payoffs. First, it yields an explicit L^2-moment formula:

$$\mathbb{E}\Big[u(t,x)^2\Big] = \sum_{n=0}^{\infty} n!\, \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}, \tag{4.1}$$

where f_n are the chaos kernels derived below. This representation is at the heart of our moment estimates below. Second, the convergence of (4.1) under Dalang's condition (2.2) provides an independent proof of the L^2-finiteness of $u(t,x)$.

4.1.1 Derivation of the chaos kernels

Since the solution $u(s,y)$ is adapted to the filtration (3.4), Proposition 3.1.10 shows that the Itô–Walsh integral in (3.1) coincides with the Skorohod integral δ (Definition 2.1.8): for any adapted $v \in \mathcal{H}_\omega$, one can write

$$\int_0^t \int_{\mathbb{R}^d} v(s,y)\, W(ds,dy) \;=\; \delta\Big(v\, \mathbf{1}_{[0,t]}\Big)\,. \tag{4.2}$$

We therefore work with the Skorohod-sense mild equation

$$u(t,x) \;=\; P_t u_0(x) \;+\; \beta \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, u(s,y)\, \delta W_{s,y}, \tag{4.3}$$

which is just another notation for (3.1) in case of an adapted solution.

By Theorem 2.1.5, for each fixed (t,x) the random variable $u(t,x)$ has a Wiener chaos decomposition

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x))\,, \tag{4.4}$$

where $f_n(\cdot,t,x) \in \mathcal{H}^{\odot n}$ is a symmetric kernel depending on (t,x) as parameters, and $f_0(t,x) = \mathbb{E}[u(t,x)]$. The content of the kernels f_n is not yet specified by (4.4); they must be identified by substituting the expansion back into the mild equation (4.3).

Lemma 4.1.1: Chaos kernels

The kernels $f_n(\,\cdot\,;\,t,x) \in \mathcal{H}^{\odot n}$ in (4.4) are given by the initial value

$$f_0(t,x) = P_t u_0(x), \tag{4.5}$$

and, for $n \geq 1$, by the explicit symmetric kernel

$$\begin{aligned} f_n(s_1,y_1,\ldots,s_n,y_n;\,t,x) = \frac{\beta^n}{n!}\, & p_{t-s_{\sigma(n)}}(x-y_{\sigma(n)})\, p_{s_{\sigma(n)}-s_{\sigma(n-1)}}(y_{\sigma(n)}-y_{\sigma(n-1)}) \\ & \times\cdots\times p_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)}-y_{\sigma(1)})\, P_{s_{\sigma(1)}}u_0(y_{\sigma(1)}), \end{aligned} \tag{4.6}$$

where σ is the unique permutation of $\{1,\ldots,n\}$ such that

$$0 < s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(n)} < t. \tag{4.7}$$

Proof. The strategy is to substitute the chaos expansion (4.4) into the mild equation (4.3) and match chaos levels using identity (2.23). This yields a recursion on the kernels (Step 1), which is then iterated explicitly for $n = 1$ and $n = 2$ (Steps 2–3) before being resolved in general by induction (Step 4).

Step 1: Derivation of the recursion. Substitute the chaos expansion (4.4) of $u(s,y)$ into the right-hand side of (4.3). The integrand $v(s,y) := p_{t-s}(x-y)\,u(s,y)$ then admits the chaos decomposition

$$v(s,y) = \sum_{n=0}^{\infty} I_n(g_n(\,\cdot\,;\,s,y))\,, \tag{4.8}$$

where the kernel g_n is defined by

$$g_n(s_1,y_1,\ldots,s_n,y_n;\,s,y) = p_{t-s}(x-y)\, f_n(s_1,y_1,\ldots,s_n,y_n;\,s,y),$$

and where (s,y) plays the role of the free $\mathcal{H}$-variable $\star$ in (2.23). Applying (2.23) to v we end up with

$$\delta(v) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{g}_n)\,, \tag{4.9}$$

where, writing (s_{n+1}, y_{n+1}) for the free variable (s, y), $\tilde{g}_n$ is the full symmetrization of $g_n(s_1, y_1, \ldots, s_n, y_n;\, s_{n+1}, y_{n+1})$ in all $n+1$ pairs (s_k, y_k):

$$\begin{aligned}&\tilde{g}_n(s_1, y_1, \ldots, s_{n+1}, y_{n+1})\\ &\qquad = \frac{1}{n+1}\sum_{k=1}^{n+1} p_{t-s_k}(x-y_k)\, f_n\Big(s_1, y_1, \ldots, \widehat{(s_k, y_k)}, \ldots, s_{n+1}, y_{n+1};\, s_k, y_k\Big)\,. \end{aligned}\tag{4.10}$$

The right-hand side of (4.3) therefore has chaos decomposition

$$P_t u_0(x) + \beta\,\delta(v) = I_0(P_t u_0(x)) + \sum_{n=1}^{\infty} I_n(\beta\,\tilde{g}_{n-1})\,. \tag{4.11}$$

By uniqueness of the chaos decomposition in Theorem 2.1.5, matching level $n = 0$ gives (4.5), and matching level $n \geq 1$ gives $f_n = \beta\,\tilde{g}_{n-1}$. Substituting (4.10) with n replaced by $n-1$ yields the recursion, for $n \geq 1$:

$$\begin{aligned}&f_n(s_1, y_1, \ldots, s_n, y_n;\, t, x)\\ &\qquad = \frac{\beta}{n}\sum_{k=1}^{n} p_{t-s_k}(x-y_k)\, f_{n-1}\Big(s_1, y_1, \ldots, \widehat{(s_k, y_k)}, \ldots, s_n, y_n;\, s_k, y_k\Big)\,,\end{aligned}\tag{4.12}$$

where $\widehat{\ }$ indicates the omitted pair. For sake of clarity, we will now detail the cases $n = 1$ and $n = 2$.

Step 2: Case $n = 1$. In (4.12) with $n = 1$, the sum reduces to a single term (there is one variable, the hat omits nothing, and no averaging is needed):

$$f_1(s_1, y_1;\, t, x) = \beta\, p_{t-s_1}(x-y_1)\, f_0(s_1, y_1) = \beta\, p_{t-s_1}(x-y_1)\, P_{s_1} u_0(y_1). \tag{4.13}$$

This is (4.6) for $n = 1$ (with the trivial permutation $\sigma = \mathrm{id}$).

Step 3: Case $n = 2$. Applying (4.12) with $n = 2$ and substituting (4.13) we obtain

$$\begin{aligned}&f_2(s_1, y_1, s_2, y_2;\, t, x)\\ &= \frac{\beta}{2}\Big[p_{t-s_1}(x-y_1)\, f_1(s_2, y_2;\, s_1, y_1) + p_{t-s_2}(x-y_2)\, f_1(s_1, y_1;\, s_2, y_2)\Big]\\ &= \frac{\beta^2}{2}\Big[p_{t-s_1}(x-y_1)\, p_{s_1-s_2}(y_1-y_2)\, P_{s_2} u_0(y_2) + p_{t-s_2}(x-y_2)\, p_{s_2-s_1}(y_2-y_1)\, P_{s_1} u_0(y_1)\Big].\end{aligned}\tag{4.14}$$

The two summands correspond to the two orderings of (s_1, y_1) and (s_2, y_2): the first is nonzero only when $s_2 < s_1 < t$ (permutation $\sigma(1) = 2$, $\sigma(2) = 1$), and the second only when $s_1 < s_2 < t$ (identity permutation). For Lebesgue-almost every (s_1, y_1, s_2, y_2), exactly one summand is nonzero, and the factor $\frac{1}{2} = \frac{1}{2!}$ is the reciprocal of the number of orderings. This is (4.6) for $n = 2$.

Step 4: General formula by induction. An induction on n, using (4.12) and the inductive hypothesis for f_{n-1}, shows that the double sum over $k \in \{1, \ldots, n\}$ and permutations

$\tau \in \mathfrak{S}_{n-1}$ (ordering the remaining $n-1$ variables) is in bijection with $\mathfrak{S}_n$, yielding

$$f_n(s_1, y_1, \ldots, s_n, y_n; t, x) = \frac{\beta^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} p_{t-s_{\sigma(n)}}(x - y_{\sigma(n)}) \prod_{j=1}^{n-1} p_{s_{\sigma(j+1)}-s_{\sigma(j)}}(y_{\sigma(j+1)} - y_{\sigma(j)}) \, P_{s_{\sigma(1)}} u_0(y_{\sigma(1)}), \quad (4.15)$$

where $\mathfrak{S}_n$ denotes the symmetric group on $\{1, \ldots, n\}$. Since $p_{t'}(\cdot) = 0$ for $t' \leq 0$, the summand for σ is nonzero only on $\{s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t\}$, and for Lebesgue-almost every $(s_1, \ldots, s_n)$ at most one σ satisfies this. Hence (4.15) reduces to the single-term formula (4.6). ∎

Remark 4.1.2 (Relation with directed polymer models)**.** *Formula* (4.6) *has a transparent probabilistic reading: one places n spacetime marks (s_i, y_i) in the slab $[0,t] \times \mathbb{R}^d$, orders them by time, and weights each consecutive pair by the heat kernel. This is precisely the n-th term in the partition function of a directed polymer in a random environment: the marks play the role of the polymer's intermediate vertices, the heat kernels are the transition weights of the underlying random walk, and the sum over chaos levels corresponds to summing over all possible numbers of contacts with the environment. We refer to [16] for a detailed account of directed polymers and their connection with the PAM.*

4.1.2 Convergence and the L^2 moment formula

Lemma 4.1.1 identifies the kernels f_n but says nothing about the convergence of the series (4.4) in $L^2(\Omega)$. This requires controlling the growth of the norms $n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$, which is where Dalang's condition (2.2) enters. Once convergence is established, the L^2-moment formula (4.17) follows immediately from the orthogonality of distinct chaos levels (2.12).

Theorem 4.1.3: Existence-uniqueness and L^2 moments via chaos

Assume Dalang's condition (2.2) and let $u_0 \in L^\infty(\mathbb{R}^d)$. Then, for every $(t,x) \in (0,\infty) \times \mathbb{R}^d$,

$$\sum_{n=0}^{\infty} n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} < \infty. \quad (4.16)$$

Consequently, the series (4.4) converges in $L^2(\Omega)$, defining a unique mild solution to (3.1) with L^2-moment

$$\mathbb{E}\Big[u(t,x)^2\Big] = \sum_{n=0}^{\infty} n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}. \quad (4.17)$$

Proof. The L^2-moment formula (4.17) follows from the orthogonality of different chaos levels (2.12), so it suffices to establish (4.16). Throughout the proof we write $\mu^{\otimes n}(d\xi) := \prod_{i=1}^n \mu(d\xi_i)$ for the n-fold product spectral measure.

Step 1: Preliminary considerations.

Before passing to Fourier variables, we record a trivial bound on the kernel f_n itself in direct variables. The heat semigroup $(P_s)_{s\geq 0}$ is a contraction on $L^\infty(\mathbb{R}^d)$, since p_s (2.40) is a probability density for every $s > 0$. Namely we have

$$|P_s u_0(y)| \;\leq\; \|u_0\|_\infty, \qquad s \geq 0,\; y \in \mathbb{R}^d. \tag{4.18}$$

Let us isolate the pure chain of heat-kernel transitions appearing in (4.6) by setting, on the ordered simplex (4.7) (with σ as in (4.7)),

$$K_n(s_1, y_1, \dots, s_n, y_n;\, t, x) := p_{t-s_{\sigma(n)}}(x - y_{\sigma(n)}) \prod_{j=1}^{n-1} p_{s_{\sigma(j+1)}-s_{\sigma(j)}}(y_{\sigma(j+1)} - y_{\sigma(j)}), \tag{4.19}$$

and $K_n := 0$ off the simplex (4.7). Substituting (4.18) into (4.6), and using that the heat kernel p_t is non-negative, gives the direct (pre-Fourier) bound

$$\Big|f_n(s_1, y_1, \dots, s_n, y_n;\, t, x)\Big| \;\leq\; \frac{\beta^n}{n!}\, \|u_0\|_\infty\, K_n(s_1, y_1, \dots, s_n, y_n;\, t, x). \tag{4.20}$$

This trivial bound isolates the role of the initial datum u_0 from the chain of transitions K_n, and it is exactly the bound on u_0 that we will need to pull out of the Fourier transform of f_n in order to control $\|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$ below.

As a second preliminary step, we reduce the integral defining $\|K_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$ (and hence, through the bound (4.20), the one defining $\|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$) to a single ordered region in time. Both K_n (4.19) and f_n (which lies in $\mathcal{H}^{\odot n}$ by Lemma 4.1.1) are symmetric functions of the n pairs $(s_1, y_1), \dots, (s_n, y_n)$: relabeling the pairs by a permutation $\pi \in \mathfrak{S}_n$ only relabels which index plays the role of $\sigma(1), \dots, \sigma(n)$ in (4.7), and leaves the sorted chain of heat-kernel transitions (hence the value of K_n, and of f_n) unchanged. Consequently, the function

$$\Psi_n(s_1, \dots, s_n) := \int_{\mathbb{R}^{nd}} \Big|\mathcal{F}_y\, K_n(s, \cdot;\, t, x)(\xi)\Big|^2 \mu^{\otimes n}(d\xi),$$

which is the s-density of $\|K_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$ under the inner product (2.4), is itself symmetric in $s = (s_1, \dots, s_n)$: permuting the s_i amounts to permuting the dummy variables y_i, equivalently ξ_i, inside the integral. This leaves its value unchanged since $\mu^{\otimes n}(d\xi)$ is itself a symmetric product measure. Now decompose the cube $[0, t]^n$ into the $n!$ disjoint images $\pi \cdot T_n(t)$ (for a generic $\pi \in \mathfrak{S}_n$) of the ordered simplex

$$T_n(t) = \Big\{(s_1, \dots, s_n) \in [0, \infty)^n : 0 < s_1 < \cdots < s_n < t\Big\}, \tag{4.21}$$

so that symmetry of Ψ_n gives

$$\|K_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \;=\; \int_{[0,t]^n} \Psi_n(s)\, ds \;=\; n! \int_{T_n(t)} \Psi_n(s)\, ds. \tag{4.22}$$

We can perform the same manipulation for f_n: writing

$$\Phi_n(s_1, \dots, s_n) := \int_{\mathbb{R}^{nd}} \Big|\mathcal{F}_y\, f_n(s, \cdot;\, t, x)(\xi)\Big|^2 \mu^{\otimes n}(d\xi)$$

for the analogous s-density of $\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}$, an identical symmetry argument gives

$$\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} = n! \int_{T_n(t)} \Phi_n(s)\, ds.$$

We may therefore restrict our attention, in what follows, to the single ordered region $T_n(t)$. On that set we have $\sigma = \mathrm{id}$ in (4.7) and the chain (4.19) takes the simple, unsymmetrized form

$$K_n(s_1,y_1,\dots,s_n,y_n;\, t,x) = p_{t-s_n}(x-y_n) \prod_{i=1}^{n-1} p_{s_{i+1}-s_i}(y_{i+1}-y_i), \qquad (s_1,\dots,s_n) \in T_n(t). \tag{4.23}$$

Let us now record, in terms of Ψ_n, the bound towards (4.16) that the two preliminary considerations above already yield. Since $\Lambda \geq 0$ pointwise (by definition of the covariance (2.1)), the $\mathcal{H}^{\otimes n}$ inner product is monotone under pointwise domination of non-negative kernels: if $|h| \leq g$ pointwise with $g \geq 0$, then $h(y)h(y') \leq |h(y)||h(y')| \leq g(y)g(y')$ for all y, y', and multiplying by the non-negative weight $\prod_i \Lambda(y_i - y_i')$ and integrating preserves the inequality, so $\|h\|^2_{\mathcal{H}^{\otimes n}} \leq \|g\|^2_{\mathcal{H}^{\otimes n}}$ Applying this, for each fixed s, to the direct bound (4.20) gives

$$\Phi_n(s) \;\leq\; \frac{\beta^{2n}}{(n!)^2}\, \|u_0\|^2_\infty\, \Psi_n(s).$$

Integrating over $T_n(t)$, using $\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} = n! \int_{T_n(t)} \Phi_n(s)\, ds$ above, and multiplying through by $n!$, we arrive at

$$n!\, \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;\leq\; \beta^{2n}\, \|u_0\|^2_\infty \int_{T_n(t)} \Psi_n(s)\, ds, \tag{4.24}$$

where $\Psi_n(s)$ will be evaluated thanks to (4.23). This is precisely the n-th term on the left-hand side of (4.16). To summarize: the two preliminary steps have reduced the proof of (4.16) to estimating, for each n, the single integral $\int_{T_n(t)} \Psi_n(s)\, ds$ built from the explicit, u_0-free heat-kernel chain K_n on the ordered simplex (4.23). This is the task of Step 2.

Step 2: Fourier bound on each term.

By (4.24), it remains to compute $\Psi_n(s)$ for $s \in T_n(t)$, where, by definition,

$$\Psi_n(s) = \int_{\mathbb{R}^{nd}} \left| \mathcal{F}_y\, K_n(s,\cdot;\, t,x)(\xi) \right|^2 \mu^{\otimes n}(d\xi).$$

We compute the spatial Fourier transform of K_n explicitly, from its simplified form (4.23) on $T_n(t)$. That is, for $s \in T_n(t)$ and $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^{nd}$, by definition of the Fourier transform we have

$$\mathcal{F}_y\, K_n(s,\cdot;\, t,x)(\xi) = \int_{\mathbb{R}^{nd}} p_{t-s_n}(x-y_n) \prod_{i=1}^{n-1} p_{s_{i+1}-s_i}(y_{i+1}-y_i)\, e^{-\imath \sum_{j=1}^n \xi_j \cdot y_j}\, dy_1 \cdots dy_n. \tag{4.25}$$

To evaluate this integral, set $u_i := y_{i+1} - y_i$ for $1 \le i \le n-1$ and $u_n := x - y_n$, so that $K_n(s, y;\, t, x) = \prod_{i=1}^n p_{s_{i+1}-s_i}(u_i)$ (where we set $s_{n+1} := t$). Note that this change of variables has unit Jacobian. Furthermore, since $y_i = x - \sum_{k=i}^n u_k$, we have

$$\sum_{i=1}^n \xi_i \cdot y_i = x \cdot \Big(\sum_{i=1}^n \xi_i\Big) - \sum_{k=1}^n u_k \cdot (\xi_1 + \cdots + \xi_k).$$

Substituting into (4.25) (the Jacobian is 1, and $K_n(s, u;\, t, x) = \prod_{i=1}^n p_{s_{i+1}-s_i}(u_i)$), we obtain

$$\mathcal{F}_y\, K_n(s, \cdot;\, t, x)(\xi) = e^{-\imath x \cdot \sum_{i=1}^n \xi_i} \int_{\mathbb{R}^{nd}} \prod_{i=1}^n p_{s_{i+1}-s_i}(u_i)\, e^{\imath\, u_i \cdot (\xi_1 + \cdots + \xi_i)}\, du_1 \cdots du_n. \tag{4.26}$$

Since the integrand in (4.26) factorizes over i, we thus get

$$\mathcal{F}_y\, K_n(s, \cdot;\, t, x)(\xi) = e^{-\imath x \cdot \sum_{i=1}^n \xi_i} \prod_{i=1}^n \Big[\int_{\mathbb{R}^d} p_{s_{i+1}-s_i}(u_i)\, e^{\imath\, u_i \cdot (\xi_1 + \cdots + \xi_i)}\, du_i \Big].$$

Using the Fourier representation of the heat kernel (2.40), that is $\mathcal{F} p_w(\zeta) = e^{-w|\zeta|^2/2}$, we end up with

$$\mathcal{F}_y\, K_n(s, \cdot;\, t, x)(\xi) = e^{-\imath x \cdot \sum_i \xi_i} \prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\xi_1 + \cdots + \xi_i|^2}, \qquad s \in T_n(t). \tag{4.27}$$

Taking the modulus square (the phase $e^{-\imath x \cdot \sum_i \xi_i}$ has modulus 1) and integrating against $\mu^{\otimes n}(d\xi)$ gives the *exact* value of Ψ_n on $T_n(t)$:

$$\Psi_n(s) = \int_{\mathbb{R}^{nd}} \prod_{i=1}^n e^{-(s_{i+1}-s_i)|\xi_1 + \cdots + \xi_i|^2}\, \mu^{\otimes n}(d\xi), \qquad s \in T_n(t). \tag{4.28}$$

Combining (4.28) with the bound (4.24) from Step 1, we conclude this step with the upper bound

$$n!\, \|f_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \;\le\; \|u_0\|^2_\infty\, \beta^{2n} \int_{T_n(t)} \int_{\mathbb{R}^{nd}} \prod_{i=1}^n e^{-(s_{i+1}-s_i)|\xi_1 + \cdots + \xi_i|^2}\, \mu^{\otimes n}(d\xi)\, ds. \tag{4.29}$$

Step 3: Reduction to gap variables.

The integral in (4.29) is over the ordered simplex $T_n(t)$, and the exponent contains the nested spectral sums $|\xi_1 + \cdots + \xi_i|^2$ that couple all n variables. This step performs two operations: a change of time variables that maps $T_n(t)$ to a product-like domain, and a decoupling of the spectral variables.

Change of time variables. Set $w_i := s_{i+1} - s_i$ for $1 \le i \le n-1$ and $w_n := t - s_n$. The map $s \mapsto w$ is a bijection from $T_n(t)$ onto

$$S_{t,n} := \{w \in [0, \infty)^n : w_1 + \cdots + w_n \le t\} \tag{4.30}$$

with unit Jacobian (it is a lower-triangular shear: $s_i = t - w_n - \cdots - w_i$). The bound (4.29) becomes

$$n! \, \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;\leq\; \|u_0\|^2_\infty \, \beta^{2n} \int_{S_{t,n}} \int_{\mathbb{R}^{nd}} \prod_{i=1}^{n} e^{-w_i|\xi_1+\cdots+\xi_i|^2} \, \mu^{\otimes n}(d\xi) \, dw. \tag{4.31}$$

Decoupling of spectral variables. We claim that for any $w > 0$ and $\eta \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{-w|\xi+\eta|^2} \, \mu(d\xi) \;\leq\; \int_{\mathbb{R}^d} e^{-w|\xi|^2} \, \mu(d\xi). \tag{4.32}$$

In order to prove (4.32), we use the Fourier representation (2.40), namely $e^{-w|\zeta|^2} = \mathcal{F}p_{2w}(\zeta) = \int_{\mathbb{R}^d} p_{2w}(x) \, e^{-\imath\zeta\cdot x} \, dx$, as well as Fubini. This enables to write

$$\int_{\mathbb{R}^d} e^{-w|\xi+\eta|^2} \, \mu(d\xi) \;=\; \int_{\mathbb{R}^d} p_{2w}(x) \, e^{-\imath\eta\cdot x} \Big[\int_{\mathbb{R}^d} e^{-\imath\xi\cdot x} \, \mu(d\xi) \Big] dx. \tag{4.33}$$

Set $\Phi(x) := \int_{\mathbb{R}^d} e^{\imath\xi\cdot x} \, \mu(d\xi)$. By Fourier inversion of $\mu = \mathcal{F}\Lambda$ we have $\Phi(x) = (2\pi)^d \Lambda(x)$, which is non-negative since $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ in (2.1). By the symmetry of μ (substitute $\xi \to -\xi$), the bracket satisfies

$$\int_{\mathbb{R}^d} e^{-\imath\xi\cdot x} \, \mu(d\xi) = \int_{\mathbb{R}^d} e^{\imath\xi\cdot x} \, \mu(d\xi) = \Phi(x) \geq 0.$$

Plugging this information in (4.33), we obtain

$$\int_{\mathbb{R}^d} e^{-w|\xi+\eta|^2} \, \mu(d\xi) \;=\; \Big| \int_{\mathbb{R}^d} p_{2w}(x) \, e^{-\imath\eta\cdot x} \, \Phi(x) \, dx \Big| \;\leq\; \int_{\mathbb{R}^d} p_{2w}(x) \, \Phi(x) \, dx. \tag{4.34}$$

To identify the right-hand side, substitute $\Phi(x) = (2\pi)^d \Lambda(x)$ and apply Parseval's identity with $\widehat{p_{2w}}(\xi) = e^{-w|\xi|^2}$ (from (2.40)), $\hat{\Lambda}(\xi) \, d\xi = \mu(d\xi)$, and the evenness of Λ:

$$\int_{\mathbb{R}^d} p_{2w}(x) \, \Phi(x) \, dx \;=\; (2\pi)^d \int_{\mathbb{R}^d} p_{2w}(x) \, \Lambda(x) \, dx \;=\; \int_{\mathbb{R}^d} e^{-w|\xi|^2} \, \mu(d\xi). \tag{4.35}$$

Hence putting together (4.34) and (4.35) establishes (4.32).

Returning to (4.31), apply (4.32) iteratively for $i = n, n-1, \ldots, 1$. At step i, all variables $\xi_{i+1}, \ldots, \xi_n$ have already been integrated out; for fixed $\xi_1, \ldots, \xi_{i-1}$ and w, set $\eta := \xi_1 + \cdots + \xi_{i-1}$ and integrate over ξ_i:

$$\int_{\mathbb{R}^d} e^{-w_i|\xi_1+\cdots+\xi_i|^2} \, \mu(d\xi_i) \;\leq\; \int_{\mathbb{R}^d} e^{-w_i|\xi_i|^2} \, \mu(d\xi_i).$$

After n such steps the spectral variables are fully decoupled, and (4.31) gives

$$n! \, \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;\leq\; \|u_0\|^2_\infty \, \beta^{2n} \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2} \, dw \, \mu^{\otimes n}(d\xi), \tag{4.36}$$

where we recall that $S_{t,n}$ is the simplex (4.30), i.e. $S_{t,n} = \{w \in [0,\infty)^n : w_1 + \cdots + w_n \leq t\}$. This is the key estimate from which Step 4 will produce a convergent series.

Step 4: Summation via Lemma 4.1.4.

To evaluate the right-hand side of (4.36), fix $N \geq 1$ and decompose the spectral measure into a low-frequency and a high-frequency part by setting

$$D_N := \mu\Big(\{\xi : |\xi| \leq N\}\Big) \qquad \text{and} \qquad C_N := \int_{|\xi|>N} \frac{\mu(d\xi)}{|\xi|^2}. \tag{4.37}$$

Dalang's condition (2.2) ensures that $C_N < \infty$ for all N, and $C_N \to 0$ as $N \to \infty$ by dominated convergence. With D_N and C_N at hand, Lemma 4.1.4 below applied to the right-hand side of (4.36) gives

$$n!\,\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;\leq\; \|u_0\|^2_\infty\,\beta^{2n}\sum_{k=0}^{n}\binom{n}{k}\frac{t^k}{k!}\,D_N^k\,C_N^{n-k}. \tag{4.38}$$

Since $C_N \to 0$ as $N \to \infty$, choose N large enough so that $2\beta^2 C_N < 1$. Using $\binom{n}{k} \leq 2^n$, we sum over $n \geq 0$:

$$\begin{aligned}
\sum_{n=0}^{\infty} n!\,\|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;&\leq\; \|u_0\|^2_\infty \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\beta^{2n}\cdot 2^n\,\frac{t^k}{k!}\,D_N^k\,C_N^{n-k} \\
&=\; \|u_0\|^2_\infty \sum_{k=0}^{\infty}\frac{(2\beta^2 t\,D_N)^k}{k!}\sum_{n=k}^{\infty}(2\beta^2 C_N)^{n-k} \\
&=\; \frac{\|u_0\|^2_\infty}{1-2\beta^2 C_N}\exp\Big(2\beta^2 t\,D_N\Big) \;<\; \infty.
\end{aligned} \tag{4.39}$$

This proves (4.16) and concludes our proof. ∎

It remains to establish the technical estimate invoked in Step 4 above. The lemma shows how Dalang's condition (2.2) allows one to split the integration over $S_{t,n}$ and $\mathbb{R}^{nd}$ into a tractable product: high-frequency modes are controlled by C_N (the tail of μ weighted by $|\xi|^{-2}$), while low-frequency modes contribute only a finite mass D_N.

Lemma 4.1.4: Simplex integral estimate

Let μ satisfy Dalang's condition (2.2), and let D_N and C_N be as in (4.37). For any $n \geq 1$ and $t > 0$, with $S_{t,n}$ the simplex defined in (4.30), it holds

$$\int_{\mathbb{R}^{nd}}\int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2}\,dw\,\mu^{\otimes n}(d\xi) \;\leq\; \sum_{k=0}^{n}\binom{n}{k}\frac{t^k}{k!}\,D_N^k\,C_N^{n-k}. \tag{4.40}$$

Proof. Write $1 = \mathbf{1}_{\{|\xi_i|\leq N\}} + \mathbf{1}_{\{|\xi_i|>N\}}$ inside each factor $e^{-w_i|\xi_i|^2}$, expand the resulting product, and collect terms according to the subset $I \subset \{1,\dots,n\}$ of low-frequency indices (those with $|\xi_i| \leq N$). This yields

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}}\int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2}\,dw\,\mu^{\otimes n}(d\xi) \\
&= \sum_{I\subset\{1,\dots,n\}}\int_{S_{t,n}}\int_{\mathbb{R}^{nd}}\prod_{i\in I} e^{-w_i|\xi_i|^2}\mathbf{1}_{\{|\xi_i|\leq N\}}\prod_{j\in I^c} e^{-w_j|\xi_j|^2}\mathbf{1}_{\{|\xi_j|>N\}}\,\mu^{\otimes n}(d\xi)\,dw.
\end{aligned} \tag{4.41}$$

For each term in (4.41), bound $e^{-w_i|\xi_i|^2} \leq 1$ for $i \in I$, and for $j \in I^c$ integrate in w_j over $[0,t]$ (bounding the simplex $S_{t,n}$ (4.30) by the product simplex $S_t^I \times S_t^{I^c}$, where $S_t^I = \{(w_i)_{i\in I} : w_i \geq 0, \sum_{i\in I} w_i \leq t\}$). We get

$$\int_{S_t^{I^c}} \prod_{j\in I^c} e^{-w_j|\xi_j|^2}\, dw \;\leq\; \prod_{j\in I^c} \int_0^t e^{-w_j|\xi_j|^2}\, dw_j \;=\; \prod_{j\in I^c} \frac{1-e^{-t|\xi_j|^2}}{|\xi_j|^2} \;\leq\; \prod_{j\in I^c} \frac{1}{|\xi_j|^2}. \tag{4.42}$$

For the low-frequency part, we use the trivial bound

$$\int_{S_t^I} \prod_{i\in I} e^{-w_i|\xi_i|^2}\, dw \;\leq\; \int_{S_t^I} \prod_{i\in I} 1\, dw \;\leq\; \frac{t^{|I|}}{|I|!}. \tag{4.43}$$

Combining (4.42) and (4.43) and integrating over $\mu^{\otimes n}(d\xi)$, the term in (4.41) indexed by I satisfies

$$\int_{S_{t,n}} \int_{\mathbb{R}^{nd}} \prod_{i\in I} e^{-w_i|\xi_i|^2} \mathbf{1}_{\{|\xi_i|\leq N\}} \prod_{j\in I^c} e^{-w_j|\xi_j|^2} \mathbf{1}_{\{|\xi_j|>N\}}\, \mu^{\otimes n}(d\xi)\, dw \;\leq\; \frac{t^{|I|}}{|I|!}\, D_N^{|I|}\, C_N^{n-|I|}. \tag{4.44}$$

Summing (4.44) over all $I \subset \{1,\ldots,n\}$ and collecting by $|I| = k$ (there are $\binom{n}{k}$ subsets of size k) yields

$$\begin{aligned}
\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i|\xi_i|^2}\, dw\, \mu^{\otimes n}(d\xi) \;&\leq\; \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k \left(\int_{|\xi|>N} \frac{\mu(d\xi)}{|\xi|^2} \right)^{n-k} \\
&=\; \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k\, C_N^{n-k},
\end{aligned} \tag{4.45}$$

which is the claimed bound (4.40). ■

Remark 4.1.5. *Theorem 4.1.3 and Theorem 3.3.5 (contraction) assert the same L^2-existence result by different methods. The chaos approach requires only Dalang's condition (2.2) and $u_0 \in L^\infty$, and delivers an explicit series representation with a concrete L^2-moment formula. The contraction approach works in the larger space $A_{T,p}$ for $p \geq 2$, gives L^p-estimates, and adapts to nonlinear multiplicative noise (Corollary 3.3.8). The two routes are complementary: the chaos expansion is essential for the moment analysis of Chapter 4, while the contraction argument extends to a wider class of equations.*

4.2 Feynman-Kac representation of moments

The informal Feynman-Kac formula (1.5) in the introduction described the solution $u(t,x)$ as a Brownian expectation of an exponential functional of the noise. For the *moments* of the solution, a rigorous and explicit version of this formula is available, and it is one of the possible key tools for the quantitative analysis of intermittency. The approach goes back to Carmona–Molchanov [12], who were dealing with discrete noises or smooth noises in space. Our setting will be derived by spatial smooth approximations, as we will see below. Since the moment formula involves k independent Brownian paths (one for each factor of u), we fix notation for this double randomness.

Notation 4.2.1

Let $B^1, \dots, B^k$ be independent standard d-dimensional Brownian motions starting at the origin, independent of the noise W. We write $\mathbb{E}_B$ for the expectation with respect to the Brownian motions only (holding W fixed), and $\mathbb{E}_W$ for the expectation with respect to W only (holding the paths fixed).

4.2.1 The spatially regularized process u^ε

For a Brownian path B starting at the origin and $\varepsilon > 0$, define the $\mathcal{H}$-valued functional

$$A^\varepsilon_{t,x}(s,y) := p_\varepsilon(B_{t-s} + x - y)\,\mathbf{1}_{[0,t]}(s) \qquad (s,y) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{4.46}$$

where p_ε (2.40) is the Gaussian heat kernel at time ε. Since $\widehat{p_\varepsilon}(\xi) = e^{-\varepsilon|\xi|^2/2}$ (2.40), the Fourier formula (2.4) gives

$$\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}} = t \int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2}\,\mu(d\xi),$$

which is finite for every $\varepsilon > 0$ by Dalang's condition (2.2), so $A^\varepsilon_{t,x} \in \mathcal{H}$. Next recalling that B is independent of the noise W, and $\mathbb{E}_B$ denotes expectation with respect to B alone (see Notation 4.2.1), we define the spatially regularized process u^ε by

$$u^\varepsilon(t,x) := \mathbb{E}_B\left[u_0(B_t + x)\,\exp\left(\beta\, W(A^\varepsilon_{t,x}) - \frac{\beta^2}{2}\,\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}}\right)\right]. \tag{4.47}$$

Observe that the Gaussian exponential above is well-defined path-by-path since $A^\varepsilon_{t,x} \in \mathcal{H}$ almost surely. In addition, the Itô correction $-\frac{\beta^2}{2}\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}}$ normalizes the conditional expectation. Namely, as we will see in (4.49), we have

$$\mathbb{E}_W\left[e^{\beta W(A^\varepsilon_{t,x}) - \frac{\beta^2}{2}\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}}}\right] = 1$$

for every fixed Brownian path B. We will show in Section 4.2.2 that u^ε converges to the mild solution u and that (4.47) leads to the Feynman-Kac moment formula (see (4.81) below). As a first step, we establish uniform-in-ε moment bounds.

Proposition 4.2.2: Uniform moment bounds

Let $u_0 \in L^\infty(\mathbb{R}^d)$ and suppose Dalang's condition (2.2) holds. Then for every integer $k \geq 1$, $t > 0$, and $x \in \mathbb{R}^d$, we have

$$\sup_{\varepsilon > 0} \mathbb{E}\left[|u^\varepsilon(t,x)|^k\right] < \infty. \tag{4.48}$$

Proof. Let $B^1, \ldots, B^k$ be the independent standard Brownian motions of Notation 4.2.1, and write $A^{\varepsilon,j}_{t,x}$ for the functional (4.46) built from B^j, $j = 1, \ldots, k$. We proceed in two steps.

Step 1: Reduction to inner products.

Since $B^1, \ldots, B^k$ are independent copies independent of W, substituting (4.47) for each factor and invoking Fubini's theorem gives

$$\mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big] \;=\; \mathbb{E}_B\left[\prod_{j=1}^k u_0(B^j_t + x)\, \mathbb{E}_W\left[\exp\left(\beta\sum_{j=1}^k W(A^{\varepsilon,j}_{t,x}) - \frac{\beta^2}{2}\sum_{j=1}^k \|A^{\varepsilon,j}_{t,x}\|^2_{\mathcal{H}}\right)\right]\right].$$

For fixed paths, $\beta\sum_j W(A^{\varepsilon,j}_{t,x}) = W\big(\beta\sum_j A^{\varepsilon,j}_{t,x}\big)$ is Gaussian with variance $\beta^2\|\sum_j A^{\varepsilon,j}_{t,x}\|^2_{\mathcal{H}}$ by the isometry (2.5), so

$$\mathbb{E}_W\left[\exp\left(\beta\sum_j W(A^{\varepsilon,j}_{t,x})\right)\right] \;=\; \exp\left(\frac{\beta^2}{2}\Big\|\sum_j A^{\varepsilon,j}_{t,x}\Big\|^2_{\mathcal{H}}\right). \tag{4.49}$$

Expanding $\|\sum_j A^{\varepsilon,j}_{t,x}\|^2_{\mathcal{H}} = \sum_j \|A^{\varepsilon,j}_{t,x}\|^2_{\mathcal{H}} + 2\sum_{i<j}\langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}}$ and canceling the diagonal terms gives

$$\mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big] \;=\; \mathbb{E}_B\left[\prod_{i=1}^k u_0(B^i_t + x)\,\exp\left(\beta^2\sum_{1\le i<j\le k}\langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}}\right)\right]. \tag{4.50}$$

Since $u_0 \in L^\infty$, it suffices to bound $\mathbb{E}_B[\exp(\beta^2\sum_{i<j}\langle A^{\varepsilon,i}, A^{\varepsilon,j}\rangle_{\mathcal{H}})]$ uniformly in ε.

Step 2: Uniform exponential bound.

By the Fourier formula (2.4) and $\widehat{p_\varepsilon}(\xi) = e^{-\varepsilon|\xi|^2/2}$ (2.40),

$$\langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}} \;=\; \int_0^t\int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2}\, e^{i\xi\cdot(B^j_r - B^i_r)}\,\mu(d\xi)\,dr \;=:\; \int_0^t \Lambda_\varepsilon(B^j_r - B^i_r)\,dr, \tag{4.51}$$

where the change of variable $r = t - s$ was used and $\Lambda_\varepsilon(z) := \int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2} e^{i\xi\cdot z}\mu(d\xi)$ is real and non-negative (as a spatial average of $\Lambda \ge 0$ (2.1) against a Gaussian weight). In particular each inner product $\langle A^{\varepsilon,i}, A^{\varepsilon,j}\rangle_{\mathcal{H}}$ is positive.

Set $M := \binom{k}{2}$ and $\lambda_0 := M\beta^2$. Since every inner product $\langle A^{\varepsilon,i}, A^{\varepsilon,j}\rangle_{\mathcal{H}} \ge 0$, the trivial bound $\sum_{i<j} a_{ij} \le M\max_{i<j} a_{ij}$ and the inequality $e^{M\max_{i<j} a_{ij}} \le \sum_{i<j} e^{Ma_{ij}}$ together give

$$\mathbb{E}_B\left[\exp\left(\beta^2\sum_{i<j}\langle A^{\varepsilon,i}, A^{\varepsilon,j}\rangle_{\mathcal{H}}\right)\right] \;\le\; \sum_{1\le i<j\le k}\mathbb{E}_{B^i,B^j}\Big[\exp\Big(\lambda_0\langle A^{\varepsilon,i}, A^{\varepsilon,j}\rangle_{\mathcal{H}}\Big)\Big].$$

Therefore by symmetry it suffices to bound $\mathbb{E}[\exp(\lambda_0\langle A^{\varepsilon,1}, A^{\varepsilon,2}\rangle_{\mathcal{H}})]$ uniformly in ε. To this aim, expanding the exponential and using (4.51) we can write

$$\mathbb{E}\Big[\langle A^{\varepsilon,1}_{t,x}, A^{\varepsilon,2}_{t,x}\rangle^n_{\mathcal{H}}\Big] \;=\; \int_{[0,t]^n}\int_{\mathbb{R}^{nd}} e^{-\varepsilon\sum_{l=1}^n|\xi_l|^2}\,\mathbb{E}\Big[e^{i\sum_{l=1}^n \xi_l\cdot(B^2_{r_l} - B^1_{r_l})}\Big]\,\mu^{\otimes n}(d\xi)\,dr. \tag{4.52}$$

Furthermore, since B^1 and B^2 are independent standard Brownian motions, their characteristic function can be expressed as

$$\mathbb{E}\Big[e^{i\sum_l \xi_l\cdot(B^2_{r_l}-B^1_{r_l})}\Big] \;=\; e^{-\sum_{l,m}(r_l\wedge r_m)\,\xi_l\cdot\xi_m}.$$

Dropping $e^{-\varepsilon\sum|\xi_l|^2}\le 1$, restricting the integral in (4.52) by symmetry to the simplex $T_n(t)$ defined by (4.21) (with factor $n!$), and writing $w_l = r_l - r_{l-1}$, the standard identity

$$\sum_{l,m}(r_l\wedge r_m)\,\xi_l\cdot\xi_m \;=\; \sum_{l=1}^n w_l\,|\xi_l+\cdots+\xi_n|^2 \quad \text{on } T_n(t) \tag{4.53}$$

yields

$$\mathbb{E}\Big[\langle A^{\varepsilon,1},A^{\varepsilon,2}\rangle^n_{\mathcal{H}}\Big] \;\le\; n!\int_{S_{t,n}}\int_{\mathbb{R}^{nd}}\exp\left(-\sum_{l=1}^n w_l|\xi_l+\cdots+\xi_n|^2\right)\mu^{\otimes n}(d\xi)\,dw.$$

From this point the argument is the same as in the proof of Theorem 4.1.3 (Steps 3–4). Applying (4.32) iteratively (integrate first over ξ_1 with $\xi_2,\dots,\xi_n$ fixed, then over ξ_2, etc.) decouples the spectral variables:

$$\int_{\mathbb{R}^{nd}}\exp\left(-\sum_{l=1}^n w_l|\xi_l+\cdots+\xi_n|^2\right)\mu^{\otimes n}(d\xi) \;\le\; \prod_{l=1}^n\int_{\mathbb{R}^d} e^{-w_l|\xi|^2}\,\mu(d\xi).$$

Next by Lemma 4.1.4 (4.40) we obtain:

$$\mathbb{E}\Big[\langle A^{\varepsilon,1},A^{\varepsilon,2}\rangle^n_{\mathcal{H}}\Big] \;\le\; n!\sum_{j=0}^n\binom{n}{j}\frac{t^j}{j!}\,D_N^j\,C_N^{n-j}.$$

Choose N large enough so that $2\lambda_0 C_N < 1$. By the same summation as (4.39), we end up with

$$\mathbb{E}\Big[\exp\Big(\lambda_0\langle A^{\varepsilon,1},A^{\varepsilon,2}\rangle_{\mathcal{H}}\Big)\Big] \;=\; \sum_{n=0}^\infty\frac{\lambda_0^n}{n!}\mathbb{E}\Big[\langle A^{\varepsilon,1},A^{\varepsilon,2}\rangle^n_{\mathcal{H}}\Big] \;\le\; \frac{\exp(2\lambda_0 t\,D_N)}{1-2\lambda_0 C_N} \;<\; \infty,$$

uniformly in $\varepsilon>0$ (since we dropped $e^{-\varepsilon\sum|\xi_l|^2}$ before applying the lemma). Combined with (4.50) and $u_0\in L^\infty$, this proves (4.48). ■

With the uniform moment bounds of Proposition 4.2.2 in hand, we can now identify the mild SHE that u^ε satisfies. It turns out that the Feynman-Kac representation (4.47) forces u^ε to solve the mild SHE driven by the spatially regularized noise $W^\varepsilon := p_\varepsilon * W$.

Lemma 4.2.3: Mild SHE with regularized noise

Let u^ε be as in (4.47). Define the spatially regularized noise W^ε by setting, for any adapted process $\phi \in \mathcal{H}$,

$$\int_0^T \int_{\mathbb{R}^d} \phi(s,y)\, W^\varepsilon(ds,dy) \;:=\; \int_0^T \int_{\mathbb{R}^d} (p_\varepsilon *_y \phi_s)(y)\, W(ds,dy), \tag{4.54}$$

where $(p_\varepsilon *_y \phi_s)(y) := \int_{\mathbb{R}^d} p_\varepsilon(y-z)\,\phi(s,z)\,dz$ is the spatial convolution of ϕ_s with p_ε. Then for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$u^\varepsilon(t,x) \;=\; p_t u_0(x) \;+\; \beta \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, u^\varepsilon(s,y)\, W^\varepsilon(ds,dy). \tag{4.55}$$

Proof. The argument proceeds in two steps. In Step 1, we derive a mild Itô–Walsh equation for u^ε by expanding the Feynman-Kac exponential via Itô's formula, averaging over the Brownian motion B, and identifying the resulting kernel via a conditional independence argument. In Step 2, we rewrite that equation as the regularized mild SHE (4.55) by recognizing the Walsh kernel as an integral against the regularized noise W^ε, using stochastic Fubini.

Step 1: Mild Itô–Walsh equation for u^ε. In order to obtain a mild equation, introduce for $(s,y) \in [0,t] \times \mathbb{R}^d$ the kernel

$$Z^{\varepsilon,t,x}(s,y) \;:=\; \int_{\mathbb{R}^d} p_{t-s}(x-z)\, p_\varepsilon(z-y)\, u^\varepsilon(s,z)\, dz. \tag{4.56}$$

We claim that u^ε satisfies the mild equation

$$u^\varepsilon(t,x) \;=\; p_t u_0(x) \;+\; \beta \int_0^t \int_{\mathbb{R}^d} Z^{\varepsilon,t,x}(s,y)\, W(ds,dy). \tag{4.57}$$

Step 1-a: Stochastic exponential. For a fixed Brownian path B, set

$$N_s^B \;:=\; \beta \int_0^s \int_{\mathbb{R}^d} p_\varepsilon(B_{t-r}+x-y)\, W(dr,dy), \qquad s \in [0,t]. \tag{4.58}$$

For fixed B, the space-time function $(r,y) \mapsto p_\varepsilon(B_{t-r}+x-y)\,\mathbf{1}_{[0,t]}(r)$ is a deterministic element of $\mathcal{H}$ (by the same norm computation as for $A^\varepsilon_{t,x}$ in (4.46)), so N^B is a W-martingale. By the Walsh isometry (3.11), its quadratic variation is

$$\langle N^B \rangle_s \;=\; \beta^2 \int_0^s \int_{\mathbb{R}^{2d}} p_\varepsilon(B_{t-r}+x-y_1)\, p_\varepsilon(B_{t-r}+x-y_2)\, \Lambda(y_1-y_2)\, dy_1\, dy_2\, dr.$$

The change of variables $z_i := B_{t-r}+x-y_i$ and the symmetry $\Lambda(y_1-y_2) = \Lambda(z_1-z_2)$ (2.1) give

$$\int_{\mathbb{R}^{2d}} p_\varepsilon(B_{t-r}+x-y_1)\, p_\varepsilon(B_{t-r}+x-y_2)\, \Lambda(y_1-y_2)\, dy_1\, dy_2 \\ = \int_{\mathbb{R}^{2d}} p_\varepsilon(z_1)\, p_\varepsilon(z_2)\, \Lambda(z_1-z_2)\, dz_1\, dz_2. \tag{4.59}$$

By the Fourier identity (2.4) (restricted to a single time slice) and $\widehat{p_\varepsilon}(\xi) = e^{-\varepsilon|\xi|^2/2}$ (2.40), the right-hand side of (4.59) equals $\int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2}\mu(d\xi) = \Lambda_\varepsilon(0)$ (the same computation as in (4.51) with $i = j$ and $B^i_r = B^j_r$). Hence

$$\langle N^B\rangle_s \;=\; \beta^2 \int_0^s \Lambda_\varepsilon(0)\,dr \;=\; \beta^2 s\,\Lambda_\varepsilon(0). \tag{4.60}$$

The Doléans–Dade stochastic exponential of N^B is thus

$$M^B_s \;:=\; \exp\Big(N^B_s - \tfrac{1}{2}\langle N^B\rangle_s\Big). \tag{4.61}$$

At $s = t$ this coincides with the Feynman-Kac exponential of (4.47). Indeed, evaluating (4.58) at $s = t$ and recalling from (4.46) that $A^\varepsilon_{t,x}(r, y) = p_\varepsilon(B_{t-r} + x - y)\,\mathbf{1}_{[0,t]}(r)$ we obtain

$$N^B_t \;=\; \beta \int_0^t \int_{\mathbb{R}^d} p_\varepsilon(B_{t-r} + x - y)\,W(dr, dy) \;=\; \beta\,W(A^\varepsilon_{t,x}).$$

For the correction term, setting $i = j$ in (4.51) gives

$$\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}} = \int_0^t \Lambda_\varepsilon(0)\,dr = t\,\Lambda_\varepsilon(0) = \frac{\langle N^B\rangle_t}{\beta^2},$$

where the last identity stems from (4.59). Therefore

$$M^B_t \;=\; \exp\Big(N^B_t - \tfrac{1}{2}\langle N^B\rangle_t\Big) \;=\; \exp\Big(\beta\,W(A^\varepsilon_{t,x}) - \tfrac{\beta^2}{2}\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}}\Big). \tag{4.62}$$

Setting $X_s := N^B_s - \frac{1}{2}\langle N^B\rangle_s$, so that $M^B_s = e^{X_s}$ (4.61), Itô's formula applied to $f(x) = e^x$ and the semimartingale X gives

$$\begin{aligned} dM^B_s &= M^B_s\,dX_s + \tfrac{1}{2}M^B_s\,d\langle X\rangle_s \\ &= M^B_s\Big(dN^B_s - \tfrac{1}{2}d\langle N^B\rangle_s\Big) + \tfrac{1}{2}M^B_s\,d\langle N^B\rangle_s \;=\; M^B_s\,dN^B_s. \end{aligned}$$

Substituting the differential $dN^B_s = \beta\int_{\mathbb{R}^d} p_\varepsilon(B_{t-s} + x - y)\,W(ds, dy)$ from (4.58) then yields

$$dM^B_s \;=\; \beta\,M^B_s \int_{\mathbb{R}^d} p_\varepsilon(B_{t-s} + x - y)\,W(ds, dy). \tag{4.63}$$

so integrating from 0 to t and using $M^B_0 = 1$ we end up with:

$$M^B_t = 1 + \beta \int_0^t \int_{\mathbb{R}^d} M^B_s\,p_\varepsilon(B_{t-s} + x - y)\,W(ds, dy). \tag{4.64}$$

Step 1-b: Averaging over B. By (4.47) and (4.62), observe that

$$u^\varepsilon(t, x) \;=\; \mathbb{E}_B\Big[u_0(B_t + x)\,M^B_t\Big].$$

Substituting (4.64) for M^B_t, using that $u_0(B_t + x)$ is W-independent for fixed B, and applying stochastic Fubini to exchange $\mathbb{E}_B$ with the Walsh integral, we thus get

$$u^\varepsilon(t, x) = p_t u_0(x) + \beta \int_0^t \int_{\mathbb{R}^d} \mathbb{E}_B\Big[u_0(B_t + x)\,M^B_s\,p_\varepsilon(B_{t-s} + x - y)\Big]\,W(ds, dy). \tag{4.65}$$

Step 1-c: Identification of the kernel. Our next objective is to show that

$$\mathbb{E}_B[u_0(B_t + x)\, M_s^B\, p_\varepsilon(B_{t-s} + x - y)] = Z^{\varepsilon,t,x}(s, y), \tag{4.66}$$

with $Z^{\varepsilon,t,x}$ as defined in (4.56). To this aim, condition on B_{t-s} and set $b := B_{t-s} + x$. Write $\tilde{B}_v := B_{t-s+v} - B_{t-s}$ for $v \in [0, s]$. This is a Brownian motion starting at 0, independent of B_{t-s}. Then for $r \in [0, s]$ one can write

$$B_{t-r} + x = b + \tilde{B}_{s-r}, \qquad B_t + x = b + \tilde{B}_s, \qquad p_\varepsilon(B_{t-s} + x - y) = p_\varepsilon(b - y), \tag{4.67}$$

Substituting the first identity of (4.67) into the integrand of N_s^B in (4.58), and using (4.60) for the quadratic variation, the definition (4.61) of M_s^B gives

$$M_s^B\Big|_{B_{t-s}=b-x} = \exp\left(\beta \int_0^s \int_{\mathbb{R}^d} p_\varepsilon(b + \tilde{B}_{s-r} - y)\, W(dr, dy) - \tfrac{\beta^2}{2} s\, \Lambda_\varepsilon(0)\right), \tag{4.68}$$

where we recall that $(\tilde{B}_v)_{v\in[0,s]}$ is now a standard Brownian motion starting at 0. Therefore comparing (4.68) with the Feynman-Kac formula (4.47) applied at (s, b), we obtain

$$\mathbb{E}_{\tilde{B}}\left[u_0(b + \tilde{B}_s)\, M_s^B\Big|_{B_{t-s}=b-x}\right] \;=\; u^\varepsilon(s, b). \tag{4.69}$$

Integrating over the law $\mathbb{P}_B(B_{t-s} + x \in db) = p_{t-s}(x - b)\, db$ and using (4.69), as well as the definition (4.56) of $Z^{\varepsilon,t,x}$, we obtain

$$\mathbb{E}_B\Big[u_0(B_t + x)\, M_s^B\, p_\varepsilon(B_{t-s} + x - y)\Big] = \int_{\mathbb{R}^d} p_{t-s}(x - b)\, p_\varepsilon(b - y)\, u^\varepsilon(s, b)\, db = Z^{\varepsilon,t,x}(s, y),$$

which proves (4.66). Substituting (4.66) back into (4.65) replaces the kernel $\mathbb{E}_B\Big[u_0(B_t + x)\, M_s^B\, p_\varepsilon(B_{t-s} + x - y)\Big]$ by $Z^{\varepsilon,t,x}(s, y)$, yielding (4.57).

Step 2: Identification as the regularized mild SHE (4.55). Substituting the definition (4.56) of $Z^{\varepsilon,t,x}$ into the Walsh integral in (4.57), and using the symmetry $p_\varepsilon(z - y) = p_\varepsilon(y - z)$, gives

$$\int_0^t \int_{\mathbb{R}^d} Z^{\varepsilon,t,x}(s, y)\, W(ds, dy) \;=\; \int_0^t \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} p_\varepsilon(y - z)\, p_{t-s}(x - z)\, u^\varepsilon(s, z)\, dz\Big)\, W(ds, dy).$$

By stochastic Fubini (which justifies exchanging the deterministic dz integral with the martingale measure $W(ds, dy)$) and the definition (4.54) of W^ε (applied with $\phi(s, y) := p_{t-s}(x - y)\, u^\varepsilon(s, y)$), it is readily checked that

$$\int_0^t \int_{\mathbb{R}^d} Z^{\varepsilon,t,x}(s, y)\, W(ds, dy) \;=\; \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y)\, u^\varepsilon(s, y)\, W^\varepsilon(ds, dy).$$

Substituting back into (4.57) yields (4.55). ■

The mild equation (4.57) also admits a reformulation in terms of Malliavin calculus: testing against any functional $F \in \mathbb{D}^{1,2}$ and applying the Skorohod duality relation replaces the Walsh integral by an $\mathcal{H}$-inner product involving the Malliavin derivative DF.

Lemma 4.2.4: Mild equation in duality form

Let u^ε be as in (4.47) and let $Z^{\varepsilon,t,x}$ be the kernel of (4.56). Then $Z^{\varepsilon,t,x} \in \mathrm{Dom}(\delta)$, and for every $F \in \mathbb{D}^{1,2}$,

$$\mathbb{E}\Big[F\, u^\varepsilon(t,x)\Big] \;=\; \mathbb{E}[F]\, p_t u_0(x) \;+\; \beta\, \mathbb{E}\Big[\langle DF, Z^{\varepsilon,t,x}\rangle_{\mathcal{H}}\Big]. \tag{4.70}$$

Proof. Multiply (4.57) by F and take expectations:

$$\mathbb{E}\Big[F\, u^\varepsilon(t,x)\Big] \;=\; \mathbb{E}[F]\, p_t u_0(x) \;+\; \beta\, \mathbb{E}\Big[F \int_0^t \int_{\mathbb{R}^d} Z^{\varepsilon,t,x}(s,y)\, W(ds,dy)\Big].$$

Since $u^\varepsilon(s,z)$ is $\mathcal{F}_s$-measurable by (4.47), the kernel $Z^{\varepsilon,t,x}(s,y)$ of (4.56) is also $\mathcal{F}_s$-measurable for each (s,y), hence adapted to $(\mathcal{F}_s)_{s\geq 0}$ of (3.4). Proposition 3.1.10 (formula (3.22)) therefore gives

$$\int_0^t \int_{\mathbb{R}^d} Z^{\varepsilon,t,x}(s,y)\, W(ds,dy) \;=\; \delta\Big(Z^{\varepsilon,t,x}\Big), \tag{4.71}$$

so in particular $Z^{\varepsilon,t,x} \in \mathrm{Dom}(\delta)$. Applying the Malliavin duality relation (2.19) gives $\mathbb{E}[F\,\delta(Z^{\varepsilon,t,x})] = \mathbb{E}[\langle DF, Z^{\varepsilon,t,x}\rangle_{\mathcal{H}}]$, which yields (4.70). ■

Remark 4.2.5 (Chaos expansion of u^ε)**.** *Iterating the mild equation (4.57) exactly as in the identification of the chaos kernels of u (Lemma 4.1.1) shows that u^ε has its own chaos expansion. Specifically one can write*

$$u^\varepsilon(t,x) = p_t u_0(x) + \sum_{n\geq 1} I_n\Big(f_n^\varepsilon(\cdot,t,x)\Big), \tag{4.72}$$

where f_n^ε is obtained from the kernel f_n of (4.6) by a convolution with p_ε in each of its n space variables. Namely, suppressing the time variables s_i and the parameters t,x in the kernels and writing the spatial arguments only for notational sake, it holds

$$f_n^\varepsilon(y_1,\ldots,y_n) := \int_{\mathbb{R}^{nd}} f_n(z_1,\ldots,z_n) \prod_{i=1}^n p_\varepsilon(y_i - z_i)\, dz_i. \tag{4.73}$$

Indeed, each iteration of (4.57) inserts the kernel $Z^{\varepsilon,t,x}$ of (4.56), which contains one convolution with p_ε in the noise variable y, while the zeroth-order term $p_t u_0(x)$ is the same as for u. Taking Fourier transforms in the space variables and recalling that $\widehat{p_\varepsilon}(\xi) = e^{-\varepsilon|\xi|^2/2}$ (2.40), relation (4.73) can be recast as

$$\mathcal{F}_x^{\otimes n} f_n^\varepsilon(s,\xi) = \prod_{i=1}^n e^{-\varepsilon|\xi_i|^2/2}\, \mathcal{F}_x^{\otimes n} f_n(s,\xi), \qquad (s,\xi) \in \mathbb{R}_+^n \times \mathbb{R}^{nd}, \tag{4.74}$$

where $\mathcal{F}_x^{\otimes n}$ denotes the Fourier transform in the n space variables, in coherence with the notation of (2.4).

The expansions (4.4) and (4.72) live on the same chaos levels, so that $u^\varepsilon(t,x) - u(t,x)$ can be compared kernel by kernel. Combined with Nelson's hypercontractivity inequality (2.25), this observation yields the following convergence result.

Proposition 4.2.6: L^k convergence of u^ε

Let $u_0 \in L^\infty(\mathbb{R}^d)$ and suppose that Dalang's condition (2.2) holds. Let u be the mild solution of (3.1) from Theorem 4.1.3, with chaos kernels f_n of (4.6). Let also u^ε be the regularized process (4.47). Then for every integer $k \geq 2$, every $t > 0$ and every $x \in \mathbb{R}^d$, it holds:

$$u^\varepsilon(t,x) \;\longrightarrow\; u(t,x) \qquad \text{in } L^k(\Omega), \text{ as } \varepsilon \to 0.$$

Proof. Owing to (4.4) and (4.72), the difference $u^\varepsilon(t,x) - u(t,x)$ admits the chaos decomposition $\sum_{n\geq 1} I_n(f_n^\varepsilon - f_n)$. We bound its $L^k(\Omega)$ norm in three steps: we first record a contraction property of the kernels f_n^ε, then prove a k-dependent summability estimate for the series of chaos norms, and finally conclude by hypercontractivity.

Step 1: Contraction and convergence at fixed chaos level. Since $0 < e^{-\varepsilon|\xi|^2/2} \leq 1$, the Fourier expression (4.74) together with the Fourier formula (2.4) for the norm in $\mathcal{H}^{\otimes n}$ yields

$$\|f_n^\varepsilon(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}} \;\leq\; \|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}}, \qquad \text{for all } \varepsilon > 0,\; n \geq 1. \tag{4.75}$$

Resorting to (4.74) and (2.4) again, and dropping the arguments $(\cdot, t, x)$ for notational sake, we also get

$$\|f_n^\varepsilon - f_n\|^2_{\mathcal{H}^{\otimes n}} = \int_{\mathbb{R}_+^n}\int_{\mathbb{R}^{nd}} \Big(1 - \prod_{i=1}^n e^{-\varepsilon|\xi_i|^2/2}\Big)^2 \Big|\mathcal{F}_x^{\otimes n} f_n(s,\xi)\Big|^2 \prod_{i=1}^n \mu(d\xi_i)\, ds. \tag{4.76}$$

The factor $(1 - \prod_{i=1}^n e^{-\varepsilon|\xi_i|^2/2})^2$ in (4.76) is bounded by 1 and converges pointwise to 0 as $\varepsilon \to 0$, while $|\mathcal{F}_x^{\otimes n} f_n|^2$ is integrable (we have $\|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}} < \infty$ by (4.39)). Hence dominated convergence in (4.76) gives, for each fixed $n \geq 1$,

$$\lim_{\varepsilon\to 0} \|f_n^\varepsilon(\cdot,t,x) - f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}} = 0. \tag{4.77}$$

Step 2: Summability of the weighted chaos norms. We now claim that for every integer $k \geq 2$,

$$\sum_{n=0}^\infty (k-1)^{n/2}\sqrt{n!}\, \|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}} \;<\; \infty. \tag{4.78}$$

To this aim, recall the bound (4.38), that is

$$n!\, \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;\leq\; \|u_0\|_\infty^2\, \beta^{2n} \sum_{m=0}^n \binom{n}{m} \frac{t^m}{m!}\, D_N^m\, C_N^{n-m},$$

where D_N and C_N are defined by (4.37). Multiplying by $(k-1)^n$ and using $\binom{n}{m} \leq 2^n$, we obtain

$$(k-1)^n\, n!\, \|f_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \;\leq\; \|u_0\|_\infty^2 \sum_{m=0}^n \frac{\Big(2(k-1)\beta^2 t\, D_N\Big)^m}{m!}\, \theta_N^{n-m}, \tag{4.79}$$

where we have set

$$\theta_N := 2(k-1)\beta^2 C_N.$$

Recall that $C_N \to 0$ as $N \to \infty$ (see the discussion after (4.37)). We can thus choose N large enough so that $\theta_N < 1$. Taking square roots in (4.79) and invoking the subadditivity of $r \mapsto \sqrt{r}$, this yields

$$(k-1)^{n/2}\sqrt{n!}\,\|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}} \;\le\; \|u_0\|_\infty \sum_{m=0}^{n} \frac{\big(2(k-1)\beta^2 t\, D_N\big)^{m/2}}{\sqrt{m!}}\,\theta_N^{(n-m)/2}.$$

Summing over $n \ge 0$ and exchanging the order of summation, we end up with

$$\sum_{n=0}^{\infty}(k-1)^{n/2}\sqrt{n!}\,\|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}} \;\le\; \frac{\|u_0\|_\infty}{1-\theta_N^{1/2}} \sum_{m=0}^{\infty} \frac{\big(2(k-1)\beta^2 t\, D_N\big)^{m/2}}{\sqrt{m!}},$$

and the series in m above converges by the ratio test. This proves the claim (4.78).

Step 3: Conclusion by hypercontractivity. Applying Nelson's inequality (2.25) to $u^\varepsilon(t,x) - u(t,x) = \sum_{n\ge1} I_n(f_n^\varepsilon - f_n)$, we get

$$\|u^\varepsilon(t,x) - u(t,x)\|_{L^k(\Omega)} \;\le\; \sum_{n=1}^{\infty}(k-1)^{n/2}\sqrt{n!}\,\|f_n^\varepsilon(\cdot,t,x) - f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}}. \tag{4.80}$$

Thanks to the contraction property (4.75) we have $\|f_n^\varepsilon - f_n\|_{\mathcal{H}^{\otimes n}} \le 2\|f_n\|_{\mathcal{H}^{\otimes n}}$, so that twice the series (4.78) dominates the right-hand side of (4.80) uniformly in ε. Moreover each summand in (4.80) vanishes as $\varepsilon \to 0$, due to (4.77). By dominated convergence for series, the right-hand side of (4.80) thus converges to 0 as $\varepsilon \to 0$, which finishes the proof. ■

4.2.2 Feynman-Kac formula for moments

The mild SHE (4.55) satisfied by u^ε, combined with the explicit moment formula of Proposition 4.2.2 and the $L^k(\Omega)$ convergence of Proposition 4.2.6, allows us to pass to the limit $\varepsilon \to 0$ and obtain a closed-form expression for all integer moments of the true solution u. The result expresses $\mathbb{E}[u(t,x)^k]$ as an expectation over k independent Brownian motions: the k-th moment is controlled by the mutual occupation times $\int_0^t \Lambda(B_s^i - B_s^j)\,ds$ of the pairs (B^i, B^j), reflecting the fact that high moments are driven by paths that spend long times close together.

Theorem 4.2.7: Feynman-Kac moment formula

Let $u_0 \in L^\infty(\mathbb{R}^d)$ and suppose Dalang's condition (2.2) holds. Let u be the mild solution of (3.1) from Theorem 4.1.3. Then for every integer $k \geq 2$, every $t > 0$, and every $x \in \mathbb{R}^d$,

$$\mathbb{E}\Big[u(t,x)^k\Big] = \mathbb{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x)\, \exp\left(\beta^2 \sum_{1\leq i<j\leq k} \int_0^t \Lambda(B_s^i - B_s^j)\, ds\right)\right], \tag{4.81}$$

where $B^1, \ldots, B^k$ are as in Notation 4.2.1.

Proof. We proceed in three steps, and then present in Step 4 an alternative self-contained proof of the convergence $u^\varepsilon \to u$.

Step 1: Moment formula for the regularized solution. By Proposition 4.2.2 (formula (4.50) with the uniform bound (4.48)), for every $\varepsilon > 0$ and integer $k \geq 2$ we have

$$\mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big] = \mathbb{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x)\, \exp\left(\beta^2 \sum_{1\leq i<j\leq k} \langle A_{t,x}^{\varepsilon,i}, A_{t,x}^{\varepsilon,j}\rangle_{\mathcal{H}}\right)\right],$$

and this quantity is bounded uniformly in $\varepsilon > 0$.

Step 2: Inner product limit. The inner product formula (4.51) from Proposition 4.2.2 enables to write

$$\langle A_{t,x}^{\varepsilon,i}, A_{t,x}^{\varepsilon,j}\rangle_{\mathcal{H}} = \int_0^t \int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2}\, e^{i\xi\cdot(B_r^j - B_r^i)}\, \mu(d\xi)\, dr \xrightarrow{\varepsilon\to 0} \int_0^t \Lambda(B_r^j - B_r^i)\, dr \tag{4.82}$$

in $L^1(\Omega)$, by dominated convergence ($|e^{i\xi\cdot z}| \leq 1$ and Dalang's condition (2.2)).

Step 3: Convergence of moments. By (4.82), the exponential in (4.50) converges $\mathbb{P}_B$-almost surely to the right-hand side of (4.81) as $\varepsilon \to 0$. Combined with the uniform bound (4.48) from Proposition 4.2.2 and the $L^1(\mathbb{P}_B)$ convergence in (4.82), dominated convergence gives

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big] = \mathbb{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x)\, \exp\left(\beta^2 \sum_{1\leq i<j\leq k} \int_0^t \Lambda(B_r^i - B_r^j)\, dr\right)\right].$$

On the other hand, Proposition 4.2.6 gives $u^\varepsilon(t,x) \to u(t,x)$ in $L^k(\Omega)$ as $\varepsilon \to 0$. Since L^k-convergence of $X_n \to X$ implies $\mathbb{E}[X_n^k] \to \mathbb{E}[X^k]$, we get $\mathbb{E}[(u^\varepsilon(t,x))^k] \to \mathbb{E}[u(t,x)^k]$. Combining both limits yields (4.81).

Step 4: Alternative proof of $u^\varepsilon \to u$ via the cross-moment formula. The argument in Step 3 identifies $\mathbb{E}[(u^\varepsilon)^k] \to \mathbb{E}[u^k]$ through Proposition 4.2.6, whose proof relies on the chaos expansion of u^ε from Remark 4.2.5. Here we give a self-contained route that stays at the level of the Feynman-Kac formula. In Step 4-a we use a cross-moment version of (4.47) to show that (u^ε) is Cauchy in $L^2(\Omega)$, with some limit v. In Step 4-b we pass to the limit in the mild equation (4.57) and invoke uniqueness to conclude $v = u$.

Step 4-a: u^ε is $L^2(\Omega)$-Cauchy. The same Gaussian computation as in Proposition 4.2.2, applied with $k = 2$ and with *distinct* regularization parameters ε and ε' for the two independent Brownian copies B^1 and B^2 (Notation 4.2.1), gives

$$\mathbb{E}\Big[u^\varepsilon(t,x)\,u^{\varepsilon'}(t,x)\Big] \;=\; \mathbb{E}_{B^1,B^2}\Big[u_0(B_t^1+x)\,u_0(B_t^2+x)\,\exp\Big(\beta^2\langle A_{t,x}^{\varepsilon,1}, A_{t,x}^{\varepsilon',2}\rangle_{\mathcal{H}}\Big)\Big], \tag{4.83}$$

where $A_{t,x}^{\varepsilon',2}(r,y) := p_{\varepsilon'}(B_{t-r}^2 + x - y)\,\mathbf{1}_{[0,t]}(r)$. By the Fourier formula (2.4):

$$\langle A_{t,x}^{\varepsilon,1}, A_{t,x}^{\varepsilon',2}\rangle_{\mathcal{H}} \;=\; \int_0^t\int_{\mathbb{R}^d} e^{-(\varepsilon+\varepsilon')|\xi|^2/2}\,e^{i\xi\cdot(B_r^2-B_r^1)}\,\mu(d\xi)\,dr,$$

which converges to $\int_0^t \Lambda(B_r^2 - B_r^1)\,dr$ as $\varepsilon, \varepsilon' \to 0$ by the same dominated convergence as in (4.82). Hence the right-hand side of (4.83) converges to the same limit $\ell := \lim_{\varepsilon\to 0}\mathbb{E}[(u^\varepsilon(t,x))^2]$ obtained in Steps 1–2 with $k = 2$, so that

$$\mathbb{E}\Big[(u^\varepsilon(t,x) - u^{\varepsilon'}(t,x))^2\Big] \;=\; \mathbb{E}\Big[(u^\varepsilon)^2\Big] - 2\,\mathbb{E}\Big[u^\varepsilon u^{\varepsilon'}\Big] + \mathbb{E}\Big[(u^{\varepsilon'})^2\Big] \;\longrightarrow\; \ell - 2\ell + \ell \;=\; 0.$$

Therefore $u^\varepsilon(t,x)$ is Cauchy in $L^2(\Omega)$; denote its L^2-limit by $v(t,x)$.

Step 4-b: Identifying $v = u$ via (4.57). Since $Z^{\varepsilon,t,x}(s,y) = (p_\varepsilon *_y [p_{t-s}(x-\cdot)\,u^\varepsilon(s,\cdot)])(y)$ (definitions (4.54) and (4.56)), the $L^2(\Omega)$ convergence $u^\varepsilon \to v$ together with the approximate-identity property $\|p_\varepsilon * f - f\|_{L^2} \to 0$ give

$$Z^{\varepsilon,t,x} \;\longrightarrow\; Z^{t,x}(s,y) := p_{t-s}(x-y)\,v(s,y) \quad \text{in } L^2(\Omega;\mathcal{H}).$$

By the Walsh isometry (3.11) applied to adapted integrands,

$$\Big\|\int_0^t\int_{\mathbb{R}^d}\Big(Z^{\varepsilon,t,x} - Z^{t,x}\Big)\,W\Big\|_{L^2(\Omega)}^2 \;=\; \mathbb{E}\Big[\|Z^{\varepsilon,t,x} - Z^{t,x}\|_{\mathcal{H}}^2\Big] \;\longrightarrow\; 0.$$

Taking $L^2(\Omega)$ limits on both sides of (4.57) therefore gives the mild equation

$$v(t,x) \;=\; p_t u_0(x) \;+\; \beta\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)\,v(s,y)\,W(ds,dy).$$

By uniqueness of mild solutions (Theorem 4.1.3), $v = u$, which yields (4.81). This concludes our proof, either by Step 3 or Step 4. ■

Remark 4.2.8. *The integral $\int_0^t \Lambda(B_s^i - B_s^j)\,ds$ measures the time the paths B^i and B^j spend near each other, weighted by Λ. When $\Lambda = \delta_0$ (space white noise, $d = 1$) it becomes the* intersection local time $\ell_t(B^i - B^j)$ *of the two paths at the origin. The exponential growth of $\mathbb{E}[u(t,x)^k]$ as $t \to \infty$ — that is, of the $\mathbb{E}_B$-expectation in (4.81) — is then governed by the growth of intersection local times, and the formula (4.81) is the starting point for the Lyapunov exponent estimates of Section 4.3.*

4.3 Moment estimates

The Feynman-Kac formula (4.81) expresses $\mathbb{E}[u(t,x)^k]$ as an expectation over k independent Brownian paths, and the chaos expansion of Section 4.1 expresses it via the norms $\|f_n(\cdot;t,x)\|_{\mathcal{H}^{\otimes n}}$. We now make these expressions quantitative under a power-law assumption on the spatial covariance Λ, using Theorem 2.1.9 as the main analytic tool.

4.3.1 Assumption and upper bound

Assumption 4.3.1: Power-law spatial correlation

There exist positive constants $c_\Lambda \le C_\Lambda$ and an exponent $0 < a < \min(2,d)$ such that

$$c_\Lambda\,|x|^{-a} \;\le\; \Lambda(x) \;\le\; C_\Lambda\,|x|^{-a}, \qquad x \in \mathbb{R}^d. \tag{4.84}$$

Assumption 4.3.1 implies Dalang's condition (2.2): for the Riesz spectral measure $\mu(d\xi) \asymp |\xi|^{a-d}d\xi$ one has $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-1}\mu(d\xi) < \infty$ if and only if $a < 2$. The Riesz kernel $\Lambda(x) = |x|^{-a}$ of Example 2.3.1 satisfies (4.84) with $c_\Lambda = C_\Lambda = 1$.

Theorem 4.3.2: Upper bound on moments

Let $u_0 \in L^\infty(\mathbb{R}^d)$ and suppose Assumption 4.3.1 holds. Then for every integer $k \ge 2$, every $t \ge 0$, and every $x \in \mathbb{R}^d$,

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\le\; C^k \exp\Big(C\,k^{\frac{4-a}{2-a}}\,t\Big), \tag{4.85}$$

where $C > 0$ depends only on β, $\|u_0\|_\infty$, C_Λ, a, and d.

Proof of Theorem 4.3.2. Set

$$\alpha \;:=\; 1 - a/2 \;\in\; (0,1) \tag{4.86}$$

and $M := \|u_0\|_\infty$.

Step 1: Reduction to chaos norms via hypercontractivity. By the chaos expansion (4.4), $u(t,x) = \sum_{n=0}^\infty I_n(f_n(\cdot;t,x))$. Theorem 2.1.9 and the isometry (2.12) give

$$\mathbb{E}\Big[u(t,x)^k\Big]^{1/k} \;\le\; \sum_{n=0}^\infty (k-1)^{n/2}\sqrt{n!}\,\|f_n(\cdot;t,x)\|_{\mathcal{H}^{\otimes n}}. \tag{4.87}$$

Step 2: Bound on $n!\,\|f_n\|^2_{\mathcal{H}^{\otimes n}}$. The key input is the decoupled bound (4.36):

$$n!\,\|f_n\|^2_{\mathcal{H}^{\otimes n}} \;\le\; M^2\,\beta^{2n}\int_{\mathbb{R}^{nd}}\int_{S_{t,n}}\prod_{i=1}^n e^{-w_i|\xi_i|^2}\,dw\,\mu^{\otimes n}(d\xi). \tag{4.88}$$

We now bound each factor in the spectral integral. Since $\widehat{p_{2w}}(\xi) = e^{-w|\xi|^2}$ by (2.40) and $\mu(d\xi) = \hat{\Lambda}(\xi)\, d\xi$, Parseval's identity and the evenness of Λ give (exactly as in (4.35)):

$$\int_{\mathbb{R}^d} e^{-w|\xi|^2}\, \mu(d\xi) \;=\; \int_{\mathbb{R}^d} \widehat{p_{2w}}(\xi)\, \hat{\Lambda}(\xi)\, d\xi \;=\; (2\pi)^d \int_{\mathbb{R}^d} p_{2w}(x)\, \Lambda(x)\, dx. \tag{4.89}$$

We now resort to the upper bound $\Lambda(x) \le C_\Lambda |x|^{-a}$ from (4.84) and the scaling identity

$$\int_{\mathbb{R}^d} p_{2w}(x)\, |x|^{-a}\, dx = (2w)^{-a/2}\mathbb{E}[|Z|^{-a}],$$

where $Z \sim \mathcal{N}(0, I_d)$ and the expectation is finite since $a < d$. This yields

$$\int_{\mathbb{R}^d} e^{-w|\xi|^2}\, \mu(d\xi) \;\le\; c_a\, w^{-a/2}, \qquad c_a := (2\pi)^d C_\Lambda 2^{-a/2}\mathbb{E}\Big[|Z|^{-a}\Big]. \tag{4.90}$$

Substituting (4.90) into (4.88), the ξ-integrals factor and each contributes $c_a\, w_i^{-a/2} = c_a\, w_i^{\alpha-1}$, leaving

$$n!\, \|f_n\|^2_{\mathcal{H}^{\otimes n}} \;\le\; M^2\, \beta^{2n}\, c_a^n \int_{S_{t,n}} \prod_{i=1}^n w_i^{\alpha-1}\, dw.$$

We now evaluate the simplex integral (defined in (4.30)). The substitution $w_i = tv_i$ maps $S_{t,n}$ to the unit simplex $S_{1,n} := \{v \ge 0 :\ v_1 + \cdots + v_n \le 1\}$ with Jacobian t^n, giving

$$\int_{S_{t,n}} \prod_{i=1}^n w_i^{\alpha-1}\, dw \;=\; t^{n\alpha} \int_{S_{1,n}} \prod_{i=1}^n v_i^{\alpha-1}\, dv.$$

The integral over $S_{1,n}$ is the classical Dirichlet integral, which we evaluate by induction on n using the Beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. The base case $n = 1$ gives $\int_0^1 v^{\alpha-1}\, dv = \Gamma(\alpha)/\Gamma(\alpha + 1)$. For the inductive step, suppose that

$$\int_{S_{1,n-1}} \prod_{i=1}^{n-1} v_i^{\alpha-1}\, dv \;=\; \frac{\Gamma(\alpha)^{n-1}}{\Gamma((n-1)\alpha + 1)} \tag{4.91}$$

holds for $n - 1$, and propagate it to n. Fix $v_1 \in [0, 1]$; then $(v_2, \ldots, v_n)$ ranges over the scaled simplex $S_{1-v_1,n-1}$, so by (4.91) (rescaled to the simplex of size $1 - v_1$) the inner integral equals $(1 - v_1)^{(n-1)\alpha}\, \Gamma(\alpha)^{n-1}/\Gamma((n - 1)\alpha + 1)$. Integrating out v_1 via $B(\alpha, (n - 1)\alpha + 1) = \Gamma(\alpha)\Gamma((n - 1)\alpha + 1)/\Gamma(n\alpha + 1)$ yields the Dirichlet formula

$$\int_{S_{t,n}} \prod_{i=1}^n w_i^{\alpha-1}\, dw \;=\; \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha + 1)}\, t^{n\alpha}. \tag{4.92}$$

We conclude

$$n!\, \|f_n(\cdot; t, x)\|^2_{\mathcal{H}^{\otimes n}} \;\le\; \frac{M^2\Big(\beta^2\, c_a\, \Gamma(\alpha)\Big)^n}{\Gamma(n\alpha + 1)}\, t^{n\alpha}. \tag{4.93}$$

Step 3: Summation. Set $\kappa := \beta^2 c_a \Gamma(\alpha)$. Inserting (4.93) into (4.87), it is easily seen that

$$\mathbb{E}\Big[u(t, x)^k\Big]^{1/k} \;\le\; M \sum_{n=0}^{\infty} \frac{r^n}{\sqrt{\Gamma(n\alpha + 1)}}, \qquad r := \Big[(k - 1)\kappa\Big]^{1/2} t^{\alpha/2}. \tag{4.94}$$

We bound the series in (4.94) using the Mittag-Leffler function of index α, defined classically by

$$E_\alpha(z) \ := \ \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \qquad z \geq 0. \tag{4.95}$$

Resorting to this function and the elementary inequality $\sqrt{AB} \leq (A+B)/2$ applied to $A_n := 2^n r^{2n}/\Gamma(n\alpha+1)$ and $B_n := 2^{-n}$, we obtain $\sqrt{A_n B_n} = r^n/\sqrt{\Gamma(n\alpha+1)}$, and hence

$$\frac{r^n}{\sqrt{\Gamma(n\alpha+1)}} \ \leq \ \frac{A_n+B_n}{2} \ = \ \frac{2^n r^{2n}}{2\,\Gamma(n\alpha+1)} + \frac{1}{2^{n+1}}.$$

Summing over $n \geq 0$, using $\sum_{n=0}^{\infty} 2^{-n} = 2$ and (4.95),

$$\sum_{n=0}^{\infty} \frac{r^n}{\sqrt{\Gamma(n\alpha+1)}} \ \leq \ \tfrac{1}{2}\, E_\alpha(2r^2) + 1. \tag{4.96}$$

The Mittag-Leffler function of index $\alpha \in (0,1)$ satisfies $E_\alpha(z) \leq C \exp(c\, z^{1/\alpha})$ for all $z \geq 0$ (see e.g. [3, Lem. A.1]). Applying this to (4.96) we discover that

$$\sum_{n=0}^{\infty} \frac{r^n}{\sqrt{\Gamma(n\alpha+1)}} \ \leq \ C_\alpha \exp\!\Big(c_\alpha\, r^{2/\alpha}\Big) \tag{4.97}$$

for constants $C_\alpha, c_\alpha > 0$ depending only on α. From the definition of r in (4.94) we have $r^2 = (k-1)\kappa\, t^\alpha$, hence $r^{2/\alpha} = (k-1)^{1/\alpha}\kappa^{1/\alpha} t$. In addition, from (4.86) we derive $1/\alpha = 2/(2-a)$. Combining (4.94) and (4.97), we get

$$\mathbb{E}\Big[u(t,x)^k\Big]^{1/k} \ \leq \ M\, C_\alpha \exp\!\Big(c'_\alpha\, (k-1)^{2/(2-a)}\, t\Big).$$

Raising to the k-th power and using $k(k-1)^{2/(2-a)} \ \leq \ 2\, k^{(4-a)/(2-a)}$ (valid for $k \geq 2$) yields (4.85). ■

4.3.2 Lower bound

The following theorem shows that the exponential rate in Theorem 4.3.2 is sharp when the initial condition is bounded away from zero.

Theorem 4.3.3: Lower bound on moments

Suppose Assumption 4.3.1 holds and $u_0 \in L^\infty(\mathbb{R}^d)$ with $m_0 := \inf_{x\in\mathbb{R}^d} u_0(x) > 0$. Then for every integer $k \geq 2$, every $t \geq 0$, and every $x \in \mathbb{R}^d$, we have

$$\mathbb{E}\Big[u(t,x)^k\Big] \ \geq \ C^k \exp\!\Big(c\, k^{\frac{4-a}{2-a}}\, t\Big), \tag{4.98}$$

where $C, c > 0$ depend only on m_0, c_Λ, β, a, and d.

Proof. We divide this proof in several steps.

Step 1: Setting. Since $u_0 \geq m_0 > 0$, the Feynman-Kac formula (4.81) gives

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\geq\; m_0^k\, \mathbb{E}_B\left[\exp\left(\beta^2 \sum_{1\leq i<j\leq k} \int_0^t \Lambda\Big(B_s^i - B_s^j\Big)\, ds\right)\right]. \tag{4.99}$$

Fix $\varepsilon > 0$. Denote by $B_s^{i,l}$ the l-th coordinate of B_s^i (Notation 4.2.1), and introduce the small ball event

$$\mathcal{A}_{\varepsilon,t} := \bigcap_{i=1}^{k} \bigcap_{l=1}^{d} \Big\{ \sup_{0\leq s\leq t} \Big|B_s^{i,l}\Big| \leq \tfrac{\varepsilon}{2} \Big\}. \tag{4.100}$$

On $\mathcal{A}_{\varepsilon,t}$, every pair (i,j) satisfies $|B_s^i - B_s^j| \leq \varepsilon$ for all $s \in [0,t]$, so the lower bound in (4.84) gives $\Lambda(B_s^i - B_s^j) \geq c_\Lambda \varepsilon^{-a}$. Using $k(k-1)/2 \geq k^2/4$, restricting the expectation in (4.99) to $\mathcal{A}_{\varepsilon,t}$ yields

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\geq\; m_0^k\, \exp\Big(\tfrac{1}{4}\beta^2 c_\Lambda k^2 t \varepsilon^{-a}\Big)\, \mathbb{P}\Big(\mathcal{A}_{\varepsilon,t}\Big). \tag{4.101}$$

Step 2: Small ball estimate. Since $B^1, \ldots, B^k$ start at the origin, the kd coordinate processes $B^{i,l}$ are independent standard one-dimensional Brownian motions, so

$$\mathbb{P}\Big(\mathcal{A}_{\varepsilon,t}\Big) = \left[\mathbb{P}\left(\sup_{0\leq s\leq t} |b_s| \leq \tfrac{\varepsilon}{2}\right)\right]^{kd}, \tag{4.102}$$

where b is a one-dimensional standard Brownian motion starting at 0. Furthermore, invoking a classical small ball estimate [43], we obtain

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |b_s| \leq \tfrac{\varepsilon}{2}\right) \;\geq\; C_0 \exp\left(-\frac{C_0 t}{\varepsilon^2}\right) \tag{4.103}$$

for a constant $C_0 > 0$. Setting $c_1 := \frac{1}{4}\beta^2 c_\Lambda$ and inserting (4.102)-(4.103) into (4.101), we obtain

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\geq\; (m_0 C_0^d)^k\, \exp\Big(c_1 k^2 t \varepsilon^{-a} - C_0 d k t \varepsilon^{-2}\Big). \tag{4.104}$$

Step 3: Optimization. We maximize the exponent in (4.104) over $\varepsilon > 0$. Setting the derivative with respect to ε to zero gives

$$\varepsilon_* = \left(\frac{2C_0 d}{a c_1 k}\right)^{1/(2-a)}. \tag{4.105}$$

At ε_* the optimality condition reads $C_0 d k \varepsilon_*^{-2} = \frac{a}{2} c_1 k^2 \varepsilon_*^{-a}$, so the exponent in (4.104) reduces to

$$c_1 k^2 t \varepsilon_*^{-a} - \tfrac{a}{2} c_1 k^2 t \varepsilon_*^{-a} = c_1 \Big(1 - a/2\Big)\, k^2 t \varepsilon_*^{-a}.$$

Substituting $\varepsilon_*^{-a} = (a c_1 k/(2C_0 d))^{a/(2-a)}$ and using $2 + a/(2-a) = (4-a)/(2-a)$, this enables to write

$$c_1(1-a/2)\, k^2 t \varepsilon_*^{-a} = c_1(1-a/2)\left(\frac{a c_1}{2C_0 d}\right)^{a/(2-a)} k^{(4-a)/(2-a)}\, t \;=:\, c'\, k^{\frac{4-a}{2-a}}\, t,$$

where $c' > 0$ since $a < 2$. Inserting into (4.104),

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\geq\; (m_0 C_0^d)^k \,\exp\Big(c'\,k^{\frac{4-a}{2-a}}\,t\Big)\,.$$

Setting $C := m_0 C_0^d$ and $c := c'$ yields (4.98). ■

Remark 4.3.4 (Comments on the moment estimates)**.** *Theorems 4.3.2 and 4.3.3 admit several complementary readings. In the discussion below, a always designates the exponent of the power-law relation* (4.84)*.*

(i) *Lyapunov exponents. Combining* (4.85) *and* (4.98)*, the k-th Lyapunov exponent*

$$\gamma(k) \;:=\; \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[u(t,x)^k\Big] \tag{4.106}$$

exists and satisfies $c\,k^{(4-a)/(2-a)} \leq \gamma(k) \leq C\,k^{(4-a)/(2-a)}$ for constants $0 < c \leq C < \infty$ independent of k. The two theorems are therefore sharp up to the values of the constants.

(ii) *Intermittency. The super-linear growth $\gamma(k) \asymp k^{(4-a)/(2-a)}$ with $(4-a)/(2-a) > 1$ for all $a \in (0,2)$ is the hallmark of* intermittency*: the normalized Lyapunov exponent $\gamma(k)/k$ is strictly increasing in k, in contrast to what would hold for a log-normal distribution (where $\gamma(k)$ is linear in k). Physically, the solution concentrates on rare, exponentially tall "islands", and the high moments are dominated by these extreme peaks rather than by a typical realization.*

(iii) *Influence of the spatial covariance. The exponent $(4-a)/(2-a)$ is a strictly increasing function of $a \in (0,2)$: as the singularity of Λ at the origin grows (larger a), the intermittency becomes stronger. In the mildest case $a \to 0$ (smooth covariance) the exponent approaches 2; near the Dalang borderline $a \to 2$ it diverges. The dimension d enters only through the admissibility constraint $a < \min(2,d)$.*

(iv) *Weighted intersection local times. As noted in Remark 4.2.8, the Feynman-Kac formula* (4.81) *expresses $\mathbb{E}[u(t,x)^k]$ in terms of the quantities $\int_0^t \Lambda(B_s^i - B_s^j)\,ds$, which are the Λ-weighted intersection local times of the paths B^i and B^j. For $\Lambda(x) = |x|^{-a}$, paths that spend time within distance r of each other contribute a factor r^{-a}, so a more singular Λ (larger a) rewards closer-running paths more heavily and drives stronger moment growth. In the degenerate case $\Lambda = \delta_0$ (space-time white noise, $d = 1$) these reduce to genuine intersection local times.*

(v) *Universality of the exponent. The exponent $(4-a)/(2-a)$ depends only on a, the singularity order of Λ at the origin. In particular it is independent of the noise intensity β, the specific profile of u_0, and the dimension d. The parameters β, u_0, and d only affect the constants C and c in* (4.85)–(4.98)*.*

(vi) *Connection to directed polymers. The $\mathbb{E}_B$-expectation in* (4.81) *is the partition function of a system of k Brownian paths with pairwise Λ-weighted interactions. This is precisely the directed polymer model in a Λ-correlated random environment. The exponential growth of $\mathbb{E}[u(t,x)^k]$ with t reflects the localization of the polymer measure on configurations where all k paths spend extended time near one another, exploiting the favorable regions of the noise.*

CHAPTER 5

A general geometric setting

This chapter and the next one carry the program developed in Chapters 2–4 over from $\mathbb{R}^d$ to a wide class of geometric structures. The current chapter focuses on the underlying geometric and analytic framework. Namely Section 5.1 introduces the metric measure spaces we shall work with, together with the related heat semigroups and fractional Laplace operators on domains. Then Section 5.2 illustrates this framework with three model examples. The parabolic Anderson model on those structures will be analyzed in Chapter 6.

5.1 General geometric and analytic settings

This section introduces the geometric framework in which the PAM will be studied. We work on a metric measure space (X, d, μ) equipped with a Dirichlet form whose heat kernel satisfies sub-Gaussian estimates. This general setting covers a wide range of situations of interest: the classical Euclidean case $\mathbb{R}^d$ (recovered when $d_h = d$ and $d_w = 2$), compact Riemannian manifolds, metric graphs, and fractal spaces such as the Sierpiński gasket for which the walk dimension d_w is strictly greater than 2. We first describe the standing assumptions on the ambient space, and then specialize to bounded domains with Dirichlet and Neumann boundary conditions.

5.1.1 Recurrent metric measure spaces

The ambient space throughout this chapter and Chapter 6 is a metric measure space satisfying several structural properties, which we collect into a single definition.

Definition 5.1.1: Metric measure space

A *locally compact complete geodesic metric measure space* is a triple (X, d, μ) where:

(i) *Metric structure.* (X, d) is a complete metric space that is locally compact (every point has a compact neighborhood) and geodesic (any two points $x, y \in X$ are joined by a curve of length $d(x, y)$).

(ii) *Measure.* μ is a Radon measure on X with full support, i.e., $\mu(U) > 0$ for every non-empty open set $U \subset X$.

We write $\operatorname{diam}(X) \in (0, +\infty]$ for the diameter of X, and $B(x, r) := \{y \in X : d(x, y) < r\}$ for the open ball of radius r centered at x.

The analytic structure on X is encoded by a Dirichlet form, which replaces the classical gradient and plays the role of the Laplacian in our setting. Most of the notions introduced below are standard and can be found in [23].

Definition 5.1.2: Dirichlet form

A *Dirichlet form* on $L^2(X, \mu)$ is a pair $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F} := \operatorname{dom} \mathcal{E}$ is a dense subspace of $L^2(X, \mu)$ and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ is a bilinear form satisfying the following properties.

(i) *Closed and symmetric.* $\mathcal{E}$ is symmetric, and $\mathcal{F}$ is complete for the $\mathcal{E}_1$-norm

$$\|f\|_{\mathcal{E}_1} := \Big(\|f\|^2_{L^2(X,\mu)} + \mathcal{E}(f, f)\Big)^{1/2}. \tag{5.1}$$

(ii) *Markovian.* If $u \in \mathcal{F}$ and v is a normal contraction of u (i.e., $|v(x) - v(y)| \leq |u(x) - u(y)|$ and $|v(x)| \leq |u(x)|$ μ-a.e.), then $v \in \mathcal{F}$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.

(iii) *Regular.* There exists a core $\mathcal{C} \subseteq C_c(X) \cap \mathcal{F}$ that is dense in $C_c(X)$ in the sup-norm and dense in $\mathcal{F}$ in the $\mathcal{E}_1$-norm.

(iv) *Strongly local.* $\mathcal{E}(u, v) = 0$ whenever $u, v \in \mathcal{F}$ have compact support and u is constant on a neighborhood of $\operatorname{supp}(v)$.

A regular strongly local Dirichlet form carries two fundamental objects: an energy measure and a generator.

Notation 5.1.3: Energy measure and generator

Let $(\mathcal{E}, \mathcal{F})$ be a regular strongly local Dirichlet form on $L^2(X, \mu)$.

(i) *Energy measure.* For every $u, v \in \mathcal{F} \cap L^\infty(X, \mu)$, the *energy measure* $\Gamma(u, v)$ is the unique signed Radon measure satisfying

$$\int_X \phi \, d\Gamma(u, v) = \frac{1}{2}\Big[\mathcal{E}(\phi u, v) + \mathcal{E}(\phi v, u) - \mathcal{E}(\phi, uv)\Big], \qquad \phi \in \mathcal{F} \cap C_c(X). \tag{5.2}$$

(ii) *Generator and semigroup.* The *generator* Δ is the self-adjoint operator on $L^2(X, \mu)$ defined on the dense domain $\mathrm{dom}(\Delta) \subset \mathcal{F}$ by the integration-by-parts formula

$$\mathcal{E}(f, g) = -\int_X f \, \Delta g \, d\mu, \qquad f \in \mathcal{F}, \; g \in \mathrm{dom}(\Delta). \tag{5.3}$$

The associated strongly continuous semigroup is $P_t = e^{t\Delta}$ [23].

A key identity links the energy measure back to the form itself. Its proof can be found in [23, Thm. 3.2.1].

Proposition 5.1.4: Beurling–Deny

Under the conditions of Notation 5.1.3,

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v), \qquad u, v \in \mathcal{F}. \tag{5.4}$$

Remark 5.1.5. *We wish to stress the following facts about the measure Γ and the operator Δ.*

(i) Interpretation of Γ. *On $\mathbb{R}^d$ with $\mathcal{E}(f, g) = \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, dx$, one has $d\Gamma(f, g) = \nabla f \cdot \nabla g \, dx$, so Γ is the analogue of the squared gradient. In particular, $\Gamma(f, f)(A)$ measures the energy of f concentrated on the set A. On a fractal, no pointwise gradient exists, but Γ is still well-defined and captures the local energy content of a function.*

(ii) Definition of $e^{t\Delta}$. *Since Δ is a non-positive self-adjoint operator on $L^2(X, \mu)$, the spectral theorem provides a projection-valued measure $E(\cdot)$ on $(-\infty, 0]$ such that $\Delta = \int_{-\infty}^0 \lambda \, dE(\lambda)$. The semigroup is then defined spectrally by $P_t = e^{t\Delta} = \int_{-\infty}^0 e^{t\lambda} \, dE(\lambda)$, which is a contraction on $L^2(X, \mu)$ for each $t \geq 0$. Under Assumption 5.1.6 below, P_t extends to a bounded operator on $L^\infty(X, \mu)$ and admits a continuous integral kernel $p_t(x, y)$, the* heat kernel.

We can now state the precise conditions on $(X, d, \mu, \mathcal{E}, \mathcal{F})$ that will be in force throughout this chapter and the next one. They amount to requiring that balls have a polynomial volume growth (controlled by a Hausdorff dimension d_h) and that the heat kernel satisfies sub-Gaussian off-diagonal estimates governed by a walk dimension $d_w \geq 2$. The condition $d_h < d_w$, absent in the Euclidean setting ($d_h = d$, $d_w = 2$, $d \geq 2$), is what forces recurrence of the associated Brownian motion and is characteristic of fractal-like spaces.

Assumption 5.1.6: Sub-Gaussian framework

Let (X, d, μ) be the locally compact complete geodesic metric measure space of Definition 5.1.1, equipped with the regular strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Definition 5.1.2. We assume:

(i) *Ahlfors regularity.* There exist $c_1, c_2, d_h \in (0, \infty)$ such that for all $x \in X$ and $r \in (0, \mathrm{diam}(X))$,

$$c_1\, r^{d_h} \;\le\; \mu\Big(B(x,r)\Big) \;\le\; c_2\, r^{d_h}. \tag{5.5}$$

The constant d_h is the *Hausdorff dimension* of (X, d).

(ii) *Regularity and strong locality.* $(\mathcal{E}, \mathcal{F})$ is a regular and strongly local Dirichlet form in the sense of Definition 5.1.2.

(iii) *Stochastic completeness.* $P_t 1 = 1$ for every $t \ge 0$.

(iv) *Sub-Gaussian heat kernel.* The semigroup $\{P_t\}_{t\ge 0}$ has a continuous heat kernel $p_t(x, y)$ satisfying, for some $c_3, c_4, c_5, c_6 \in (0, \infty)$ and $d_w \in [2, +\infty)$,

$$\begin{aligned} c_5\, t^{-d_h/d_w} \exp\left(-c_6\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\right) &\le p_t(x,y) \\ &\le c_3\, t^{-d_h/d_w} \exp\left(-c_4\Big(\frac{d(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\right) \end{aligned} \tag{5.6}$$

for $\mu\times\mu$-a.e. $(x, y) \in X \times X$ and each $t \in (0, \mathrm{diam}(X)^{d_w})$. The constant d_w is called the *walk dimension.*

(v) *Recurrence condition.* $d_h < d_w$.

Remark 5.1.7. *When $d_w = 2$, the sub-Gaussian estimates (5.6) reduce to the classical Gaussian estimates. The Euclidean space $\mathbb{R}^n$ satisfies Assumption 5.1.6 with $d_h = n$ and $d_w = 2$ (and the recurrence condition $d_h < d_w$ fails for $n \ge 2$, consistently with the transience of Brownian motion in $\mathbb{R}^n$, $n \ge 2$). Fractal spaces such as the Sierpiński gasket satisfy $d_w > 2$; see Section 5.2.4.*

By a fundamental theorem on Dirichlet forms [23, Thm. 7.2.1], Assumption 5.1.6 guarantees the existence of a canonical Markov process associated with $(\mathcal{E}, \mathcal{F})$, which we call the Brownian motion on X.

Definition 5.1.8: Brownian motion on X

A *Brownian motion* on (X, d, μ) is a continuous X-valued process $(B_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega^B, \mathcal{F}^B, (\mathcal{F}_t^B)_{t\geq 0})$, together with a family of probability measures $(\mathbb{P}_x)_{x\in X}$ on $(\Omega^B, \mathcal{F}^B)$, such that:

(i) *Initial condition.* $\mathbb{P}_x(B_0 = x) = 1$ for every $x \in X$.

(ii) *Markov property.* For every bounded measurable $f : X \to \mathbb{R}$, every $s, t \geq 0$, and every $x \in X$,

$$\mathbb{E}_x\Big[f(B_{s+t})\,\Big|\,\mathcal{F}_t^B\Big] \;=\; \mathbb{E}_{B_t}\Big[f(B_s)\Big] \qquad \mathbb{P}_x\text{-a.s.} \tag{5.7}$$

(iii) *Semigroup.* The semigroup of $(B_t)_{t\geq 0}$ coincides with P_t of Notation 5.1.3: for every $f \in L^2(X, \mu)$ bounded,

$$P_t f(x) \;=\; \mathbb{E}_x\Big[f(B_t)\Big], \qquad x \in X,\; t \geq 0. \tag{5.8}$$

We write $\mathbb{E}_x$ for expectation under $\mathbb{P}_x$, and $\mathbb{E}_B$ when the starting point is clear from context or held fixed. In joint computations with the noise W defined on $(\Omega, \mathcal{F}, \mathbb{P})$, we treat B and W as independent by working on the product space $(\Omega \times \Omega^B, \mathcal{F} \otimes \mathcal{F}^B, \mathbb{P} \otimes \mathbb{P}_x)$.

Remark 5.1.9. *The exponents d_h and d_w in Assumption 5.1.6 have the following geometric and probabilistic interpretations.*

(i) *The Hausdorff dimension d_h describes the volume growth of metric balls. Namely we have $\mu(B(x,r)) \asymp r^{d_h}$.*

(ii) *The walk dimension d_w describes the spreading of the Brownian motion $(B_t)_{t\geq 0}$. That is it holds $\mathbb{E}_x[d(x, B_t)] \asymp t^{1/d_w}$.*

The condition $d_h < d_w$ therefore means that the process spreads more slowly than the volume grows, which forces recurrence.

As mentioned in Remark 5.1.9, The recurrence of the Brownian motion B is a consequence of the condition $d_h < d_w$ in Assumption 5.1.6. We state this property as a lemma, and refer to [4, Cor. 3.29] for a complete proof.

Lemma 5.1.10: Recurrence

Under Assumption 5.1.6, the Brownian motion $(B_t)_{t\geq 0}$ is strongly recurrent: for every $x, y \in X$,

$$\mathbb{P}_x\Big(\exists\, t \geq 0 : B_t = y\Big) = 1.$$

5.1.2 Dirichlet boundary conditions

We now specialize to a bounded open set $U \subset X$ with compact closure $\overline{U}$. This subsection introduces the heat semigroup on U with Dirichlet boundary conditions, its heat kernel, and the associated fractional powers of the Laplacian.

5.1.2.1 Heat semigroup

Definition 5.1.11: Localized domains

Let $U \subset X$ be a non-empty bounded open set with compact closure. Set

$$\mathcal{F}(U) := \Big\{u \in L^2(U,\mu) : \exists\, f \in \mathcal{F},\ f|_U = u\ \mu\text{-a.e.}\Big\}, \tag{5.9}$$

and

$$\mathcal{F}_c(U) := \Big\{u \in \mathcal{F}(U) : \text{ess-supp}(u) \text{ is compact in } U\Big\}. \tag{5.10}$$

The *Dirichlet domain* $\mathcal{F}^D(U)$ is the closure of $\mathcal{F}_c(U)$ in the $\mathcal{E}_1$-norm (5.1). Functions in $\mathcal{F}^D(U)$ satisfy the Dirichlet boundary condition.

Restricting the global form $(\mathcal{E},\mathcal{F})$ to $\mathcal{F}^D(U)$ yields a well-defined Dirichlet form on U, whose generator provides the Laplacian with Dirichlet boundary conditions. We label this as a proposition (borrowed from [23] for further use).

Proposition 5.1.12: Dirichlet form on U

The restricted form

$$\mathcal{E}^D_U(u,v) := \mathcal{E}(u,v), \qquad u,v \in \mathcal{F}^D(U), \tag{5.11}$$

is a strongly local regular Dirichlet form on $L^2(U,\mu)$. Its generator is denoted $\Delta_{D,U}$ and called the *Dirichlet Laplacian on U*.

The associated semigroup $P_t^{D,U}$ admits a probabilistic representation via the Brownian motion $(B_t)_{t\geq 0}$ of Lemma 5.1.10. Let $T_U := \inf\{t \geq 0 : B_t \notin U\}$ be the first exit time from U. Then, for $f \in L^2(U,\mu)$ bounded and $x \in U$, the following is shown in [23, Thm. 4.4.2]:

$$P_t^{D,U} f(x) = \mathbb{E}_x\Big[f(B_t)\,\mathbf{1}_{t<T_U}\Big]. \tag{5.12}$$

Now to obtain sharp heat kernel bounds on U we restrict to *inner uniform domains*, defined as follows. Recall that a curve $\gamma : [0,1] \to X$ is *rectifiable* if its length is finite, where the length is defined as

$$\text{length}(\gamma) := \sup \sum_i d(\gamma(t_i),\gamma(t_{i+1})),$$

the supremum above being over all finite subdivisions of $[0,1]$.

Definition 5.1.13: Inner metric and inner uniform domain

For a connected open set $U \subset X$, the *inner metric* is

$$d_U(x,y) := \inf\Big\{\text{length}(\gamma) : \gamma : [0,1] \to U \text{ rectifiable}, \gamma(0) = x, \gamma(1) = y\Big\}. \tag{5.13}$$

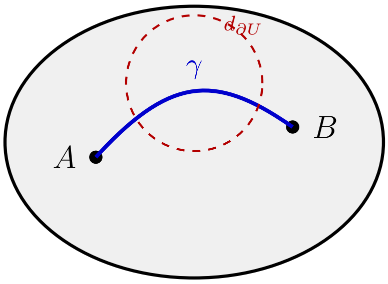
We denote by $\widetilde{U}$ the completion of U for d_U.

Fix $c \in (0,1)$ and $C > 1$. A rectifiable curve $\gamma : [\alpha, \beta] \to U$ is a (c, C) *inner uniform curve* if:

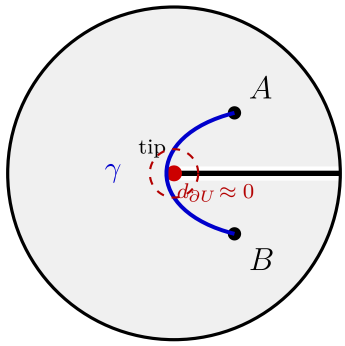
(i) for all $t \in [\alpha, \beta]$, $d_U(\gamma(t), \widetilde{U} \setminus U) \geq c \min\{d_U(\gamma(\alpha), \gamma(t)), d_U(\gamma(t), \gamma(\beta))\}$;

(ii) $\text{length}(\gamma) \leq C\, d_U(\gamma(\alpha), \gamma(\beta))$.

The domain U is called *inner uniform* if any two points in U can be joined by a (c, C) inner uniform curve for some fixed c, C.



(a) inner uniform



(b) non-inner-uniform

Figure 5.1 – (a) An inner uniform curve γ in a smooth convex domain: it stays at distance $d_{\partial U}$ from ∂U, proportional to the shorter arc length to each endpoint (condition (i) of Definition 5.1.13), and its length is comparable to $d_U(A, B)$ (condition (ii)). (b) A slit disk (disk with a radial cut shown as a thick segment). Any curve from A to B in U must cross $\{y = 0\}$ to the left of the slit tip (red dot), since the slit blocks all crossings for $x \geq 0$. A curve that passes close to the tip (as drawn) violates condition (i) of Definition 5.1.13, while a curve that swings far to the left to avoid the tip incurs excess length, violating condition (ii). No fixed (c, C) accommodates both constraints simultaneously, so the domain is not inner uniform.

Remark 5.1.14. *In Euclidean space, every bounded convex domain is inner uniform, as is every bounded domain with piecewise smooth boundary and non-zero interior angles. The class of inner uniform domains also includes sets with fractal boundaries such as the Koch snowflake [44]. On the other hand, domains with inward cusps or slits are not* *inner uniform. The slit disk of Figure 5.1 is one example. Another is the cusp domain $U = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ |y| < x^2\}$: two points $(x_0, \pm x_0^2/2)$ symmetric about the x-axis lie at Euclidean distance x_0^2, yet any path in U joining them must travel around the cusp tip at the origin, incurring length of order x_0; as $x_0 \to 0$ no fixed constant C can bound the ratio $\mathrm{length}(\gamma)/d_U(A, B)$, so condition (ii) of Definition 5.1.13 fails.*

The following theorem gives sharp heat kernel estimates for $P_t^{D,U}$ on inner uniform domains. Let Φ_1 be the first (positive) eigenfunction of $-\Delta_{D,U}$ with eigenvalue $\lambda_1 > 0$ (see Notation 5.1.20 below).

Theorem 5.1.15: [44, Thm. 3.3]

Let $U \subset X$ be an inner uniform domain. Then $(\mathcal{E}_U^D, \mathcal{F}^D(U))$ is a strongly local regular Dirichlet form admitting a continuous symmetric heat kernel $p_t^{D,U}(x, y)$. For all $(x, y) \in U \times U$ and $t > 0$,

$$p_t^{D,U}(x, y) \;\le\; \frac{c_1\, e^{-\lambda_1 t}}{\min(1, t^{d_h/d_w})} \exp\left(-c_2\Big(\frac{d(x, y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\right), \tag{5.14}$$

and

$$p_t^{D,U}(x, y) \;\ge\; \frac{c_3\, \Phi_1(x)\Phi_1(y)\, e^{-\lambda_1 t}}{\min(1, t^{d_h/d_w})} \exp\left(-c_4\Big(\frac{d_U(x, y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\right), \tag{5.15}$$

for constants $c_1, c_2, c_3, c_4 > 0$.

Remark 5.1.16 (Comments on the bounds of Theorem 5.1.15)**.** *The upper bound (5.14) for small t follows from the global estimate (5.6) and $p_t^{D,U}(x, y) \le p_t(x, y)$, which is immediate from (5.12). The lower bound involves the first eigenfunction Φ_1, which is positive on U by [44]. In particular, for every compact $A \subset U$ one has $\inf_{y \in A} \Phi_1(y) > 0$.*

The positivity of Φ_1 invoked in Remark 5.1.16 relies on the inner uniformity of U. The next remark records a positivity property of the ground state which holds on general connected open sets.

Remark 5.1.17 (Positivity of the ground state on connected sets)**.** *Let $V \subset X$ be a connected bounded open set with compact closure, and assume that $-\Delta_{D,V}$ admits a discrete spectrum together with an orthonormal basis of eigenfunctions, as in Notation 5.1.20 below. Then the smallest eigenvalue $\lambda_1(V)$ is simple, and the associated ground state Φ_1 can be chosen to be strictly positive on V. This is the classical Perron-Frobenius argument for positivity improving semigroups, for which we refer to [50, Thm. XIII.43 and XIII.44]. Let us briefly recall the mechanism of the proof. First, owing to the Markov property of the Dirichlet form $\mathcal{E}_V^D$, replacing Φ_1 by $|\Phi_1|$ does not increase the quotient*

$$\frac{\mathcal{E}_V^D(f, f)}{\|f\|^2_{L^2(V,\mu)}}.$$

Hence, up to a change of sign, one can assume $\Phi_1 \geq 0$ on V. Second, since V is connected the killed semigroup $P_t^{D,V}$ is positivity improving. That is, its kernel $p_t^{D,V}(y,z)$ is strictly positive for all $t > 0$ and $y, z \in V$. Therefore the eigenfunction relation

$$\Phi_1 = e^{\lambda_1(V)t}\, P_t^{D,V} \Phi_1 \tag{5.16}$$

yields $\Phi_1(y) > 0$ for every $y \in V$. Indeed, since $\Phi_1 \geq 0$ and $\|\Phi_1\|_{L^2(V,\mu)} = 1$, the set $A := \{z \in V : \Phi_1(z) > 0\}$ has positive measure. Thus writing (5.16) at a point $y \in V$ we get

$$\Phi_1(y) = e^{\lambda_1(V)t} \int_V p_t^{D,V}(y,z)\, \Phi_1(z)\, d\mu(z) \;\geq\; e^{\lambda_1(V)t} \int_A p_t^{D,V}(y,z)\, \Phi_1(z)\, d\mu(z) \;>\; 0,$$

where the strict inequality is due to the fact that the integrand is strictly positive on A and $\mu(A) > 0$. Summarizing, for a connected set V the ground state Φ_1 can always be chosen strictly positive on V. This property will be applied to small metric balls in the proof of Lemma 6.3.12 in Section 6.3.3.

The following corollaries of Theorem 5.1.15 will be used repeatedly.

Corollary 5.1.18: L^2 bound

Under the conditions of Theorem 5.1.15, there exists $C > 0$ such that for all $x \in U$ and $t > 0$,

$$\int_U p_t^{D,U}(x,y)^2\, d\mu(y) \;\leq\; C\, \frac{e^{-2\lambda_1 t}}{\min(1, t^{d_h/d_w})}. \tag{5.17}$$

Proof. By symmetry of $p_t^{D,U}$ (Theorem 5.1.15) and the semigroup property, we have

$$\int_U p_t^{D,U}(x,y)^2\, d\mu(y) = p_{2t}^{D,U}(x,x).$$

The bound then follows from (5.14). ■

Corollary 5.1.19: L^∞ bound

Under the same conditions, there exists $C > 0$ such that for every $t \geq 0$ and $u_0 \in L^\infty(U,\mu)$,

$$\|P_t^{D,U} u_0\|_\infty \;\leq\; C\, e^{-\lambda_1 t} \|u_0\|_\infty. \tag{5.18}$$

Proof. For $0 \leq t \leq 1$ use (5.12) to write $\|P_t^{D,U} u_0\|_\infty \leq \|u_0\|_\infty$. For $t > 1$, write $P_t^{D,U} u_0(x) = \int_U p_t^{D,U}(x,y)\, u_0(y)\, d\mu(y)$ and apply Cauchy–Schwarz:

$$|P_t^{D,U} u_0(x)| \;\leq\; \left(\int_U p_t^{D,U}(x,y)^2\, d\mu(y)\right)^{1/2} \left(\int_U u_0(y)^2\, d\mu(y)\right)^{1/2}. \tag{5.19}$$

Since $\mu(U) < \infty$ we have $\|u_0\|_{L^2(U)} \leq \mu(U)^{1/2} \|u_0\|_\infty$. For $t > 1$ we have $\min(1, t^{d_h/d_w}) = 1$, so (5.17) gives $\int_U p_t^{D,U}(x,y)^2\, d\mu(y) \leq C\, e^{-2\lambda_1 t}$. Substituting into (5.19) and taking the supremum over $x \in U$ yields (5.18) with constant $C^{1/2} \mu(U)^{1/2}$. ■

5.1.2.2 Fractional Laplacian

With the heat-kernel bounds of the previous subsection in hand, we turn to the *fractional powers* $(-\Delta_{D,U})^{-\alpha}$ of the Dirichlet Laplacian and the associated function spaces. These operators are defined by spectral calculus. We will see that this type of operator admits a Riesz kernel $G_\alpha^{D,U}$, obtained as a Laplace-type transform of the heat kernel $p_t^{D,U}$ (Lemma 5.1.25), and we derive sharp two-sided bounds on $G_\alpha^{D,U}$. These bounds yield the L^2-estimate of Lemma 5.1.27 and motivate the Dirichlet Sobolev space $\mathcal{W}_D^{-\alpha}(U)$. Those objects are of prime importance for us, since they provide the natural function-space setting for the noise in the PAM on U.

Notation 5.1.20: Dirichlet spectrum

The operator $-\Delta_{D,U}$ admits a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ together with an orthonormal basis $\{\Phi_j\}_{j\geq 1}$ of $L^2(U,\mu)$ such that $\Delta_{D,U}\Phi_j = -\lambda_j \Phi_j$ for each $j \geq 1$.

With the spectrum fixed by Notation 5.1.20, we record the spectral expansion of the heat kernel and a summability condition on the eigenvalues; both will be needed to show that $(-\Delta_{D,U})^{-\alpha}$ is a bounded operator on $L^2(U,\mu)$.

Lemma 5.1.21: Spectral expansion and eigenvalue series

Under Notation 5.1.20, the Dirichlet heat kernel admits the uniformly convergent expansion

$$p_t^{D,U}(x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\Phi_j(x)\Phi_j(y), \qquad t > 0,\ x,y \in U. \tag{5.20}$$

Moreover, for every $\alpha > d_h/d_w$,

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^\alpha} < +\infty. \tag{5.21}$$

Proof. The expansion (5.20) is classical; see [4, Thm. 3.44]. The series (5.21) follows from diagonal heat kernel estimates and Tauberian theorems [29]. ■

The spectral expansion (5.20) suggests a natural way to define fractional powers of $-\Delta_{D,U}$: replace the heat-semigroup weights $e^{-\lambda_j t}$ by $\lambda_j^{-\alpha}$ in the eigenfunction series.

Definition 5.1.22: Dirichlet fractional Laplacian

Working under Notation 5.1.20, for $\alpha \geq 0$ and $f \in L^2(U,\mu)$ define

$$(-\Delta_{D,U})^{-\alpha} f(x) := \sum_{j=1}^{\infty} \frac{1}{\lambda_j^\alpha}\left(\int_U \Phi_j(y) f(y)\, d\mu(y)\right)\Phi_j(x). \tag{5.22}$$

The formula (5.22) defines an L^2 series; the following proposition confirms that it converges and yields a bounded operator.

Proposition 5.1.23: L^2 boundedness of $(-\Delta_{D,U})^{-\alpha}$

Under Notation 5.1.20, for every $\alpha \geq 0$ the operator $(-\Delta_{D,U})^{-\alpha}$ defined in (5.22) is bounded on $L^2(U, \mu)$, with

$$\|(-\Delta_{D,U})^{-\alpha}\|_{L^2(U,\mu)\to L^2(U,\mu)} \leq \lambda_1^{-\alpha}. \tag{5.23}$$

Proof. Set $\hat{f}_j := \int_U \Phi_j(y) f(y)\, d\mu(y)$ for each $j \geq 1$. Since $\{\Phi_j\}$ is an orthonormal basis of $L^2(U, \mu)$, Parseval's identity gives $\sum_{j=1}^{\infty} |\hat{f}_j|^2 = \|f\|^2_{L^2(U,\mu)}$ and

$$\|(-\Delta_{D,U})^{-\alpha} f\|^2_{L^2(U,\mu)} = \sum_{j=1}^{\infty} \lambda_j^{-2\alpha} |\hat{f}_j|^2.$$

Since $\lambda_j \geq \lambda_1 > 0$ for all $j \geq 1$, we have $\lambda_j^{-\alpha} \leq \lambda_1^{-\alpha}$, and therefore

$$\|(-\Delta_{D,U})^{-\alpha} f\|^2_{L^2(U,\mu)} = \sum_{j=1}^{\infty} \lambda_j^{-2\alpha} |\hat{f}_j|^2 \;\leq\; \lambda_1^{-2\alpha} \sum_{j=1}^{\infty} |\hat{f}_j|^2 = \lambda_1^{-2\alpha} \|f\|^2_{L^2(U,\mu)}.$$

Taking square roots gives (5.23). ■

To identify a kernel for $(-\Delta_{D,U})^{-\alpha}$, we introduce a class of smooth test functions.

Definition 5.1.24: Dirichlet–Schwartz space

$$\mathcal{S}_D(U) := \left\{ f \in L^2(U, \mu) : \forall\, k \in \mathbb{N},\ \lim_{n\to\infty} n^k \left| \int_U \Phi_n(y) f(y)\, d\mu(y) \right| = 0 \right\}.$$

On $\mathcal{S}_D(U)$, the spectral definition (5.22) can be rewritten as convolution against an explicit kernel: the identity $\lambda_j^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda_j t}\, dt$ allows one to exchange the sum over j with the time integral and recover the heat kernel via (5.20).

Lemma 5.1.25: Riesz kernel

For $\alpha > 0$ the operator $(-\Delta_{D,U})^{-\alpha}$ admits a kernel $G_\alpha^{D,U}$ given by

$$G_\alpha^{D,U}(x,y) := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} p_t^{D,U}(x,y)\,dt, \qquad x,y \in U,\ x \neq y, \tag{5.24}$$

such that for $f \in \mathcal{S}_D(U)$ and $y \in U$ we have

$$(-\Delta_{D,U})^{-\alpha} f(y) = \int_U f(x)\, G_\alpha^{D,U}(x,y)\, d\mu(x), \tag{5.25}$$

and for $f, g \in \mathcal{S}_D(U)$ it holds

$$\int_U (-\Delta_{D,U})^{-\alpha} f \cdot (-\Delta_{D,U})^{-\alpha} g \, d\mu = \int_U \int_U f(x) g(y)\, G_{2\alpha}^{D,U}(x,y)\, d\mu(x)\, d\mu(y). \tag{5.26}$$

Proof. See [6, Lem. 2.9 and Cor. 2.11] for full details. Here we will just spell out the main steps.

Step 1: Convergence of (5.24). For $x \neq y$, the sub-Gaussian upper bound (5.14) implies that $p_t^{D,U}(x,y)$ decays faster than any power of t as $t \to 0^+$, while the spectral expansion (5.20) gives $p_t^{D,U}(x,y) \leq Ce^{-\lambda_1 t}$ for $t \geq 1$. Hence the integrand $t^{\alpha-1} p_t^{D,U}(x,y)$ is integrable on $(0,+\infty)$ for every $\alpha > 0$, and (5.24) is well defined.

Step 2: Proof of (5.25). Set $\hat{f}_j := \int_U \Phi_j(y) f(y)\, d\mu(y)$. Since $f \in \mathcal{S}_D(U)$ is bounded, the sub-probability bound (5.12) for $t \in (0,1]$ and the exponential decay of (5.14) for $t > 1$ give

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} \int_U |f(x)|\, p_t^{D,U}(x,y)\, d\mu(x)\, dt < +\infty,$$

Therefore applying Fubini's theorem to $G_\alpha^{D,U}$ defined by (5.24), we get

$$\int_U f(x)\, G_\alpha^{D,U}(x,y)\, d\mu(x) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} P_t^{D,U} f(y)\, dt.$$

Moreover substituting the spectral expansion (5.20), one can write

$$P_t^{D,U} f(y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \hat{f}_j \Phi_j(y).$$

Hence interchanging sum and time integral (the rapid decay $|\hat{f}_j| = o(j^{-k})$ for every $k \geq 1$, guaranteed by Definition 5.1.24, makes this interchange valid) then yield

$$\frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} P_t^{D,U} f(y)\, dt = \sum_{j=1}^{\infty} \hat{f}_j \Phi_j(y) \cdot \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-\lambda_j t}\, dt = \sum_{j=1}^{\infty} \frac{\hat{f}_j}{\lambda_j^\alpha}\, \Phi_j(y),$$

where the second equality uses $\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\lambda t}\, dt = \lambda^{-\alpha}$. By (5.22) the right-hand side equals $(-\Delta_{D,U})^{-\alpha} f(y)$, which gives (5.25).

Step 3: Proof of (5.26). Let $\hat{g}_j := \int_U \Phi_j(y) g(y)\, d\mu(y)$. By Parseval's identity applied twice,

$$\int_U (-\Delta_{D,U})^{-\alpha} f \cdot (-\Delta_{D,U})^{-\alpha} g \, d\mu = \sum_{j=1}^{\infty} \frac{\hat{f}_j \, \hat{g}_j}{\lambda_j^{2\alpha}} = \int_U f \cdot (-\Delta_{D,U})^{-2\alpha} g \, d\mu.$$

Applying (5.25) with 2α in place of α to $(-\Delta_{D,U})^{-2\alpha} g$ then gives (5.26). ■

The following proposition gives sharp two-sided bounds on $G_\alpha^{D,U}$ in three regimes.

Proposition 5.1.26: Bounds on $G_\alpha^{D,U}$

Let $U \subset X$ be an inner uniform domain and $G_\alpha^{D,U}$ be as in (5.24). Then:

(i) If $0 < \alpha < d_h/d_w$: there exist $c, C > 0$ such that for all $x, y \in U$, $x \neq y$,

$$c\, \frac{\Phi_1(x)\Phi_1(y)}{d_U(x,y)^{d_h - \alpha d_w}} \;\le\; G_\alpha^{D,U}(x,y) \;\le\; \frac{C}{d(x,y)^{d_h - \alpha d_w}}. \tag{5.27}$$

(ii) If $\alpha = d_h/d_w$: there exist $c, C > 0$ such that for all $x, y \in U$, $x \neq y$,

$$c\, \Phi_1(x)\Phi_1(y) \max\Big(1, |\ln d_U(x,y)|\Big) \;\le\; G_\alpha^{D,U}(x,y) \;\le\; C \max\Big(1, |\ln d(x,y)|\Big). \tag{5.28}$$

(iii) If $\alpha > d_h/d_w$: there exist $c, C > 0$ such that for all $x, y \in U$,

$$c\, \Phi_1(x)\Phi_1(y) \;\le\; G_\alpha^{D,U}(x,y) \;\le\; C. \tag{5.29}$$

Proof. We give the main computations; see [7, Prop. 2.6] for complete details.

Step 1: Reduction of the upper bounds to $t \in (0,1]$. Split (5.24) at $t = 1$. For $t \ge 1$, the bound (5.14) gives $p_t^{D,U}(x,y) \le c_1 e^{-\lambda_1 t}$. Therefore we have

$$\frac{1}{\Gamma(\alpha)} \int_1^{+\infty} t^{\alpha-1} p_t^{D,U}(x,y)\, dt \;\le\; \frac{c_1}{\Gamma(\alpha)} \int_1^{+\infty} t^{\alpha-1} e^{-\lambda_1 t}\, dt \;=:\; C_\alpha \;<\; +\infty.$$

For $t \le 1$, we have $\min(1, t^{d_h/d_w}) = t^{d_h/d_w}$, so (5.14) gives

$$\frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} p_t^{D,U}(x,y)\, dt \;\le\; \frac{c_1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1-d_h/d_w} \exp\left(-c_2 \left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) dt.$$

The substitution $t = u\, d(x,y)^{d_w}$ transforms the right-hand side into

$$\frac{c_1}{\Gamma(\alpha)}\, d(x,y)^{\alpha d_w - d_h} \int_0^{d(x,y)^{-d_w}} u^{\alpha-1-d_h/d_w} \exp\left(-c_2 \left(\frac{1}{u}\right)^{\frac{1}{d_w-1}}\right) du. \tag{5.30}$$

Step 2: Upper bounds in the three regimes.

In case (i), $\alpha < d_h/d_w$: extend the upper limit in (5.30) to $+\infty$. The resulting integral is finite because the exponential factor controls $u \to 0$ and the power $u^{\alpha-1-d_h/d_w}$ is integrable at $+\infty$ (its exponent equals $-1-(d_h/d_w-\alpha) < -1$). This gives $G_\alpha^{D,U}(x,y) \le C\, d(x,y)^{\alpha d_w - d_h} = C/d(x,y)^{d_h-\alpha d_w}$.

In case (iii), $\alpha > d_h/d_w$: drop the exponential in the t-integral (it is ≤ 1) to get

$$\int_0^1 t^{\alpha-1-d_h/d_w}\, dt \;=\; \frac{d_w}{\alpha d_w - d_h} \;<\; +\infty,$$

which yields $G_\alpha^{D,U}(x,y) \le C$.

In case (ii), $\alpha = d_h/d_w$: the integrand in (5.30) reduces to $u^{-1}\exp(-c_2(1/u)^{1/(d_w-1)})$. For $u \in (0, R^{-d_w})$ (where $R = \operatorname{diam}(U)$) the super-exponential decay contributes a finite constant. For $u \in (R^{-d_w}, d(x,y)^{-d_w})$ bound the exponential by 1 and integrate:

$$\int_{R^{-d_w}}^{d(x,y)^{-d_w}} u^{-1}\, du \;=\; d_w |\ln d(x,y)| + C,$$

giving $G_\alpha^{D,U}(x,y) \le C \max(1, |\ln d(x,y)|)$.

Step 3: Lower bounds. Inserting (5.15) into (5.24) gives

$$G_\alpha^{D,U}(x,y) \;\ge\; \frac{c_3\, \Phi_1(x)\Phi_1(y)}{\Gamma(\alpha)} \int_0^{+\infty} \frac{t^{\alpha-1}\, e^{-\lambda_1 t}}{\min(1, t^{d_h/d_w})} \exp\left(-c_4\Big(\frac{d_U(x,y)^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\right) dt. \quad (5.31)$$

For case (ii) and for small $d_U(x,y)$ in case (i) we apply the substitution $r = t/d_U(x,y)^{d_w}$ to (5.31); for case (iii) and for large $d_U(x,y)$ in case (i) we restrict directly to $t \ge 1$, where $\min(1, t^{d_h/d_w}) = 1$. More precisely, one can argue as follows:

In case (i): restrict to $t \in (0,1]$ and substitute $r = t/d_U(x,y)^{d_w}$ to get

$$\begin{aligned} G_\alpha^{D,U}(x,y) \;\ge\; & \frac{c_3 \Phi_1(x)\Phi_1(y)}{\Gamma(\alpha)}\, d_U(x,y)^{\alpha d_w - d_h} \\ & \times \int_0^{d_U(x,y)^{-d_w}} r^{\alpha-1-d_h/d_w} e^{-\lambda_1 r\, d_U(x,y)^{d_w}} \exp\left(-c_4\Big(\frac{1}{r}\Big)^{\frac{1}{d_w-1}}\right) dr. \quad (5.32) \end{aligned}$$

Note that for all $r \ge 1$,

$$\exp\left(-c_4\Big(\frac{1}{r}\Big)^{\frac{1}{d_w-1}}\right) \ge e^{-c_4} > 0. \quad (5.33)$$

Since the integrand is non-negative, we may restrict the integration to any subinterval of $(0, d_U(x,y)^{-d_w}]$. For $d_U(x,y) \le 2^{-1/d_w}$ the interval $[1,2]$ lies in $(0, d_U(x,y)^{-d_w}]$, and on $[1,2]$ each factor in (5.32) is bounded below by a positive constant: $r^{\alpha-1-d_h/d_w}$ is bounded away from 0 on a compact interval; $\exp(-c_4(1/r)^{1/(d_w-1)}) \ge e^{-c_4} > 0$ by (5.33); and $e^{-\lambda_1 r\, d_U(x,y)^{d_w}} \ge e^{-2\lambda_1 d_U(x,y)^{d_w}}$, which tends to 1 as $d_U(x,y) \to 0$. Hence the integral over $[1,2]$ is bounded below by a positive constant, giving the lower bound in (5.27) for small $d_U(x,y)$. As mentioned above, for $d_U(x,y) \ge 2^{-1/d_w}$ we don't use the above substitution and we argue directly from (5.31). Namely, since $d_U(x,y) \geq 2^{-1/d_w}$, the lower-bound side of (5.27) satisfies

$$\frac{c\,\Phi_1(x)\Phi_1(y)}{d_U(x,y)^{d_h-\alpha d_w}} \leq c\,2^{(d_h-\alpha d_w)/d_w}\Phi_1(x)\Phi_1(y),$$

so it suffices to show $G_\alpha^{D,U}(x,y) \geq c'\Phi_1(x)\Phi_1(y)$ for some $c' > 0$. This follows from (5.31) restricted to $t \geq 1$: for $t \geq 1$ we have $\min(1, t^{d_h/d_w}) = 1$, and since $d_U(x,y) \leq \operatorname{diam}(U)$,

$$\left(\frac{d_U(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}} \leq \operatorname{diam}(U)^{\frac{d_w}{d_w-1}}, \qquad t \geq 1, \tag{5.34}$$

so the exponential in (5.31) satisfies $\exp(-c_4(\cdots)) \geq c_5 > 0$ uniformly. Gathering this into (5.31) gives

$$G_\alpha^{D,U}(x,y) \geq c'\Phi_1(x)\Phi_1(y)\int_1^{+\infty} t^{\alpha-1}e^{-\lambda_1 t}\,dt > 0,$$

which is the desired inequality. This concludes the lower bound for case (i).

In case (iii): restrict the integration in (5.31) to $t \geq 1$. For $t \geq 1$ we have $\min(1, t^{d_h/d_w}) = 1$, and (5.34) gives $\exp(-c_4(\cdots)) \geq c_5 > 0$ uniformly. Hence

$$G_\alpha^{D,U}(x,y) \;\geq\; c\,\Phi_1(x)\Phi_1(y)\int_1^{+\infty} t^{\alpha-1}e^{-\lambda_1 t}\,dt \;=\; c'\Phi_1(x)\Phi_1(y).$$

In case (ii): since $\alpha = d_h/d_w$, for $t \in (0,1]$ we have $\min(1, t^{d_h/d_w}) = t^{d_h/d_w}$, so the integrand in (5.31) reduces to $t^{-1}e^{-\lambda_1 t}\exp(-c_4(d_U(x,y)^{d_w}/t)^{1/(d_w-1)})$. Restricting (5.31) to $t \in (0,1]$ and applying the substitution $r = t/d_U(x,y)^{d_w}$ gives

$$G_\alpha^{D,U}(x,y) \;\geq\; \frac{c_3\,\Phi_1(x)\Phi_1(y)}{\Gamma(d_h/d_w)}\int_0^{d_U(x,y)^{-d_w}} r^{-1}\,e^{-\lambda_1 r\,d_U(x,y)^{d_w}}\,\exp\left(-c_4\left(\frac{1}{r}\right)^{\frac{1}{d_w-1}}\right)dr. \tag{5.35}$$

Restricting to $r \in [1, d_U(x,y)^{-d_w}]$ in (5.35) and using (5.33) together with $e^{-\lambda_1 r d_U(x,y)^{d_w}} \geq e^{-\lambda_1} > 0$ (since $r d_U(x,y)^{d_w} \leq 1$ on this range), one obtains

$$G_\alpha^{D,U}(x,y) \;\geq\; c\,\Phi_1(x)\Phi_1(y)\int_1^{d_U(x,y)^{-d_w}} r^{-1}\,dr \;=\; c\,\Phi_1(x)\Phi_1(y)\,d_w|\ln d_U(x,y)|,$$

which gives the lower bound in (5.28).

Combining the upper bounds of Steps 1–2 (obtained via (5.30) in cases (i)–(ii) and by dropping the exponential in case (iii)) with the lower bounds of Step 3 (obtained via (5.32) and (5.34) in case (i), via (5.35) in case (ii), and via (5.34) in case (iii)) establishes (5.27), (5.28), and (5.29) respectively, and completes the proof of Proposition 5.1.26. ∎

The next lemma controls the L^2-norm of $(-\Delta_{D,U})^{-\alpha}p_t^{D,U}(x,\cdot)$, which is one of the key estimates for the existence theory of the PAM on U.

Lemma 5.1.27

Let $\alpha \geq 0$. There exists $C > 0$ such that for every $t > 0$ and $x \in U$,

$$\sup_{x\in U} \|(-\Delta_{D,U})^{-\alpha} p_t^{D,U}(x,\cdot)\|_{L^2(U,\mu)} \leq \begin{cases} C\, e^{-\lambda_1 t}(1 + t^{\alpha - d_h/(2d_w)}), & 0 \leq \alpha < \frac{d_h}{2d_w}, \\ C\, e^{-\lambda_1 t}|\ln\min(\frac{1}{2}, t)|, & \alpha = \frac{d_h}{2d_w}, \\ C\, e^{-\lambda_1 t}, & \alpha > \frac{d_h}{2d_w}. \end{cases} \tag{5.36}$$

Proof. Using (5.24) and the semigroup property one writes

$$(-\Delta_{D,U})^{-\alpha} p_t^{D,U}(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} p_{s+t}^{D,U}(x,y)\, ds, \qquad t > 0,\ x, y \in U. \tag{5.37}$$

Minkowski's inequality then reduces the estimate to $\int_0^\infty s^{\alpha-1}\|p_{s+t}^{D,U}(x,\cdot)\|_{L^2}\, ds$, which is bounded using (5.17) together with elementary calculus. ■

The kernel bounds of Proposition 5.1.26 and the L^2-estimate of Lemma 5.1.27 together suggest a natural Hilbert space of distributions on U: its inner product is given by the Riesz kernel $G_{2\alpha}^{D,U}$, or equivalently, by pairing through $(-\Delta_{D,U})^{-\alpha}$ in $L^2(U,\mu)$. This space provides the natural function-space setting for the noise in the PAM on U.

Definition 5.1.28: Dirichlet Sobolev space

For $\alpha > 0$, the *Dirichlet Sobolev space* $\mathcal{W}_D^{-\alpha}(U)$ is the completion of $\mathcal{S}_D(U)$ with respect to the inner product

$$\begin{aligned} \langle f, g\rangle_{\mathcal{W}_D^{-\alpha}(U)} &:= \int_U\int_U f(x)g(y)\, G_{2\alpha}^{D,U}(x,y)\, d\mu(x)\, d\mu(y) \\ &= \int_U (-\Delta_{D,U})^{-\alpha} f \cdot (-\Delta_{D,U})^{-\alpha} g\, d\mu, \end{aligned} \tag{5.38}$$

where the second equality uses (5.26).

5.1.3 Neumann boundary conditions

The Neumann setting mirrors the Dirichlet one, with two key differences: the domain of the form is the full $\mathcal{F}(U)$ (no vanishing condition at the boundary), and one requires the slightly stronger notion of a *uniform domain* to ensure that the resulting Dirichlet form is regular.

5.1.3.1 Heat semigroup

The Neumann form is defined by the same integral expression as the Dirichlet form (5.11), but on the larger function space $\mathcal{F}(U)$ (Definition 5.1.11) rather than $\mathcal{F}^D(U)$: no vanishing condition is imposed at the boundary. At the probabilistic level, this corresponds to *reflected* rather than killed Brownian motion in U (see Theorem 5.1.32 below).

Definition 5.1.29: Neumann Dirichlet form

For a non-empty bounded open set $U \subset X$ with compact closure, the *Neumann Dirichlet form* is

$$\mathcal{E}_U^N(u,v) := \int_U d\Gamma(u,v), \qquad u,v \in \mathcal{F}(U), \tag{5.39}$$

where Γ is the energy measure (5.2).

For the Dirichlet form (5.11), the inner uniform condition of Definition 5.1.13 was sufficient to guarantee sharp heat kernel bounds: the two conditions on the curve γ were stated in terms of the inner metric d_U. For the Neumann form, the reflected process can reach the boundary, and a stronger geometric condition on U is required: the same two conditions must hold with the inner metric d_U replaced throughout by the ambient metric d of X.

Definition 5.1.30: Uniform domain

A connected bounded open set $U \subset X$ (with compact closure) is called a (c, C) *uniform domain* if any two points in U can be joined by a rectifiable curve $\gamma \subset U$ satisfying:

(i) $d(\gamma(t), \partial U) \geq c \min\{d(\gamma(\alpha), \gamma(t)), d(\gamma(t), \gamma(\beta))\}$ for all t;

(ii) $\operatorname{length}(\gamma) \leq C\, d(\gamma(\alpha), \gamma(\beta))$.

Every uniform domain is inner uniform (Definition 5.1.13) [30].

Example 5.1.31: Slit annulus: inner uniform but not uniform

Let $X = \mathbb{R}^2$ with the Euclidean metric, fix $0 < R_1 < R_2$, and set $\bar{r} := (R_1 + R_2)/2$. In polar coordinates (r, θ), define

$$U \;:=\; \Big\{(r,\theta) : R_1 < r < R_2,\ 0 < \theta < 2\pi\Big\},$$

i.e., the open annulus $\{R_1 < |x| < R_2\}$ with the positive real axis removed (Figure 5.2). Then U is inner uniform but *not* uniform.

Proof. U is not uniform. Let $A = (\bar{r}\cos\varepsilon,\, \bar{r}\sin\varepsilon)$ and $B = (\bar{r}\cos\varepsilon,\, -\bar{r}\sin\varepsilon)$ for small $\varepsilon > 0$. Their Euclidean distance is $d(A, B) = 2\bar{r}\sin\varepsilon \to 0$ as $\varepsilon \to 0$. Since the slit

$\{(x, 0) : x > 0\}$ blocks any direct crossing, every path $\gamma \subset U$ from A to B must travel almost all the way around the annulus, so $\mathrm{length}(\gamma) \geq 2\pi\bar{r}(1 - \varepsilon/\pi)$. No fixed constant C can satisfy $\mathrm{length}(\gamma) \leq C\, d(A, B)$ for all $\varepsilon > 0$, and condition (ii) of Definition 5.1.30 fails.

U is inner uniform. In the completion $\widetilde{U}$ under d_U, the two sides of the slit are distinct boundary components. For any $x = (r, \theta) \in U$ the inner distance to $\widetilde{U} \setminus U$ satisfies

$$d_U(x, \widetilde{U} \setminus U) \;\geq\; \min\Big(r - R_1,\; R_2 - r,\; r\theta,\; r(2\pi - \theta)\Big), \tag{5.40}$$

since the nearest boundary component is either one of the two circles or one of the two sides of the slit (reached by an arc of length $r\theta$ or $r(2\pi - \theta)$ respectively). We verify condition (i) of Definition 5.1.13 for the representative pair $A = (\bar{r}, \varepsilon)$, $B = (\bar{r}, 2\pi - \varepsilon)$ above; the general case is analogous. The inner geodesic is the arc $\gamma(t) = (\bar{r}, \theta(t))$ with $\theta(t) = \varepsilon + (2\pi - 2\varepsilon)t$, $t \in [0, 1]$. Since $\bar{r} - R_1 = R_2 - \bar{r} = (R_2 - R_1)/2$, (5.40) gives

$$d_U(\gamma(t), \widetilde{U} \setminus U) \;\geq\; \min\Big(\tfrac{R_2 - R_1}{2},\; \bar{r}\,\theta(t),\; \bar{r}(2\pi - \theta(t))\Big).$$

The inner arc-lengths from the endpoints are $d_U(A, \gamma(t)) = \bar{r}(\theta(t) - \varepsilon)$ and $d_U(\gamma(t), B) = \bar{r}(2\pi - \varepsilon - \theta(t))$. We consider three cases.

- $\bar{r}\,\theta(t) \leq (R_2 - R_1)/2$ (near A, close to the slit): then $d_U(\gamma(t), \widetilde{U} \setminus U) \geq \bar{r}\,\theta(t) \geq \bar{r}(\theta(t) - \varepsilon) = d_U(A, \gamma(t))$, so condition (i) holds with $c = 1$.
- $\bar{r}(2\pi - \theta(t)) \leq (R_2 - R_1)/2$ (near B, close to the slit): symmetric, condition (i) holds with $c = 1$.
- Both $\bar{r}\,\theta(t) > (R_2 - R_1)/2$ and $\bar{r}(2\pi - \theta(t)) > (R_2 - R_1)/2$ (middle of the arc): then $d_U(\gamma(t), \widetilde{U} \setminus U) \geq (R_2 - R_1)/2$, while $\min\{d_U(A, \gamma(t)), d_U(\gamma(t), B)\} \leq \bar{r}\pi$ (worst at the midpoint $\theta = \pi$). Condition (i) requires $(R_2 - R_1)/2 \geq c\bar{r}\pi$, i.e., $c \leq (R_2 - R_1)/(2\pi\bar{r})$. Since $\bar{r} \leq R_2$, taking $c = (R_2 - R_1)/(2\pi R_2)$ suffices.

This proves condition (i) for our case of study. Condition (ii) is trivially satisfied since γ is a geodesic. ■

The Neumann analogue of Theorem 5.1.15 is [47, Thm. 2.8]. It requires the stronger uniform condition of Definition 5.1.30, and the resulting bounds differ from the Dirichlet case in three respects. First, there is no factor $e^{-\lambda_1 t}$: the Neumann Laplacian has 0 as an eigenvalue (the constant function), so the semigroup does not decay exponentially. Second, the lower bound carries no $\Phi_1(x)\Phi_1(y)$ factor, since the reflected Brownian motion does not vanish at the boundary. Third, the ambient metric $d(x, y)$ appears throughout (in both upper and lower bounds), whereas the Dirichlet lower bound (5.15) involves the inner metric d_U.

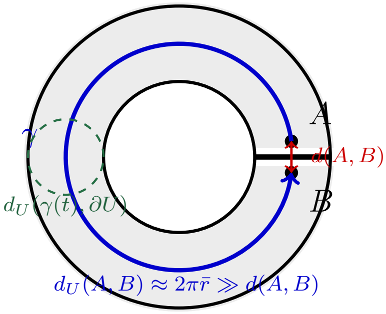


Figure 5.2 – The slit annulus U. Points A and B lie at radius $\bar{r}$ just above and below the slit; their Euclidean distance $d(A,B) = 2\bar{r}\sin\varepsilon$ is small, but every path $\gamma \subset U$ must travel almost all the way around (inner geodesic in blue), so $d_U(A,B) \approx 2\pi\bar{r} \gg d(A,B)$. The dashed circle illustrates that the midpoint of γ stays at inner distance $\approx (R_2 - R_1)/2$ from the boundary, which is proportional to the shorter arc-length to each endpoint, confirming condition (i) of Definition 5.1.13. Condition (ii) of Definition 5.1.30 fails because no fixed C satisfies $\text{length}(\gamma) \le C\, d(A,B)$ for all $\varepsilon > 0$.

Theorem 5.1.32

Let $U \subset X$ be a uniform domain. Then $(\mathcal{E}_U^N, \mathcal{F}(U))$ is a strongly local regular Dirichlet form with a continuous heat kernel $p_t^{N,U}(x,y)$ satisfying, for all $(x,y) \in U \times U$ and $t > 0$,

$$p_t^{N,U}(x,y) \;\le\; \frac{c_1}{\min(1, t^{d_h/d_w})} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right), \tag{5.41}$$

$$p_t^{N,U}(x,y) \;\ge\; \frac{c_3}{\min(1, t^{d_h/d_w})} \exp\left(-c_4\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right). \tag{5.42}$$

The associated diffusion process is the *reflected Brownian motion* in U.

The following L^2 bound is the Neumann analogue of Corollary 5.1.18. The proof is the same: symmetry of $p_t^{N,U}$ and the semigroup property give $\int_U p_t^{N,U}(x,y)^2\, d\mu(y) = p_{2t}^{N,U}(x,x)$, and the bound follows from (5.41) with $d(x,x) = 0$. Compared with (5.17), the only difference is the absence of the factor $e^{-2\lambda_1 t}$: since the Neumann spectrum contains 0, there is no exponential decay.

Corollary 5.1.33

Under the conditions of Theorem 5.1.32, there exists $C > 0$ such that

$$\int_U p_t^{N,U}(x,y)^2\, d\mu(y) \;\le\; C \min\!\Big(t^{-d_h/d_w}, 1\Big), \qquad t > 0,\ x \in U. \tag{5.43}$$

5.1.3.2 Fractional Laplacian

The spectral theory of the Neumann Laplacian $-\Delta_{N,U}$ (the generator of $(\mathcal{E}_U^N, \mathcal{F}(U))$) parallels Notation 5.1.20 and Lemma 5.1.21, with one key difference: constant functions belong to $\mathcal{F}(U)$ and have zero energy ($\mathcal{E}_U^N(c, v) = \int_U d\Gamma(c, v) = 0$ for every $v \in \mathcal{F}(U)$), so 0 is an eigenvalue. As a result, the heat kernel expansion acquires an additional constant term $1/\mu(U)$, reflecting the fact that $p_t^{N,U}(x, \cdot) \to 1/\mu(U)$ in L^2 as $t \to \infty$ (the reflected Brownian motion equidistributes over U).

Proposition 5.1.34: Neumann spectrum

Consider a uniform domain U. The operator $-\Delta_{N,U}$ has a discrete spectrum $0 = \nu_0 < \nu_1 \leq \nu_2 \leq \cdots$ with corresponding eigenfunctions: $\Psi_0 = \mu(U)^{-1/2}$ (constant), and $\{\Psi_j\}_{j \geq 1}$ an orthonormal basis of $L_0^2(U, \mu) := \{f \in L^2(U, \mu) : \int_U f \, d\mu = 0\}$. The Neumann heat kernel admits the expansion

$$p_t^{N,U}(x, y) = \frac{1}{\mu(U)} + \sum_{j=1}^{\infty} e^{-\nu_j t} \Psi_j(x) \Psi_j(y), \qquad t > 0, \; x, y \in U. \tag{5.44}$$

Moreover, $\sum_{j=1}^{\infty} \nu_j^{-\alpha} < +\infty$ for every $\alpha > d_h/d_w$.

The fractional Neumann Laplacian is defined by the same spectral formula as Definition 5.1.22, with two modifications forced by $\nu_0 = 0$. First, the sum must start at $j = 1$. Second, since Ψ_0 is a constant, $\langle f, \Psi_0 \rangle_{L^2} \Psi_0$ is the mean of f, and the sum over $j \geq 1$ automatically projects f onto its mean-zero part; accordingly, $(-\Delta_{N,U})^{-\alpha}$ maps $L^2(U, \mu)$ into $L_0^2(U, \mu)$.

Definition 5.1.35: Neumann fractional Laplacian

For $\alpha > 0$ and $f \in L^2(U, \mu)$,

$$(-\Delta_{N,U})^{-\alpha} f := \sum_{j=1}^{\infty} \frac{1}{\nu_j^{\alpha}} \Psi_j \int_U \Psi_j(y) f(y) \, d\mu(y). \tag{5.45}$$

This is a bounded operator $L^2(U, \mu) \to L_0^2(U, \mu)$ with norm at most $\nu_1^{-\alpha}$.

Defining the Neumann–Schwartz space $\mathcal{S}_N(U)$ analogously to Definition 5.1.24 (replacing Φ_n by Ψ_n), the operator $(-\Delta_{N,U})^{-\alpha}$ has a kernel.

Lemma 5.1.36: Neumann Riesz kernel

For $\alpha > 0$, $(-\Delta_{N,U})^{-\alpha}$ admits the kernel

$$G_\alpha^{N,U}(x,y) := \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1}\left(p_t^{N,U}(x,y) - \frac{1}{\mu(U)}\right) dt, \qquad x,y \in U,\ x \neq y, \tag{5.46}$$

with the representation $(-\Delta_{N,U})^{-\alpha} f(y) = \int_U f(x) G_\alpha^{N,U}(x,y)\, d\mu(x)$ for $f \in \mathcal{S}_N(U)$, and the bilinear formula analogous to (5.26).

Proof. The argument follows the same three steps as the proof of Lemma 5.1.25; see [7] for full details. We highlight the differences due to the zero eigenvalue.

Step 1: Convergence of (5.46). For small t, the sub-Gaussian upper bound (5.41) controls $p_t^{N,U}(x,y) - 1/\mu(U)$ exactly as (5.14) controls $p_t^{D,U}(x,y)$ in the Dirichlet case. For large t, the key difference appears: unlike $p_t^{D,U}$, the Neumann heat kernel does *not* decay to 0 as $t \to \infty$ (it converges to $1/\mu(U)$), so $t^{\alpha-1} p_t^{N,U}(x,y)$ would not be integrable at $+\infty$. The subtraction of $1/\mu(U)$ restores integrability: by (5.44),

$$p_t^{N,U}(x,y) - \frac{1}{\mu(U)} = \sum_{j=1}^{\infty} e^{-\nu_j t} \Psi_j(x) \Psi_j(y),$$

which decays like $e^{-\nu_1 t}$ as $t \to \infty$, making $t^{\alpha-1}(p_t^{N,U}(x,y) - 1/\mu(U))$ integrable on $(1,+\infty)$ for any $\alpha > 0$.

Step 2: Kernel representation. Set $\hat{f}_j := \int_U \Psi_j(y) f(y)\, d\mu(y)$. For $f \in \mathcal{S}_N(U)$ the same Fubini argument as in Step 2 of Lemma 5.1.25 gives

$$\int_U f(x)\, G_\alpha^{N,U}(x,y)\, d\mu(x) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1}\left(P_t^{N,U} f(y) - \frac{1}{\mu(U)} \int_U f\, d\mu\right) dt.$$

By (5.44), since the $j = 0$ terms cancel, it is readily checked that

$$P_t^{N,U} f(y) - \frac{1}{\mu(U)} \int_U f\, d\mu = \sum_{j=1}^{\infty} e^{-\nu_j t} \hat{f}_j \Psi_j(y)$$

Interchanging sum and integral via the rapid decay of $\hat{f}_j$ (Definition of $\mathcal{S}_N(U)$) and applying the gamma identity $\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\nu t}\, dt = \nu^{-\alpha}$ to each $j \geq 1$ yields

$$\int_U f(x)\, G_\alpha^{N,U}(x,y)\, d\mu(x) = \sum_{j=1}^{\infty} \frac{\hat{f}_j}{\nu_j^\alpha} \Psi_j(y) = (-\Delta_{N,U})^{-\alpha} f(y).$$

Step 3: Bilinear formula. Identical to Step 3 of Lemma 5.1.25, with ν_j and Ψ_j in place of λ_j and Φ_j, and Parseval's identity applied to the sum over $j \geq 1$. ∎

The following proposition gives sharp two-sided bounds on $G_\alpha^{N,U}$ in the same three regimes as Proposition 5.1.26. The structure is parallel: the leading singularity near $x = y$ is again governed by $d(x,y)^{\alpha d_w - d_h}$, and the three regimes are still determined by the comparison of α with d_h/d_w. Two differences appear. First, the ambient metric $d(x,y)$ replaces the inner metric $d_U(x,y)$ throughout, reflecting the fact that the reflected Brownian motion can reach the boundary (see Theorem 5.1.32). Second, the lower bounds carry a negative additive constant $-C$: because $G_\alpha^{N,U}$ is defined via the mean-subtracted kernel $p_t^{N,U} - 1/\mu(U)$, which changes sign, the kernel is not pointwise non-negative, and no $\Phi_1(x)\Phi_1(y)$ factor appears in the lower bounds.

Proposition 5.1.37: Bounds on $G_\alpha^{N,U}$

Let U be a uniform domain and $G_\alpha^{N,U}$ as in (5.46). Then:

(i) If $0 < \alpha < d_h/d_w$: there exist $C_1, C_2, C_3 > 0$ such that for all $x, y \in U$, $x \neq y$,

$$C_1\, d(x,y)^{\alpha d_w - d_h} - C_2 \;\le\; G_\alpha^{N,U}(x,y) \;\le\; C_3\, d(x,y)^{\alpha d_w - d_h}. \tag{5.47}$$

(ii) If $\alpha = d_h/d_w$: there exist $C_1, C_2, C_3 > 0$ such that for all $x, y \in U$, $x \neq y$,

$$-C_1 \ln d(x,y) - C_2 \;\le\; G_\alpha^{N,U}(x,y) \;\le\; C_3 \max\Big(|\ln d(x,y)|, 1\Big). \tag{5.48}$$

(iii) If $\alpha > d_h/d_w$: there exist $C_1, C_2 > 0$ such that for all $x, y \in U$,

$$-C_1 \;\le\; G_\alpha^{N,U}(x,y) \;\le\; C_2. \tag{5.49}$$

Proof. The argument parallels the proof of Proposition 5.1.26; see [7, Prop. 2.6] for full details. We describe the main differences.

Step 1: Upper bounds. Split (5.46) at $t = 1$:

$$G_\alpha^{N,U}(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}\left(p_t^{N,U}(x,y) - \frac{1}{\mu(U)}\right) dt + \frac{1}{\Gamma(\alpha)} \int_1^{+\infty} t^{\alpha-1}\left(p_t^{N,U}(x,y) - \frac{1}{\mu(U)}\right) dt, \tag{5.50}$$

where we recall that $p_t^{N,U}$ is defined by (5.44). Hence for $t \ge 1$, the spectral expansion (5.44) gives $|p_t^{N,U}(x,y) - 1/\mu(U)| \le Ce^{-\nu_1 t}$, so the second integral in (5.50) is bounded in absolute value by a constant C independent of x and y. Note that in the Dirichlet case, it is $p_t^{D,U}(x,y)$ itself that decays like $e^{-\lambda_1 t}$, via (5.14). Here the decay comes only after subtracting $1/\mu(U)$, via (5.44). Therefore, the large-t integral contributes only a bounded additive constant to $G_\alpha^{N,U}(x,y)$, uniformly in $(x,y) \in U \times U$; all singular dependence on the pair (x,y) comes from the integral over $[0,1]$.

Let us now handle the case $t \le 1$. In this case the term $-1/\mu(U)$ contributes

$-c_\alpha/\mu(U)$, a bounded constant, and the main contribution is the term

$$\frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}p_t^{N,U}(x,y)\,dt.$$

For this term we resort to (5.41) and the substitution $t = ud(x,y)^{d_w}$, with the *ambient* metric $d(x,y)$ in place of the inner metric $d_U(x,y)$ used in Proposition 5.1.26. This yields the upper bounds (5.47)–(5.49) by the same three-case analysis as Steps 1–2 of Proposition 5.1.26.

Step 2: Lower bounds. The large-t integral in (5.50) satisfies

$$\frac{1}{\Gamma(\alpha)}\int_1^\infty t^{\alpha-1}(p_t^{N,U} - 1/\mu(U))\,dt \geq -C,$$

by the same spectral estimate $|p_t^{N,U}(x,y) - 1/\mu(U)| \leq Ce^{-\nu_1 t}$ used in Step 1, and the small-t subtracted term contributes the bounded constant $-c_\alpha/\mu(U)$. Combining gives

$$G_\alpha^{N,U}(x,y) \;\geq\; \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}p_t^{N,U}(x,y)\,dt \;-\; C. \tag{5.51}$$

Inserting the lower bound (5.42) into (5.51) and applying the substitution $t = ud(x,y)^{d_w}$ produces the lower bounds in (5.47)–(5.49), with three differences from Step 3 of Proposition 5.1.26:

(i) The ambient metric $d(x,y)$ appears throughout, because (5.42) is stated in terms of d.

(ii) There is no factor $\Phi_1(x)\Phi_1(y)$, because the Neumann lower bound (5.42) requires no eigenfunction weight.

(iii) The additive $-C$ in (5.51) shifts all three lower bounds downward, producing the negative additive constants in (5.47)–(5.49). In particular, for case (iii) the contribution from $\int_0^1 t^{\alpha-1}p_t^{N,U}\,dt$ is bounded (not growing in $d(x,y)$), so the lower bound reduces to $G_\alpha^{N,U}(x,y) \geq -C_1$.

Combining Steps 1 and 2, the upper bounds in (5.47)–(5.49) follow from Step 1, and the lower bounds follow from Step 2 together with the three items above. This completes the proof of Proposition 5.1.37. ■

The kernel bounds of Proposition 5.1.37 likewise suggest a natural Hilbert space of distributions on U, paralleling Definition 5.1.28. There is one essential difference: because the lower bounds in (5.47)–(5.49) all carry a negative additive constant, $G_{2\alpha}^{N,U}$ is not pointwise non-negative, and the kernel alone does not define a positive inner product. Adding a sufficiently large constant C_U restores positivity; the resulting topology on $\mathcal{W}_N^{-\alpha}(U)$ is independent of the choice of C_U.

Definition 5.1.38: Neumann Sobolev space

For $\alpha > 0$ and a constant $C_U > 0$ chosen so that $G^{N,U}_{2\alpha}(x,y) + C_U \geq 0$, the *Neumann Sobolev space* $\mathcal{W}_N^{-\alpha}(U)$ is the completion of $\mathcal{S}_N(U)$ with respect to

$$\langle f, g\rangle_{\mathcal{W}_N^{-\alpha}(U)} := \int_U\int_U f(x)g(y)\Big(G^{N,U}_{2\alpha}(x,y) + C_U\Big)\, d\mu(x)\, d\mu(y). \tag{5.52}$$

The value of C_U does not affect the topology of $\mathcal{W}_N^{-\alpha}(U)$.

5.2 Examples

The abstract framework of Assumption 5.1.6 is broad enough to cover a wide variety of geometric structures, ranging from classical Euclidean domains to fractals with anomalous diffusion. We illustrate this with three families of examples. The first two (compact intervals and metric graphs) are classical in the sense that $d_h = 1$ and $d_w = 2$, so the heat kernel satisfies standard Gaussian estimates; they differ in their topology. The third example, the Sierpiński gasket, is genuinely fractal: the walk dimension $d_w = \ln 5/\ln 2 > 2$ reflects the anomalously slow diffusion caused by the fractal geometry, and the sub-Gaussian heat kernel estimates (5.6) are sharp. In each case we verify Assumption 5.1.6 and identify the dimensional parameters d_h and d_w that govern the kernel bounds of Propositions 5.1.26 and 5.1.37.

5.2.1 Compact intervals

The simplest example is $X = \mathbb{R}$ with Euclidean distance $d(x,y) = |x-y|$, Lebesgue measure $\mu = dx$, and the standard Dirichlet form $\mathcal{E}(f,g) = \int_{\mathbb{R}} f'g'\,dx$ on $\mathcal{F} = W^{1,2}(\mathbb{R})$. The heat kernel is the Gaussian kernel

$$p_t(x,y) = (4\pi t)^{-1/2} e^{-(x-y)^2/(4t)}$$

giving $d_h = 1$ and $d_w = 2$. The recurrence condition $d_h < d_w$ holds.

Let us now focus on compact sets, the prototypical example being $U = (0,1)$ (or any interval (a,b)). In this case one can separate two cases:

(i) *Dirichlet boundary conditions.* The form $(\mathcal{E}^D_U, \mathcal{F}^D(U))$ becomes

$$(\mathcal{E}^D, W^{1,2}_0([0,1])), \quad \text{where} \quad \mathcal{E}^D(f,g) = \int_0^1 f'g'\,dx.$$

Here the Sobolev space $W^{1,2}_0([0,1])$ is defined by

$$W^{1,2}_0([0,1]) := \{f \in L^2([0,1]) : f' \in L^2([0,1]),\ f(0) = f(1) = 0\}.$$

It is the concrete instance of the Dirichlet domain $\mathcal{F}^D(U)$ (see Proposition 5.1.12). By a classical Littlewood–Paley characterization of Sobolev spaces, the extension-by-zero map identifies $W_0^{1,2}([0,1])$ isometrically with the Besov space $B^1_{2,2}(\mathbb{R})$. That is in the notation of Definition 2.2.7, $B^1_{2,2}(\mathbb{R})$ coincides with $\mathcal{B}^1_{2,w_0}(\mathbb{R})$ with trivial weight $w_0 \equiv 1$. Moreover, the Dirichlet Laplacian is given by

$$\Delta_D = \frac{d^2}{dx^2}, \quad \text{with domain} \quad \{f \in W^{2,2}([0,1]) : f(0) = f(1) = 0\}.$$

Eventually, the heat kernel and eigenvalues are explicit:

$$p_t^D(x,y) = \frac{2}{\sqrt{\pi t}} \sum_{k\in\mathbb{Z}} \left(e^{-(x-y+2k)^2/(4t)} - e^{-(x+y+2k)^2/(4t)} \right),$$

and

$$\lambda_k = k^2\pi^2, \quad \text{with} \quad \Phi_k(x) = \sqrt{2}\sin(k\pi x).$$

(ii) *Neumann boundary conditions.* The form $(\mathcal{E}_U^N, \mathcal{F}(U))$ becomes

$$(\mathcal{E}^N, W^{1,2}([0,1])), \quad \text{where} \quad \mathcal{E}^N(f,g) = \int_0^1 f'g'\,dx.$$

Here the Sobolev space $W^{1,2}([0,1])$ is defined by

$$W^{1,2}([0,1]) := \{f \in L^2([0,1]) : f' \in L^2([0,1])\}.$$

It is the concrete instance of the Neumann domain $\mathcal{F}(U)$ (see Definition 5.1.29); unlike the Dirichlet case, no vanishing condition is imposed at the boundary. By restriction of $B^1_{2,2}(\mathbb{R})$ to $[0,1]$, one has $W^{1,2}([0,1]) = \mathcal{B}^1_{2,w_0}([0,1])$ with trivial weight $w_0 \equiv 1$ (Definition 2.2.7). Moreover, the Neumann Laplacian is given by

$$\Delta_N = \frac{d^2}{dx^2}, \quad \text{with domain} \quad \{f \in W^{2,2}([0,1]) : f'(0) = f'(1) = 0\}.$$

The heat kernel and eigenvalues are explicit:

$$p_t^N(x,y) = \frac{1}{\sqrt{\pi t}} \sum_{k\in\mathbb{Z}} \left(e^{-(x-y+2k)^2/(4t)} + e^{-(x+y+2k)^2/(4t)} \right),$$

and

$$\nu_k = k^2\pi^2, \quad \text{with} \quad \Psi_k(x) = \sqrt{2}\cos(k\pi x).$$

The PAM on $(0,1)$ with these boundary conditions has been studied in [20, 21].

5.2.2 Metric graphs

Metric graphs generalize the interval example of Section 5.2.1 by gluing finitely many one-dimensional edges and half-lines at a finite set of vertices. Each edge is a copy of a compact interval, so the local geometry remains Euclidean: the heat kernel exponents are $d_h = 1$ and $d_w = 2$, and the recurrence condition $d_h < d_w$ holds. The new ingredient is a *Kirchhoff vertex condition* at each junction, which encodes flux conservation and singles out the correct self-adjoint extension of the edge-wise Laplacian $-f''$.

Definition 5.2.1: Metric graph

A *metric graph* $\mathbf{G} = (V, \mathbf{E}, \mathbf{R})$ consists of:

(i) a finite set of vertices V;

(ii) a finite set of edges $\mathbf{E}$, each identified with a closed interval $I_e = [0, r(e)]$ of finite length $r(e) > 0$, with two endpoints $e^-, e^+ \in V$;

(iii) a finite set of rays $\mathbf{R}$, each identified with a half-line $[0, \infty)$, with one finite endpoint $e^- \in V$.

The graph $\mathbf{G}$ is assumed connected. It is equipped with the geodesic distance d induced by the Euclidean lengths of edges and rays, and with the measure μ that restricts to Lebesgue measure on each edge and ray.

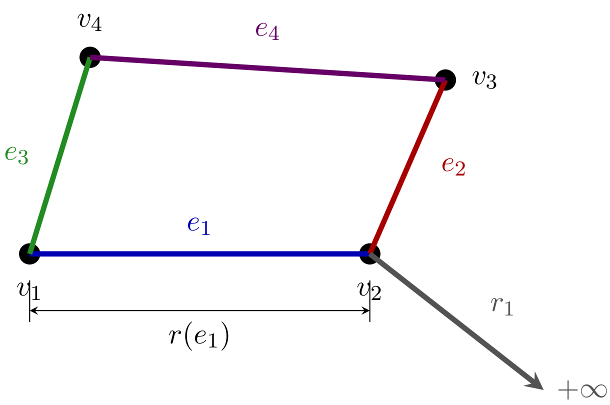


Figure 5.3 – A metric graph $\mathbf{G} = (V, \mathbf{E}, \mathbf{R})$ with vertices $V = \{v_1, v_2, v_3, v_4\}$, edges $\mathbf{E} = \{e_1, e_2, e_3, e_4\}$, and one ray $r_1 \in \mathbf{R}$ based at v_2. The length $r(e_1)$ of e_1 is indicated; the ray r_1 extends to $+\infty$.

Before defining function spaces, we fix notation for the edges incident at each vertex and their orientation, which will appear systematically in the vertex conditions.

Notation 5.2.2: Adjacent edges and orientation

Let $\mathbf{G} = (V, \mathbf{E}, \mathbf{R})$ be a metric graph. Then

(i) For each vertex $v \in V$, the set of *adjacent edges* is

$$\mathbf{E}_v = \Big\{ e \in \mathbf{E} \cup \mathbf{R} : v = e^- \text{ or } v = e^+ \Big\}.$$

(ii) The *orientation function* $U_v : \mathbf{E}_v \to \{-1, 1\}$ is defined by $U_v(e) = 1$ if $v = e^-$ and $U_v(e) = -1$ if $v = e^+$.

With this orientation in place, we define the natural function spaces on $\mathbf{G}$, assembled edge-wise from the standard L^2 and H^1 spaces on each interval I_e, with continuity enforced at vertices.

Definition 5.2.3: Function spaces on G

Let $\mathbf{G} = (V, \mathbf{E}, \mathbf{R})$ be a metric graph (Definition 5.2.1). Functions on $\mathbf{G}$ are families $f = (f_e)_{e\in\mathbf{E}\cup\mathbf{R}}$ with $f_e : I_e \to \mathbb{R}$.

(i) The space $L^2(\mathbf{G}, \mu)$ decomposes as

$$L^2(\mathbf{G}, \mu) = \bigoplus_{e\in\mathbf{E}\cup\mathbf{R}} L^2(I_e). \tag{5.53}$$

(ii) A function f is *continuous on* $\mathbf{G}$ if $f_e \in C(I_e)$ for all e and $f_{e_1}(v) = f_{e_2}(v)$ whenever $e_1, e_2 \in \mathbf{E}_v$ (see Notation 5.2.2 (i)).

(iii) The *Sobolev space* $H^1_0(\mathbf{G})$ is the natural domain for Dirichlet forms on $\mathbf{G}$:

$$H^1_0(\mathbf{G}) = \Big\{ f = (f_e)_e : f_e \in H^1(I_e) \text{ for all } e \in \mathbf{E}\cup\mathbf{R}, \\ \text{and } f \text{ continuous at every vertex} \Big\}. \tag{5.54}$$

(iv) For $f = (f_e)_e$ with $f_e \in H^1(I_e)$, the *edge derivative at a vertex* $f'_e(v)$ is the trace of f'_e at the endpoint of I_e corresponding to v:

$$f'_e(v) = \begin{cases} f'_e(0) & \text{if } v = e^-, \\ f'_e(r(e)) & \text{if } v = e^+. \end{cases} \tag{5.55}$$

(v) The *inward-pointing normal derivative* of f along edge e at vertex v is $U_v(e)f'_e(v)$, with U_v as in Notation 5.2.2 (ii) and $f'_e(v)$ as in (iv).

The Dirichlet form on $\mathbf{G}$ is obtained by summing the edge-wise H^1 energies; the vertex continuity condition in (5.54) ensures that $H^1_0(\mathbf{G})$ is a Hilbert space and makes the sum below well defined.

Definition 5.2.4: Kirchhoff Dirichlet form

Let $\mathbf{G} = (V, \mathbf{E}, \mathbf{R})$ be a metric graph. The *Kirchhoff Dirichlet form* on $L^2(\mathbf{G}, \mu)$ (see Definition 5.2.3 (i)) is

$$\mathcal{E}(f, g) = \sum_{e\in\mathbf{E}\cup\mathbf{R}} \int_{I_e} f'_e(x) g'_e(x)\, dx, \qquad f, g \in H^1_0(\mathbf{G}), \tag{5.56}$$

with domain $H^1_0(\mathbf{G})$ as in Definition 5.2.3 (iii).

The main analytical properties of $\mathcal{E}$ (its generator, the precise form of the vertex condition, and the heat kernel bounds that place $\mathbf{G}$ in the framework of Section 5.2) are recorded in the following theorem.

Theorem 5.2.5

The Kirchhoff Dirichlet form $(\mathcal{E}, H_0^1(\mathbf{G}))$ of Definition 5.2.4 is a strongly local regular Dirichlet form on $L^2(\mathbf{G}, \mu)$. Moreover:

(i) *Generator.* The generator of $\mathcal{E}$ is $\Delta f = -f''$, acting edge-wise, with domain

$$\operatorname{Dom}(\Delta) = \Big\{ f \in H_0^1(\mathbf{G}) : f_e'' \in L^2(I_e) \text{ for all } e \in \mathbf{E} \cup \mathbf{R}, \\ \text{and } \textstyle\sum_{e \in \mathbf{E}_v} U_v(e) f_e'(v) = 0 \text{ for all } v \in V \Big\}, \tag{5.57}$$

where $\mathbf{E}_v$, U_v are as in Notation 5.2.2 and $f_e'(v)$ is as in Definition 5.2.3 (iv). The condition $f_e'' \in L^2(I_e)$ ensures $f_e' \in H^1(I_e)$, so the traces in (5.55) are well defined.

(ii) *Heat kernel bounds.* The heat kernel $p_t(x, y)$ associated with $\mathcal{E}$ satisfies the Gaussian estimates (5.6) with $d_h = 1$ and $d_w = 2$ [31].

Remark 5.2.6. *A few comments on Theorem 5.2.5 are in order:*

(i) *Conservation of flux. The Kirchhoff condition in* (5.57) *requires the sum of inward-pointing normal derivatives (Definition 5.2.3 (v)) to vanish at every vertex $v \in V$. This is the graph analogue of the Neumann condition $\partial_n f = 0$ on a manifold with boundary.*

(ii) *Independence of orientation. Neither $\mathcal{E}$ nor Δ depends on the choice of orientation. If the sign convention for $U_v(e)$ is reversed for some edge e, then $f_e'(v)$ changes sign accordingly by* (5.55)*, leaving both the bilinear form* (5.56) *and the vertex condition in* (5.57) *unchanged.*

Connected bounded open subsets of $\mathbf{G}$ are uniform domains, so both the Dirichlet and Neumann setups of Sections 5.1.2.1–5.1.3.2 apply; see [49] for details on function spaces on metric graphs.

5.2.3 Riemannian manifolds

Riemannian manifolds generalize $\mathbb{R}^n$ by allowing non-trivial curvature while retaining the local Euclidean structure that makes classical analysis available. The heat kernel exponents remain $d_h = n$ (the manifold dimension) and $d_w = 2$ (standard Gaussian diffusion), exactly as for flat $\mathbb{R}^n$; the curvature governs the volume growth of geodesic balls and enters through the constants in the heat kernel bounds. Non-negative Ricci curvature is the standard assumption for these estimates, and includes the sphere $\mathbb{S}^n$ and flat tori $\mathbb{T}^n$ as prototypical examples.

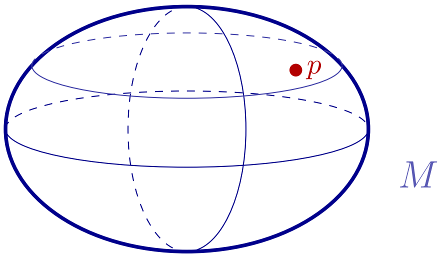


Figure 5.4 – A compact Riemannian manifold (M, g) without boundary, with positive sectional curvature (an ellipsoid with three distinct semi-axes), where $p \in M$ is a generic point.

Definition 5.2.7: Riemannian manifold

A *Riemannian manifold* (M, g) consists of a connected smooth manifold M of dimension $n \geq 1$ equipped with a Riemannian metric g. We equip M with:

(i) The *Riemannian measure* $\mu = \mathrm{vol}_g$, given in local coordinates by

$$d\mu = \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n.$$

(ii) The *Riemannian distance*, defined by

$$d(x, y) = \inf_{\gamma} \int_0^1 |\dot{\gamma}(t)|_g \, dt \,,$$

where the infimum runs over smooth curves $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$;

The derivative structure on M is encoded by the Riemannian gradient, which extends the Euclidean gradient $\partial_i f$ to curved geometry and gives rise to the Sobolev space serving as the natural domain for the Dirichlet form.

Definition 5.2.8: Riemannian gradient and Sobolev space

Let (M, g) be a Riemannian manifold.

(i) The *Riemannian gradient* $\nabla_g f$ of $f \in C^\infty(M)$ is the vector field characterized by $\langle \nabla_g f, X\rangle_g = df(X)$ for every smooth vector field X on M. In local coordinates $(\nabla_g f)^i = g^{ij}\partial_j f$, so that

$$|\nabla_g f|_g^2 = g^{ij}\partial_i f\,\partial_j f. \tag{5.58}$$

(ii) The *Sobolev space* $W^{1,2}(M)$ is the completion of $C_c^\infty(M)$ under the norm

$$\|f\|_{W^{1,2}(M)}^2 = \int_M \Big(f^2 + |\nabla_g f|_g^2\Big)\, d\mu, \tag{5.59}$$

with $|\nabla_g f|_g^2$ as in (5.58).

The Riemannian Dirichlet form is the natural generalization of $\int_{\mathbb{R}^n} |\nabla f|^2\, dx$: it integrates the pointwise squared gradient length against the Riemannian measure.

Definition 5.2.9: Riemannian Dirichlet form

Let (M, g) be a complete Riemannian manifold. The *Riemannian Dirichlet form* on $L^2(M,\mu)$ is

$$\mathcal{E}(f, h) = \int_M \langle \nabla_g f, \nabla_g h\rangle_g\, d\mu, \qquad f, h \in W^{1,2}(M), \tag{5.60}$$

with domain $W^{1,2}(M)$ as in Definition 5.2.8 (ii).

The Riemannian Dirichlet form is strongly local and regular, and its generator is the Laplace–Beltrami operator. This is a classical result of functional analysis on manifolds; see [28] for a detailed treatment.

Proposition 5.2.10

Let (M, g) be a complete Riemannian manifold of dimension n. The form $(\mathcal{E}, W^{1,2}(M))$ of Definition 5.2.9 is a strongly local regular Dirichlet form on $L^2(M,\mu)$. Its generator is the *Laplace–Beltrami operator*

$$\Delta_M f = -\mathrm{div}_g(\nabla_g f) = -\frac{1}{\sqrt{|g|}}\,\partial_i\Big(\sqrt{|g|}\, g^{ij}\partial_j f\Big), \tag{5.61}$$

with domain

$$\mathrm{Dom}(\Delta_M) = \Big\{f \in W^{1,2}(M) : \mathrm{div}_g(\nabla_g f) \in L^2(M,\mu)\Big\}. \tag{5.62}$$

The restriction of Δ_M to $C_c^\infty(M)$ is essentially self-adjoint on $L^2(M,\mu)$.

The generator is now identified. The question of how the associated heat semigroup $e^{t\Delta_M}$ spreads mass is governed by the geometry of (M, g), and specifically by its curvature.

Under non-negative Ricci curvature, Li and Yau [42] established sharp two-sided Gaussian estimates for the heat kernel, which translate directly into the sub-Gaussian framework of Assumption 5.1.6 with $d_h = n$ and $d_w = 2$.

Theorem 5.2.11: Li–Yau

Let (M, g) be a complete Riemannian manifold of dimension n with $\mathrm{Ric}_g \geq 0$. The heat kernel $p_t(x, y)$ of the semigroup generated by Δ_M (Proposition 5.2.10) satisfies the Gaussian estimates (5.6) with $d_h = n$ and $d_w = 2$ [28, 42]. In particular, $(M, d_g, \mu, \mathcal{E})$ fits into the framework of Assumption 5.1.6 whenever $n < 2$, i.e. $n = 1$.

Remark 5.2.12. *Two aspects of the Riemannian setting deserve comment: the status of the recurrence assumption, and the prototypical compact examples to which the theorem applies.*

(i) *Recurrence. The recurrence condition $d_h < d_w$ of Assumption 5.1.6 requires $n < 2$, hence $n = 1$ for a manifold. For compact Riemannian manifolds of dimension $n \geq 2$ the condition is not satisfied, so the general framework of this course does not apply directly. The PAM on such manifolds can nonetheless be studied by techniques similar to ours; see [13, 14].*

(ii) *Compact examples. The sphere $\mathbb{S}^n$ (with $\mathrm{Ric}_g = (n-1)g > 0$) and the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (with $\mathrm{Ric}_g = 0$) are compact manifolds satisfying $\mathrm{Ric}_g \geq 0$; their heat kernels satisfy the bounds in Theorem 5.2.11 for small times $t \leq \mathrm{diam}(M)^2$.*

5.2.4 Sierpiński gasket

The Sierpiński gasket is one of the most studied examples of a post-critically finite (p.c.f.) fractal [4, 38]. Unlike the interval and the metric graph, it exhibits a genuinely sub-Gaussian diffusion ($d_w = \ln 5/\ln 2 > 2$), providing a non-trivial verification of Assumption 5.1.6.

Definition 5.2.13: Sierpiński gasket

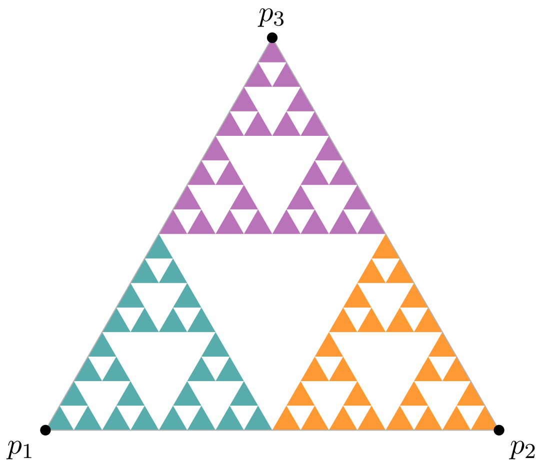
Let $V_0 = \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle of side 1 in $\mathbb{C}$. For $i = 1, 2, 3$, define the contraction maps

$$\mathfrak{f}_i(z) = \frac{z - p_i}{2} + p_i. \tag{5.63}$$

The *Sierpiński gasket* K is the unique non-empty compact set satisfying

$$K = \bigcup_{i=1}^{3} \mathfrak{f}_i(K). \tag{5.64}$$

The set $\partial K := V_0$ is called the *boundary* of K. We equip K with the Euclidean distance $d(x, y) = |x - y|$.



The self-similar structure of K determines a canonical measure and a pair of dimensional exponents.

Definition 5.2.14: Hausdorff measure and dimensions of K

The (normalized) Hausdorff measure μ on K is the unique Borel measure satisfying

$$\mu\Big(\mathfrak{f}_{i_1} \circ \cdots \circ \mathfrak{f}_{i_n}(K)\Big) = 3^{-n} \tag{5.65}$$

for every finite word $(i_1, \ldots, i_n) \in \{1, 2, 3\}^n$. The measure μ is Ahlfors d_h-regular in the sense of (5.5), with dimensional exponents

$$d_h = \frac{\ln 3}{\ln 2}, \qquad d_w = \frac{\ln 5}{\ln 2}. \tag{5.66}$$

Since $d_h \approx 1.585 < d_w \approx 2.322$, the recurrence condition of Assumption 5.1.6 holds.

Remark 5.2.15 (Intuition for $d_h = \ln 3/\ln 2$)**.** *The gasket consists of exactly 3 copies of itself, each contracted by factor $1/2$ (the maps $\mathfrak{f}_i$ in (5.63)). For a self-similar set made of N copies at scaling ratio r, the Hausdorff dimension satisfies $N \cdot r^{d_h} = 1$. With $N = 3$ and $r = 1/2$:*

$$3 \cdot \left(\frac{1}{2}\right)^{d_h} = 1, \qquad \textit{hence} \qquad d_h = \frac{\ln 3}{\ln 2}. \tag{5.67}$$

Concretely, at level m the gasket is covered by 3^m cells of diameter 2^{-m} and mass 3^{-m}, so a ball of radius $r \asymp 2^{-m}$ satisfies $\mu(B(x,r)) \asymp 3^{-m} \asymp r^{d_h}$, consistently with (5.5).

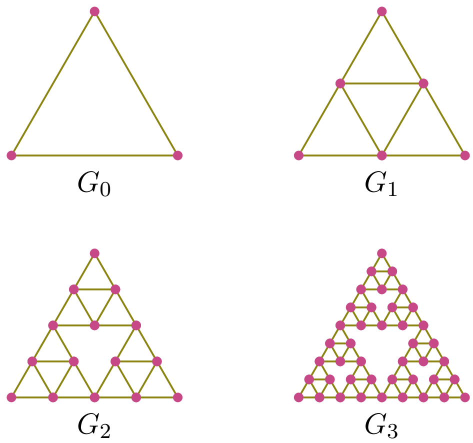
The construction of the Dirichlet form on K proceeds by passing to the limit along a sequence of finite graphs. We first set up the necessary notation.

Notation 5.2.16: Discrete approximation of K

Starting from $V_0 = \{p_1, p_2, p_3\}$, define inductively

$$V_{m+1} = \bigcup_{i=1}^{3} \mathfrak{f}_i(V_m), \qquad m \geq 0. \tag{5.68}$$

One has $\#V_m = \frac{3(3^m+1)}{2}$. Set $V_* = \bigcup_{m\geq 0} V_m$ and $V_*^0 = V_* \setminus V_0$. For each $m \geq 0$, the *pre-gasket graph* G_m has vertex set V_m and edges between all pairs at Euclidean distance 2^{-m}; write $p \sim_m q$ for adjacency in G_m.



The Dirichlet form on K (Definition 5.2.13) is built as a limit of discrete energies defined on the approximating graphs G_m of Notation 5.2.16. The key point is to choose the correct renormalization factor so that these discrete energies are compatible as m increases.

Definition 5.2.17: Discrete energy on G_m

Here K is the Sierpiński gasket of Definition 5.2.13 and $G_m = (V_m, \sim_m)$ are the pre-gasket graphs of Notation 5.2.16. For $m \geq 0$ and $f : V_m \to \mathbb{R}$, the *discrete energy* of f on G_m is

$$\mathcal{E}_m(f, f) = \left(\frac{5}{3}\right)^m \sum_{\substack{p,q \in V_m \\ p \sim_m q}} (f(p) - f(q))^2. \tag{5.69}$$

The factor $(5/3)^m$ in (5.69) is the *self-similar resistance scaling* of the gasket. It is the unique choice making the sequence $m \mapsto \mathcal{E}_m(f|_{V_m}, f|_{V_m})$ non-decreasing for every continuous $f : K \to \mathbb{R}$: since $(5/3)^m$ is exactly the ratio that preserves energy under harmonic extension from V_m to V_{m+1}, the restriction $f|_{V_{m+1}}$ (which is generally not harmonic) satisfies $\mathcal{E}_{m+1}(f|_{V_{m+1}}, f|_{V_{m+1}}) \geq \mathcal{E}_m(f|_{V_m}, f|_{V_m})$; see [38, Proposition 3.1.3 and Corollary 3.4.7]. The two following remarks explain where this factor comes from: Remark 5.2.18 sets up the resistor-network language needed for the computation, and Remark 5.2.19 carries out the explicit resistance calculation on G_1 that identifies $5/3$ as the correct ratio.

Remark 5.2.18 (Discrete Dirichlet forms and resistor networks)**.** *There is a classical correspondence between discrete Dirichlet forms and electrical resistor networks [4, 38], which underlies Remarks 5.2.19 and 5.2.20. On a finite graph $G = (V, E)$ with unit edge resistances, let $f : V \to \mathbb{R}$ be any function, interpreted as a voltage distribution on the nodes. By Ohm's law, the current on edge $\{p, q\}$ is $I_{pq} = f(p) - f(q)$, so the total power dissipated is*

$$P(f) = \sum_{\{p,q\}\in E} I_{pq}^2 = \sum_{\{p,q\}\in E} (f(p) - f(q))^2. \tag{5.70}$$

The quantity $P(f)$ is the discrete Dirichlet form on G with unit conductances; on the pre-gasket graph G_m it is exactly the unrenormalized energy $\sum_{p\sim_m q}(f(p)-f(q))^2$, i.e., the summand in (5.69) before multiplication by $(5/3)^m$.

The unique voltage distribution h minimizing $P(h)$ with prescribed boundary values $h(a) = 1$ and $h(b) = 0$ is the solution to the discrete Laplace equation (Kirchhoff's current law: total current into each node is zero)

$$\Delta_G h(p) := \sum_{q\sim p}(h(q) - h(p)) = 0, \qquad p \in V \setminus \{a, b\}, \tag{5.71}$$

supplemented by the boundary conditions $h(a) = 1$, $h(b) = 0$. This is a linear system in the values $\{h(p)\}_{p\in V\setminus\{a,b\}}$ with a unique solution (the matrix $-\Delta_G$ restricted to interior nodes is positive definite). The effective resistance *between a and b is*

$$R_{\mathrm{eff}}(a, b) = \frac{h(a) - h(b)}{I_a}, \quad \textit{where} \quad I_a = \sum_{q\sim a}(h(a) - h(q))$$

is the current injected at a. In order to give another expression for $R_{\mathrm{eff}}(a, b)$, let us compute $P(h)$. To this aim, for every edge write

$$(h(p) - h(q))^2 = h(p)(h(p) - h(q)) + h(q)(h(q) - h(p)),$$

and sum over all directed pairs (p, q) with $p \sim q$. Regrouping by base node gives

$$P(h) = \sum_{p\in V} h(p) \sum_{q\sim p}(h(p) - h(q)). \tag{5.72}$$

Using (5.71) at all nodes $p \in V \setminus \{a, b\}$, the only surviving terms in (5.72) are $h(a)I_a + h(b)(-I_a) = I_a$, giving

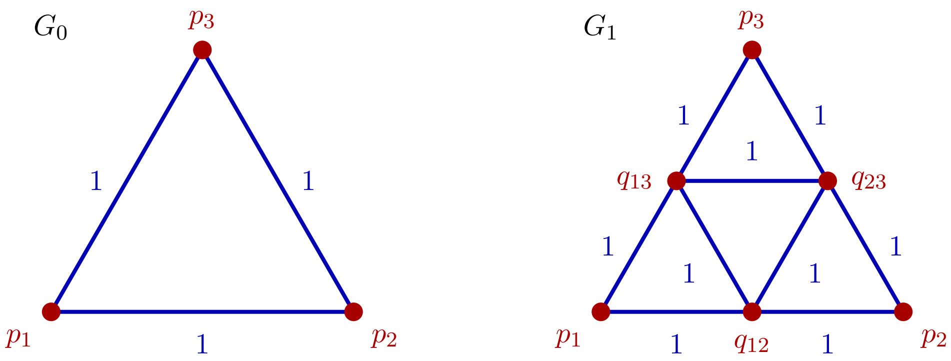
$$R_{\mathrm{eff}}(a, b) = \frac{1}{P(h)}. \tag{5.73}$$

The function h also has a probabilistic interpretation: for the simple random walk on G, $h(p) = P_p(\tau_a < \tau_b)$, the probability of hitting a before b when started at p (this follows *from the fact that the hitting probability satisfies* (5.71) *at every interior node, the same equation characterizing* h*). The effective resistance is then linked to the random walk by the* commute-time formula *[4]: denoting by* $|E|$ *the number of edges of* G*,*

$$E_a[\tau_b] + E_b[\tau_a] = 2\,|E|\,R_{\text{eff}}(a,b). \tag{5.74}$$

Larger resistance thus means a longer expected round-trip time between a *and* b*.*

Remark 5.2.19 (Origin of the renormalization factor 5/3)**.** *By Remark 5.2.18, resistor networks obey the standard rules of circuit theory: resistances in series add (*$R_{\text{series}} = R_1 + R_2$*), resistances in parallel combine as* $R_{\text{par}}^{-1} = R_1^{-1} + R_2^{-1}$*, and Ohm's law* $I = V/R$ *holds on each edge. We apply these rules to* G_0 *and* G_1*, each with unit resistance on every edge. In* G_0 *(a triangle with three unit-resistance edges), the effective resistance between any two corner vertices is* $R_0 = 2/3$ *(two parallel paths of resistance* 1 *and* 2*). To find the effective resistance in* G_1*, assign voltages* $V(p_1) = 1$*,* $V(p_2) = 0$*.*



Here q_{ij} *denotes the midpoint of the segment* p_ip_j*. Since the graph* G_1 *has a reflection symmetry swapping* p_1 *and* p_2 *(reflection across the perpendicular bisector of* p_1p_2*), and the boundary values also swap (*$V(p_1) = 1 \leftrightarrow V(p_2) = 0$*), the unique Kirchhoff solution satisfies* $V \mapsto 1 - V$ *under this reflection. Both* q_{12} *and* p_3 *lie on the symmetry axis, so* $V(q_{12}) = 1 - V(q_{12})$ *and* $V(p_3) = 1 - V(p_3)$*, giving* $V(q_{12}) = V(p_3) = \frac{1}{2}$*. Setting* $a = V(q_{13})$ *(so* $V(q_{23}) = 1 - a$ *by the same symmetry), Kirchhoff's equation* (5.71) *at* q_{13} *(which has four neighbors* p_1*,* q_{12}*,* q_{23}*,* p_3 *in* G_1*) reads*

$$(1-a) + \left(\frac{1}{2} - a\right) + (1-2a) + \left(\frac{1}{2} - a\right) = 0, \qquad \textit{giving} \qquad a = \frac{3}{5}. \tag{5.75}$$

By Remark 5.2.18, $R_1 = (V(p_1) - V(p_2))/I_{p_1}$*, where* $I_{p_1} = \sum_{q \sim_1 p_1}(V(p_1) - V(q))$ *is the current injected at* p_1*. Since* p_1 *has exactly two neighbors* q_{12} *and* q_{13} *in* G_1*,*

$$I_{p_1} = \Big(V(p_1) - V(q_{12})\Big) + \Big(V(p_1) - V(q_{13})\Big) = \left(1 - \frac{1}{2}\right) + \left(1 - \frac{3}{5}\right) = \frac{9}{10}. \tag{5.76}$$

Therefore we obtain $R_1 = (V(p_1) - V(p_2))/I_{p_1} = \frac{10}{9}$*, and*

$$\frac{R_1}{R_0} = \frac{10/9}{2/3} = \frac{5}{3}. \tag{5.77}$$

Iterating this procedure, we obtain that each step of self-similar refinement multiplies the effective resistance by $5/3$*. Hence without the factor* $(5/3)^m$ *in* (5.69)*, the raw sum* $\sum_{p\sim_m q}(f(p)-f(q))^2$ *would decay as* $(3/5)^m \to 0$ *for any fixed continuous* f*; the factor* $(5/3)^m$ *is the unique compensation making the limit energy finite and non-degenerate.*

Remark 5.2.20 (Intuition for $d_w = \ln 5/\ln 2$)**.** *As established in Remark 5.2.19, the effective resistance between two points at Euclidean distance* $r \asymp 2^{-m}$ *scales as* $(3/5)^m$*. Setting* $r \asymp 2^{-m}$ *identifies the resistance exponent* β*:*

$$R(x,y) \asymp \left(\frac{3}{5}\right)^m \asymp d(x,y)^\beta, \qquad \beta = \frac{\ln(5/3)}{\ln 2}. \tag{5.78}$$

The walk dimension is then given by the Einstein relation $d_w = d_h + \beta$*, which links the exit-time exponent to the volume-growth and resistance exponents (see [41, eq. (1.2)] and [5, 38]):*

$$d_w = d_h + \beta = \frac{\ln 3}{\ln 2} + \frac{\ln(5/3)}{\ln 2} = \frac{\ln 5}{\ln 2}. \tag{5.79}$$

The fact that $\beta > 0$ *(resistance grows with distance faster than linearly) is responsible for* $d_w > 2$*: the Brownian motion on* K *spreads anomalously slowly compared to flat space.*

Remarks 5.2.19 and 5.2.20 have established two things: the factor $(5/3)^m$ in (5.69) is the unique compensation that keeps the sequence $m \mapsto \mathcal{E}_m(f|_{V_m}, f|_{V_m})$ non-decreasing, and this same factor encodes the anomalous geometry of K through the exponents $d_h = \ln 3/\ln 2$ and $d_w = \ln 5/\ln 2$. Since the sequence is non-decreasing, the limit $\lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m})$ exists in $[0, +\infty]$ for every $f : V_* \to \mathbb{R}$, and it is finite precisely for those f that carry bounded energy across all approximating graphs. This class of functions, equipped with the limiting energy, is the Dirichlet form on K.

Definition 5.2.21: Dirichlet form on K

Recall that K is the Sierpiński gasket of Definition 5.2.13. Set

$$\mathcal{F}_* = \left\{ f : V_* \to \mathbb{R} \ : \ \sup_{m\ge 0} \mathcal{E}_m(f|_{V_m}, f|_{V_m}) < \infty \right\}, \qquad \mathcal{E}(f,f) = \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m}). \tag{5.80}$$

Every $f \in \mathcal{F}_*$ extends uniquely to a continuous function on K; let $\mathcal{F}$ denote the space of such extensions. Then $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(K, \mu)$ [4, 38].

The pair $(\mathcal{E}, \mathcal{F})$ is the ambient Dirichlet form in Assumption 5.1.6 with $X = K$. Boundary conditions at $\partial K = V_0$ are imposed by restricting the domain of the form.

Definition 5.2.22: Neumann and Dirichlet Laplacians on K

(i) *Neumann Laplacian.* The generator of $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ is the *Neumann Laplacian* Δ_N, with domain

$$\mathrm{Dom}(\Delta_N) = \Big\{ u \in \mathcal{F} \ : \ \exists\, f \in L^2(K, \mu), \quad \mathcal{E}(u, v) = \int_K f v \, d\mu \quad \forall\, v \in \mathcal{F} \Big\}, \tag{5.81}$$

and $\Delta_N u = -f$.

(ii) *Dirichlet Laplacian.* Set $\mathcal{F}_0 = \{f \in \mathcal{F} : f|_{V_0} = 0\}$. Then $(\mathcal{E}, \mathcal{F}_0)$ is itself a strongly local regular Dirichlet form on $L^2(K, \mu)$ [38, Corollary 3.4.7], and its generator is the *Dirichlet Laplacian* Δ_D, with domain

$$\mathrm{Dom}(\Delta_D) = \Big\{ u \in \mathcal{F}_0 \ : \ \exists\, f \in L^2(K, \mu), \quad \mathcal{E}(u, v) = \int_K f v \, d\mu \quad \forall\, v \in \mathcal{F}_0 \Big\}, \tag{5.82}$$

and $\Delta_D u = -f$.

Both Laplacians admit a concrete pointwise description, introduced by Kigami [37, 38], which parallels the classical notion of a Laplacian in smooth settings. For $p \in V_m$, let $V_{m,p}$ denote the set of neighbors of p in G_m; one has $\#V_{m,p} = 4$ for $p \in V_m \setminus V_0$ and $\#V_{m,p} = 2$ for $p \in V_0$.

Definition 5.2.23: Kigami Laplacian

For $m \geq 1$ and $p \in V_m \setminus V_0$, the discrete Laplacian on G_m is

$$\Delta_m f(p) = \frac{3}{2} \cdot 5^m \sum_{q \in V_{m,p}} (f(q) - f(p)). \tag{5.83}$$

The domain of the *Kigami Laplacian* is

$$\mathcal{D} = \Big\{ f \in C(K) \ : \ \exists\, g \in C(K), \quad \lim_{m \to \infty} \max_{p \in V_m \setminus V_0} |\Delta_m f(p) - g(p)| = 0 \Big\}, \tag{5.84}$$

and one sets $\Delta f := g$ for $f \in \mathcal{D}$. For $p \in V_0$, the *Neumann derivative* of f at p is

$$(df)_p = -\lim_{m \to \infty} \left(\frac{5}{3}\right)^m \sum_{q \in V_{m,p}} (f(q) - f(p)), \tag{5.85}$$

whenever the limit exists.

Remark 5.2.24 (Origin of the factor $\frac{3}{2} \cdot 5^m$ in (5.83))**.** *The factor $\frac{3}{2} \cdot 5^m$ in the Kigami Laplacian (5.83) arises from two independent contributions that we identify in turn.*

Factor 5^m: balancing energy against measure. The Kigami Laplacian is defined so that the discrete Green's identity

$$\mathcal{E}_m(f,g) = -\sum_{p\in V_m\setminus V_0} g(p)\,\Delta_m f(p)\,\mu_m(p) + \textit{boundary terms} \tag{5.86}$$

holds, in analogy with $\mathcal{E}(f,g) = -\int_K g\,\Delta f\,d\mu$. Here $\mu_m(p)$ is the natural discrete approximation to μ at level m. Since μ assigns mass 3^{-m}, comparing the right hand side of (5.86) with (5.69) yields the factor 5^m in (5.83).

Factor $3/2$: cell-sharing at interior vertices. The mass $\mu_m(p)$ is computed by distributing the μ-mass of each level-m cell equally among its three corner vertices. One can check by induction that every interior vertex $p \in V_m \setminus V_0$ is a corner of exactly two level-m cells (as is already visible in G_1 from Remark 5.2.19), so

$$\mu_m(p) = 2\cdot\frac{3^{-m}}{3} = \frac{2}{3^{m+1}}. \tag{5.87}$$

The full prefactor is then $(5/3)^m/\mu_m(p) = 5^m\cdot 3^{m+1}/(2\cdot 3^m) = \frac{3}{2}\cdot 5^m$.

Remark 5.2.25 (Another explanation for the factor 5^m in (5.83))**.** *In $\mathbb{R}^n$ on a grid of mesh size $h = 2^{-m}$ (namely $2^{-m}\mathbb{Z}$), the discrete Laplacian carries a factor $h^{-2} = 4^m$ because a random walk exits a ball of radius h in time $\asymp h^2$. On the Sierpiński gasket, the walk dimension d_w is $\ln 5/\ln 2$ (Remark 5.2.20). This means that the exit time from a ball of radius 2^{-m} scales as $(2^{-m})^{d_w} = 5^{-m}$, so the generator is renormalized by 5^m rather than 4^m.*

The Kigami Laplacian Δ is related to Δ_N and Δ_D through boundary conditions at V_0, in full analogy with the classical smooth setting.

Proposition 5.2.26: [38, Theorem 3.7.9]

Let $\mathcal{D}_N = \{f\in\mathcal{D} : (df)_p = 0\ \forall\, p\in V_0\}$ and $\mathcal{D}_D = \{f\in\mathcal{D} : f|_{V_0} = 0\}$.

(i) One has $\mathcal{D}_N = \mathrm{Dom}(\Delta_N)\cap\mathcal{D}$ and $\Delta_N|_{\mathcal{D}_N} = \Delta|_{\mathcal{D}_N}$; moreover, Δ_N is the Friedrichs extension of $\Delta|_{\mathcal{D}_N}$.

(ii) One has $\mathcal{D}_D = \mathrm{Dom}(\Delta_D)\cap\mathcal{D}$ and $\Delta_D|_{\mathcal{D}_D} = \Delta|_{\mathcal{D}_D}$; moreover, Δ_D is the Friedrichs extension of $\Delta|_{\mathcal{D}_D}$.

With the Neumann and Dirichlet Laplacians Δ_N and Δ_D of Definition 5.2.22 in place, it remains to verify that the Sierpiński gasket fits into the sub-Gaussian framework of Assumption 5.1.6. The following proposition confirms this and closes the verification of the abstract setup for $X = K$.

Theorem 5.2.27: Sub-Gaussian heat kernel on K

Here K is the Sierpiński gasket of Definition 5.2.13, μ is the self-similar measure of Definition 5.2.14, Δ_N and Δ_D are the Laplacians of Definition 5.2.22, and $d_h = \ln 3/\ln 2$, $d_w = \ln 5/\ln 2$ are the exponents of (5.66). The semigroups $e^{t\Delta_N}$ and $e^{t\Delta_D}$ each admit a continuous heat kernel $p_t(x, y)$ satisfying the sub-Gaussian estimates (5.6). In particular, $(K, d, \mu, \mathcal{E})$ satisfies Assumption 5.1.6 with $X = K$ [4].

We close this section by mentioning that open sets of the form $U = K \setminus V_0$ (the gasket with the boundary vertices removed) or $U = \mathfrak{f}_{i_1} \circ \cdots \circ \mathfrak{f}_{i_n}(K \setminus V_0)$ (scaled copies thereof) are inner uniform domains [10, Sec. 2.4.3], so all the results of Sections 5.1.2.1–5.1.3.2 apply. For a systematic account of inner uniform and uniform domains within the Sierpiński gasket, we refer to [4, 44].

CHAPTER 6

PAM on metric spaces

With the geometric framework of Chapter 5 in hand, we can now analyze the parabolic Anderson model on a metric measure space satisfying Assumption 5.1.6. Our route parallels the Euclidean one. Namely we first construct a natural family of noises on a bounded domain $U \subset X$ (Section 6.1). We then solve the equation by means of chaos expansions (Section 6.2). Finally we quantify the intermittent behavior of the solution through upper and lower bounds on its moments (Section 6.3).

6.1 Noises for PAM on metric measure spaces

In Chapter 2, the driving noise for the PAM on $\mathbb{R}^d$ was a centered Gaussian family white in time and correlated in space with covariance $\Lambda(x-y)$, and its Hilbert space $\mathcal{H}$ was built by completing the test functions under the corresponding inner product (Section 2.1.1). In the geometric setting of Chapter 5, the translation-invariant kernel $\Lambda(x-y)$ is no longer available. The role it played is taken over by the Riesz kernels $G_{2\alpha}^{D,U}$ or $G_{2\alpha}^{N,U}$ introduced in Sections 5.1.2.2 and 5.1.3.2. The parameter $\alpha \geq 0$ controls the spatial regularity of the noise: $\alpha = 0$ corresponds to space-time white noise on U, while $\alpha > 0$ introduces a spatially colored covariance produced by the fractional operator $(-\Delta_U)^{-2\alpha}$.

6.1.1 The fractional noise family

We work throughout this section under the standing framework of the previous sections: (X, d, μ) is a metric measure space satisfying Assumption 5.1.6 and $U \subset X$ is a bounded open set with compact closure. We define the noise for the Dirichlet setting first, then for the Neumann setting; the two coincide when $\alpha = 0$. More specifically, the covariance of the Dirichlet noise is governed by the Riesz kernel $G^{D,U}_{2\alpha}$ of (5.24), and the natural Hilbert space of test functions is the Dirichlet Sobolev space $\mathcal{W}^{-\alpha}_D(U)$ of Definition 5.1.28, extended to space-time functions.

Definition 6.1.1: Dirichlet fractional noise

Let $U \subset X$ be an inner uniform domain and $\alpha > 0$. The space-time Hilbert space corresponding to the Ditrichlet fractional noise is

$$\mathcal{H}^D_\alpha \;:=\; L^2\Big(\mathbb{R}_+,\; \mathcal{W}^{-\alpha}_D(U)\Big), \tag{6.1}$$

with inner product $\langle \varphi, \psi\rangle_{\mathcal{H}^D_\alpha} := \int_{\mathbb{R}_+} \langle \varphi(t,\cdot), \psi(t,\cdot)\rangle_{\mathcal{W}^{-\alpha}_D(U)}\, dt$. The *Dirichlet fractional noise* is a centered Gaussian family $\{W^D_\alpha(\varphi);\, \varphi \in \mathcal{H}^D_\alpha\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ whose covariance is

$$\mathbb{E}\Big[W^D_\alpha(\varphi)\, W^D_\alpha(\psi)\Big] \;=\; \langle \varphi, \psi\rangle_{\mathcal{H}^D_\alpha}, \qquad \varphi, \psi \in \mathcal{H}^D_\alpha. \tag{6.2}$$

Unfolding the inner product of Definition 5.1.28, the covariance structure can be spelled out as

$$\mathbb{E}\Big[W^D_\alpha(\varphi)\, W^D_\alpha(\psi)\Big] = \int_{\mathbb{R}_+}\int_{U\times U} \varphi(t,x)\,\psi(t,y)\, G^{D,U}_{2\alpha}(x,y)\, d\mu(x)\, d\mu(y)\, dt. \tag{6.3}$$

Since $G^{D,U}_{2\alpha}(x,y) \geq 0$ for all $x, y \in U$ (see (5.24)), the right-hand side of (6.3) defines a genuine inner product on $\mathcal{W}^{-\alpha}_D(U)$. The covariance identity (6.2) means that the map $\varphi \mapsto W^D_\alpha(\varphi)$ is a linear isometry from $\mathcal{H}^D_\alpha$ to $L^2(\Omega)$, of the form

$$\mathbb{E}\Big[W^D_\alpha(h)\, W^D_\alpha(k)\Big] \;=\; \langle h, k\rangle_{\mathcal{H}^D_\alpha} \qquad \text{for all } h, k \in \mathcal{H}^D_\alpha. \tag{6.4}$$

In particular, W^D_α is an isonormal Gaussian process on $\mathcal{H}^D_\alpha$ in the sense of Definition 2.1.3. Following the convention of Section 2.1.1, we write Wiener integrals as

$$W^D_\alpha(h) \;=\; \int_0^\infty \int_U h(s,x)\, W^D_\alpha(ds, dx), \qquad h \in \mathcal{H}^D_\alpha.$$

The case $\alpha = 0$ plays a special role: the covariance kernel $G^{D,U}_0$ reduces to the Dirac mass on the diagonal, and the noise becomes the space-time white noise on U.

Definition 6.1.2: Space-time white noise on U

The Hilbert space for the white noise is

$$\mathcal{H}_0 \;:=\; L^2(\mathbb{R}_+,\, L^2(U,\mu)), \tag{6.5}$$

with inner product $\langle \varphi, \psi\rangle_{\mathcal{H}_0} = \int_{\mathbb{R}_+}\int_U \varphi(t,x)\,\psi(t,x)\,d\mu(x)\,dt$. The *space-time white noise* on U is a centered Gaussian family $\{W_0(\varphi);\, \varphi \in \mathcal{H}_0\}$ with covariance

$$\mathbb{E}\Big[W_0(\varphi)\,W_0(\psi)\Big] \;=\; \int_{\mathbb{R}_+}\int_U \varphi(t,x)\,\psi(t,x)\,d\mu(x)\,dt. \tag{6.6}$$

As for $\alpha > 0$, the map $W_0 : \mathcal{H}_0 \to L^2(\Omega)$ is a linear isometry, and W_0 is an isonormal Gaussian process on $\mathcal{H}_0$.

The three regimes identified in Proposition 5.1.26 for the kernel $G^{D,U}_{2\alpha}$ have a direct probabilistic interpretation in terms of the regularity of the field W^D_α.

Remark 6.1.3 (Regularity regimes)**.** *The critical exponent $d_h/(2d_w)$ separates three regimes. The precise Sobolev and Hölder regularity of W^D_α in each regime is worked out in Section 6.1.3, but we can already summarize what we get as follows:*

(i) *Distributional regime ($0 \le \alpha < d_h/(2d_w)$). The covariance kernel $G^{D,U}_{2\alpha}(x,y)$ diverges like $d_U(x,y)^{2\alpha d_w - d_h} \to +\infty$ near the diagonal (see (5.27)), so the field W^D_α is a distribution in the space variable. This includes the white noise case $\alpha = 0$.*

(ii) *Log-correlated regime ($\alpha = d_h/(2d_w)$). The kernel grows logarithmically: $G^{D,U}_{2\alpha}(x,y) \asymp |\ln d_U(x,y)|$ near the diagonal (see (5.28)).*

(iii) *Pointwise regime ($\alpha > d_h/(2d_w)$). The kernel $G^{D,U}_{2\alpha}$ is bounded on $U \times U$ (see (5.29)), and the field W^D_α can be realized as a pointwise continuous Gaussian field on $\mathbb{R}_+ \times U$.*

We now turn to the Neumann setting. The construction mirrors Definition 6.1.1, with $\mathcal{W}^{-\alpha}_N(U)$ replacing $\mathcal{W}^{-\alpha}_D(U)$. Recall from Definition 5.1.38 that the inner product on $\mathcal{W}^{-\alpha}_N(U)$ involves the shifted kernel $G^{N,U}_{2\alpha}(x,y) + C_U$. Indeed, since $G^{N,U}_{2\alpha}$ is not pointwise non-negative (Proposition 5.1.37), the constant C_U is chosen large enough so that

$$G^{N,U}_{2\alpha} + C_U \ge 0 \quad \text{pointwise},$$

restoring positivity.

Definition 6.1.4: Neumann fractional noise

Let $U \subset X$ be a uniform domain and $\alpha > 0$. The space-time Hilbert space is

$$\mathcal{H}_\alpha^N \;:=\; L^2\Big(\mathbb{R}_+,\; \mathcal{W}_N^{-\alpha}(U)\Big), \tag{6.7}$$

with the inner product inherited from Definition 5.1.38. The *Neumann fractional noise* is a centered Gaussian family $\{W_\alpha^N(\varphi);\, \varphi \in \mathcal{H}_\alpha^N\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance

$$\mathbb{E}\Big[W_\alpha^N(\varphi)\, W_\alpha^N(\psi)\Big] \;=\; \langle \varphi, \psi \rangle_{\mathcal{H}_\alpha^N}, \qquad \varphi, \psi \in \mathcal{H}_\alpha^N. \tag{6.8}$$

Unfolding the inner product of Definition 5.1.38, the expression for the covariance function reads

$$\mathbb{E}\Big[W_\alpha^N(\varphi)\, W_\alpha^N(\psi)\Big] = \int_{\mathbb{R}_+} \int_{U \times U} \varphi(t,x)\, \psi(t,y)\, \Big(G_{2\alpha}^{N,U}(x,y) + C_U\Big)\, d\mu(x)\, d\mu(y)\, dt. \tag{6.9}$$

The map $W_\alpha^N : \mathcal{H}_\alpha^N \to L^2(\Omega)$ is a linear isometry and W_α^N is an isonormal Gaussian process on $\mathcal{H}_\alpha^N$. We write $W_\alpha^N(h) = \int_0^\infty \int_U h(s,x)\, W_\alpha^N(ds, dx)$ for $h \in \mathcal{H}_\alpha^N$.

Remark 6.1.5. *For $\alpha = 0$ both boundary conditions yield the same noise. The formal limit of $\mathcal{W}_D^{-\alpha}(U)$ and $\mathcal{W}_N^{-\alpha}(U)$ as $\alpha \to 0$ is $L^2(U, \mu)$, so both W_α^D and W_α^N degenerate to the space-time white noise W_0 of Definition 6.1.2, with $\mathcal{H}_0^D = \mathcal{H}_0^N = \mathcal{H}_0$. The choice of boundary condition is therefore relevant only for $\alpha > 0$.*

6.1.2 Wiener chaos and Malliavin calculus in the geometric setting

Since each of W_α^D and W_α^N is an isonormal Gaussian process on its respective Hilbert space $\mathcal{H}_\alpha$ (either $\mathcal{H}_\alpha^D$ or $\mathcal{H}_\alpha^N$), the full Wiener chaos framework of Section 2.1.2 carries over verbatim with the abstract $\mathcal{H}$ of that section replaced by the present $\mathcal{H}_\alpha$. We briefly record the main consequences in this subsection.

First, every $F \in L^2(\Omega, \sigma(W_\alpha), \mathbb{P})$ admits a unique Wiener chaos decomposition

$$F \;=\; \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{6.10}$$

where $f_n \in (\mathcal{H}_\alpha)^{\odot n}$ for each $n \geq 1$ and the series converges in $L^2(\Omega)$ (Theorem 2.1.5). The n-th multiple stochastic integral $I_n : \mathcal{H}_\alpha^{\odot n} \to \mathbf{H}_n$ is an isometry satisfying (see (2.12))

$$\mathbb{E}[I_n(f)\, I_m(g)] = \mathbf{1}_{n=m} \cdot n!\, \langle f, g \rangle_{\mathcal{H}_\alpha^{\otimes n}}.$$

The Malliavin derivative D and Skorohod integral δ of Section 2.1.3 are equally available in this setting. For $F = \sum_{n \geq 0} I_n(f_n) \in \mathbb{D}^{1,2}$ the Malliavin derivative acts chaos-level by chaos-level according to

$$DF = \sum_{n=1}^{\infty} n\, I_{n-1}\Big(f_n(\,\cdot\,,\star)\Big), \tag{6.11}$$

where $\star$ denotes the free $\mathcal{H}_\alpha$-variable (see (2.18)). The duality relation (2.19) between D and δ, the product rule (2.21), and Nelson's hypercontractivity estimate (2.24) all carry over without modification. For adapted integrands the Skorohod integral coincides with the Itô–Walsh integral (Proposition 3.1.10), which is the stochastic integral used in the mild formulation of the PAM in the following section.

6.1.3 Inclusion in Sobolev spaces

For this regularity study, we will focus on the Dirichlet setting for conciseness. The Neumann case can be treated along the same lines, replacing $-\Delta_{D,U}$ by $-\Delta_{N,U}$ throughout; we refer to [7] for the details.

The Dirichlet Sobolev space $\mathcal{W}_D^{-\alpha}(U)$ (Definition 5.1.28) and the noise W_α^D (Definition 6.1.1) are defined for any metric measure space satisfying Assumption 5.1.6. On the Sierpiński gasket K, the precise spectral asymptotics of $-\Delta_{D,U}$ allow a sharper analysis: one can characterize, via a classical scale of Sobolev spaces on K, exactly how singular W_α^D is in the spatial variable. We can also identify a pointwise Hölder-continuous realization of the noise when $\alpha > d_h/(2d_w)$. Throughout this subsection we work in the setting of Subsection 5.2.4: K is the Sierpiński gasket of Definition 5.2.13, $V_0 = \{p_1, p_2, p_3\}$ is the boundary, $U = K \setminus V_0$, μ is the self-similar measure of Definition 5.2.14, and $(\lambda_j, \Phi_j)_{j\geq 1}$ are the eigenpairs of $-\Delta_{D,U}$ from Notation 5.1.20.

Let us start by discussing spectral asymptotics. Lemma 5.1.21 recorded the sufficient condition $\alpha > d_h/d_w$ for the convergence of $\sum_j \lambda_j^{-\alpha}$. For the Sierpiński gasket, Fukushima and Shima [22] proved a sharp two-sided bound on the eigenvalue counting function $N(t) := \#\{j : \lambda_j \leq t\}$, turning that condition into an exact equivalence.

Lemma 6.1.6: Weyl law on K

There exist constants $0 < c \leq C < \infty$ such that

$$c\, t^{d_h/d_w} \;\leq\; N(t) \;\leq\; C\, t^{d_h/d_w}, \qquad t > 0. \tag{6.12}$$

As a consequence, for every $\beta > 0$,

$$\sum_{j=1}^{\infty} \lambda_j^{-\beta} \;<\; +\infty \quad \text{if and only if} \quad \beta > \frac{d_h}{d_w}. \tag{6.13}$$

Proof. The bound (6.12) is due to [22]. For (6.13), a Stieltjes integration gives $\sum_j \lambda_j^{-\beta} = \int_0^\infty t^{-\beta}\, dN(t)$. An integration by parts, combined with the two-sided bound (6.12), shows that this integral is finite if and only if $\int_1^\infty t^{-\beta+d_h/d_w-1}\, dt < \infty$, i.e., $\beta > d_h/d_w$. ■

As a direct application of Lemma 6.1.6, we have that $\sum_j \lambda_j^{-\beta-2\alpha} < \infty$ if and only if $\beta > d_h/d_w - 2\alpha$. This sharpens the one-directional statement (5.21) and is the key input for the Sobolev regularity analysis below. Namely we introduce the Hilbert scale on U naturally associated with the Dirichlet eigenvalues, following [6, Sec. 3.1].

Definition 6.1.7: Sobolev spaces on K

For $f \in \mathcal{S}_D(U)$ set $\hat{f}_j := \int_U f\,\Phi_j\,d\mu$, where $(\lambda_j, \Phi_j)_{j\geq 1}$ are the eigenpairs of $-\Delta_{D,U}$ from Notation 5.1.20. For $\alpha \geq 0$, the *Sobolev space* $H^\alpha(K)$ is the completion of $\mathcal{S}_D(U)$ under the norm

$$\|f\|^2_{H^\alpha(K)} := \sum_{j=1}^{\infty} \lambda_j^\alpha\, \hat{f}_j^2. \tag{6.14}$$

Its dual $H^{-\alpha}(K) := (H^\alpha(K))^*$, viewed as a subspace of $\mathcal{S}'_D(U)$, carries the canonical norm

$$\|\psi\|^2_{H^{-\alpha}(K)} := \sum_{j=1}^{\infty} \lambda_j^{-\alpha}\, \psi(\Phi_j)^2. \tag{6.15}$$

Here $\psi(\Phi_j) := \langle \psi, \Phi_j\rangle_{\mathcal{S}'_D(U),\mathcal{S}_D(U)}$ denotes the action of the distribution ψ on the test function Φ_j.

That (6.15) is indeed the dual norm is the content of the next lemma.

Lemma 6.1.8: Dual norm on $H^{-\alpha}(K)$

For every $\psi \in H^{-\alpha}(K)$, the right-hand side of (6.15) is such that

$$\|\psi\|^2_{H^{-\alpha}(K)} = \sum_{j=1}^{\infty} \lambda_j^{-\alpha}\, \psi(\Phi_j)^2 = \sup\{|\psi(f)|^2 : \|f\|_{H^\alpha(K)} \leq 1\}. \tag{6.16}$$

Proof. By the Riesz representation theorem there exists $f_\psi \in H^\alpha(K)$ satisfying $\psi(g) = (f_\psi, g)_{H^\alpha(K)}$ for all $g \in \mathcal{S}_D(U)$, and $\|\psi\|_{H^{-\alpha}} = \|f_\psi\|_{H^\alpha}$. Setting $\hat{f}_j^\psi := \int_U f_\psi \Phi_j\,d\mu$ and evaluating at $g = \Phi_j$ gives

$$\psi(\Phi_j) = (f_\psi, \Phi_j)_{H^\alpha(K)} = \sum_{k=1}^{\infty} \lambda_k^\alpha\, \hat{f}_k^\psi \int_U \Phi_j \Phi_k\,d\mu = \lambda_j^\alpha \hat{f}_j^\psi,$$

where the last step uses the $L^2(U,\mu)$-orthonormality of the functions Φ_j. Therefore we get

$$\|\psi\|^2_{H^{-\alpha}} = \|f_\psi\|^2_{H^\alpha} = \sum_{j=1}^{\infty} \lambda_j^\alpha \left(\hat{f}_j^\psi\right)^2 = \sum_{j=1}^{\infty} \lambda_j^{-\alpha} \psi(\Phi_j)^2.$$

This proves the first identity in (6.16). For the second, fix $f \in H^\alpha(K)$ with $\|f\|_{H^\alpha} \leq 1$. By the Cauchy-Schwarz inequality in $H^\alpha(K)$,

$$|\psi(f)|^2 = |(f_\psi, f)_{H^\alpha}|^2 \leq \|f_\psi\|^2_{H^\alpha}\, \|f\|^2_{H^\alpha} \leq \|f_\psi\|^2_{H^\alpha} = \|\psi\|^2_{H^{-\alpha}},$$

with equality when $f = f_\psi/\|f_\psi\|_{H^\alpha}$. Hence $\sup\{|\psi(f)|^2 : \|f\|_{H^\alpha} \leq 1\} = \|f_\psi\|^2_{H^\alpha} = \|\psi\|^2_{H^{-\alpha}}$, which is the second identity in (6.16). ∎

Remark 6.1.9 (Identification with $H^{-2\alpha}(K)$)**.** *The Sobolev spaces of Definition 6.1.7 are directly related to the Dirichlet Sobolev space of Definition 5.1.28:*

$$\mathcal{W}_D^{-\alpha}(U) = H^{-2\alpha}(K) \quad \textit{(isometrically)}. \tag{6.17}$$

In particular, $\mathcal{H}^D_\alpha = L^2(\mathbb{R}_+, H^{-2\alpha}(K))$, and the noise W^D_α can equivalently be viewed as an isonormal Gaussian process on $L^2(\mathbb{R}_+, H^{-2\alpha}(K))$.

Proof. For $f \in \mathcal{S}_D(U)$, the second line of (5.38) gives

$$\|f\|^2_{\mathcal{W}_D^{-\alpha}(U)} = \int_U \Big[(-\Delta_{D,U})^{-\alpha} f\Big]^2 \, d\mu = \sum_{j=1}^{\infty} \lambda_j^{-2\alpha} \hat{f}_j^2,$$

where the second step uses $(-\Delta_{D,U})^{-\alpha} f = \sum_j \lambda_j^{-\alpha} \hat{f}_j \Phi_j$ and the $L^2(U,\mu)$-orthonormality of (Φ_j) (Notation 5.1.20). Applying (6.15) with α replaced by 2α and noting that $f(\Phi_j) = \hat{f}_j$ for $f \in \mathcal{S}_D(U)$,

$$\|f\|^2_{H^{-2\alpha}(K)} = \sum_{j=1}^{\infty} \lambda_j^{-2\alpha} \hat{f}_j^2.$$

Since both norms coincide on the dense subspace $\mathcal{S}_D(U)$, the two completions are isometrically equal. ∎

We now turn to the Sobolev regularity of W^D_α, which will be more easily expressed thanks to another piece of notation.

Notation 6.1.10: Time-integrated noise

For $t > 0$, the *time-integrated noise* $W^D_\alpha[t]$ is the centred Gaussian family

$$W^D_\alpha[t] := \Big\{W^D_\alpha(\mathbf{1}_{[0,t]} \otimes f) : f \in \mathcal{S}_D(U)\Big\}. \tag{6.18}$$

We also set $\xi_j(t) := W^D_\alpha(\mathbf{1}_{[0,t]} \otimes \Phi_j)$.

The key consequence of Notation 6.1.10 is that the variables $\xi_j(t)$ are independent centred Gaussians, which makes the Sobolev regularity of $W^D_\alpha[t]$ tractable.

Proposition 6.1.11: Sobolev regularity of W^D_α

Let $0 \leq \alpha \leq d_h/(2d_w)$ and $t > 0$. The time-integrated noise $W^D_\alpha[t]$ (Notation 6.1.10) belongs almost surely to $H^{-\beta}(K)$ for every $\beta > d_h/d_w - 2\alpha$.

Proof. Recall that $(\lambda_j, \Phi_j)_{j\geq 1}$ are the eigenpairs of $-\Delta_{D,U}$ (Notation 5.1.20). The isometry (6.4) and the inner product formula (5.38) give

$$\mathbb{E}\Big[\xi_j(t)\,\xi_k(t)\Big] = t\,\langle \Phi_j, \Phi_k\rangle_{\mathcal{W}_D^{-\alpha}} = t\,\lambda_j^{-2\alpha}\delta_{jk}, \tag{6.19}$$

so the $\xi_j(t)$ are independent centred Gaussians with variance $t\lambda_j^{-2\alpha}$. Lemma 6.1.8 then identifies $\|W_\alpha^D[t]\|_{H^{-\beta}}^2 = \sum_j \lambda_j^{-\beta}\xi_j(t)^2$. Taking expectations and using (6.19) enables to write

$$\mathbb{E}\Big[\|W_\alpha^D[t]\|_{H^{-\beta}}^2\Big] = \sum_{j=1}^{\infty}\lambda_j^{-\beta}\,\mathbb{E}\Big[\xi_j(t)^2\Big] = t\sum_{j=1}^{\infty}\lambda_j^{-\beta-2\alpha}.$$

By Lemma 6.1.6, this series converges if and only if $\beta+2\alpha > d_h/d_w$, i.e., $\beta > d_h/d_w - 2\alpha$. For β strictly above this threshold, $\mathbb{E}[\|W_\alpha^D[t]\|_{H^{-\beta}}^2] < \infty$ and hence $W_\alpha^D[t] \in H^{-\beta}(K)$ almost surely. ■

As α increases toward $d_h/(2d_w)$, the threshold $d_h/d_w - 2\alpha$ decreases to 0. At the log-correlated value $\alpha = d_h/(2d_w)$ (Remark 6.1.3), $W_\alpha^D[t]$ belongs to $H^{-\beta}(K)$ for every $\beta > 0$, but not to $L^2(U,\mu) = H^0(K)$. The regularity at this critical value is more delicate to handle, and we skip this topic for the sake of conciseness. When $\alpha > d_h/(2d_w)$, Lemma 6.1.6 gives $\sum_j \lambda_j^{-2\alpha} < \infty$, which implies that $G_{2\alpha}^{D,U}(x,x) = \sum_j \lambda_j^{-2\alpha}\Phi_j(x)^2$ is finite for each $x \in U$, and hence $G_\alpha^{D,U}(x,\cdot) \in L^2(U,\mu)$ for each x. This allows one to evaluate W_α^D pointwise in space, leading to the following definition.

Definition 6.1.12: Density field of W_α^D

For $\alpha > d_h/(2d_w)$, the *density field* of W_α^D is the process

$$\tilde{W}_\alpha^D(t,x) := W_0\Big(\mathbf{1}_{[0,t]} \otimes G_\alpha^{D,U}(x,\cdot)\Big), \qquad t > 0,\ x \in U, \tag{6.20}$$

where W_0 is the white noise of Definition 6.1.2. By the bilinear formula (5.26), its covariance is

$$\mathbb{E}\Big[\tilde{W}_\alpha^D(t,x)\,\tilde{W}_\alpha^D(s,y)\Big] = \min(t,s)\,G_{2\alpha}^{D,U}(x,y), \tag{6.21}$$

which coincides with the covariance of W_α^D applied to point masses: the field $\tilde{W}_\alpha^D$ is the pointwise spatial density of the noise.

With the density field in hand, we turn to its spatial regularity. Since $\tilde{W}_\alpha^D(t,x) = W_0(\mathbf{1}_{[0,t]} \otimes G_\alpha^{D,U}(x,\cdot))$, the regularity of $x \mapsto \tilde{W}_\alpha^D(t,x)$ is governed by the regularity of $x \mapsto G_\alpha^{D,U}(x,\cdot)$ in $L^2(U,\mu)$, which in turn reflects the Hölder continuity of the operator $(-\Delta_{D,U})^{-\alpha} : L^2(U,\mu) \to C(U)$. The natural Hölder exponent for this operator is

$$H_\alpha := \min\Big(\alpha\, d_w - \frac{d_h}{2},\ d_w - d_h\Big), \tag{6.22}$$

which is positive for $\alpha > d_h/(2d_w)$, grows linearly in α in the subcritical regime, and saturates at the value $d_w - d_h$ once $\alpha \geq 1 - d_h/(2d_w)$. The following theorem makes this precise.

Theorem 6.1.13: Hölder regularity of $(-\Delta_{D,U})^{-\alpha}$

Let $\alpha > d_h/(2d_w)$, and let H_α be as in (6.22). There exists a constant $C > 0$ such that for every $f \in L^2(U,\mu)$ and $x, y \in U$ (where d is the metric of Definition 5.1.1, which can be reduced to the Euclidean distance in the case of the Sierpiński gasket; see Section 5.2.4):

(i) If $\alpha \neq 1 - d_h/(2d_w)$:

$$\Big|(-\Delta_{D,U})^{-\alpha} f(x) - (-\Delta_{D,U})^{-\alpha} f(y)\Big| \;\leq\; C\, d(x,y)^{H_\alpha}\, \|f\|_{L^2(U,\mu)}. \tag{6.23}$$

(ii) If $\alpha = 1 - d_h/(2d_w)$, for $d(x,y) \leq 1/2$:

$$\Big|(-\Delta_{D,U})^{-\alpha} f(x) - (-\Delta_{D,U})^{-\alpha} f(y)\Big| \;\leq\; C\, d(x,y)^{H_\alpha}\, |\ln d(x,y)|\, \|f\|_{L^2(U,\mu)}. \tag{6.24}$$

Proof. See [6, Theorem 3.4]. Throughout, the integral representation (5.24) gives

$$\Big|(-\Delta_{D,U})^{-\alpha} f(x) - (-\Delta_{D,U})^{-\alpha} f(y)\Big| \;\leq\; \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \Big| P_t^{D,U} f(x) - P_t^{D,U} f(y)\Big|\, dt.$$

We will use two bounds. The first is the following pointwise bound: by Cauchy–Schwarz and Corollary 5.1.18, it is readily checked that

$$|P_t^{D,U} f(x)| \;\leq\; \Big(\int_U p_t^{D,U}(x,z)^2\, d\mu(z)\Big)^{1/2} \|f\|_{L^2} \;\leq\; C\, \frac{e^{-\lambda_1 t}}{\min(1, t^{d_h/d_w})^{1/2}}\, \|f\|_{L^2}. \tag{6.25}$$

In particular, for $t \leq 1$ this simplifies to $|P_t^{D,U} f(x)| \leq C\, t^{-d_h/(2d_w)} \|f\|_{L^2}$. The second is the semigroup Hölder bound (see [4, 6]):

$$\Big|P_t^{D,U} f(x) - P_t^{D,U} f(y)\Big| \;\leq\; C\, e^{-\lambda_1 t/2}\, \frac{d(x,y)^{d_w - d_h}}{t^{1-d_h/(2d_w)}}\, \|f\|_{L^2}. \tag{6.26}$$

Set $\delta := d(x,y)^{d_w}$, so $\delta \leq 1$ when $d(x,y) \leq 1$. We now divide the proof in several steps.

Step 1: $\alpha < 1 - d_h/(2d_w)$. Split the time integral at δ:

$$\int_0^\infty t^{\alpha-1} \Big|P_t^{D,U} f(x) - P_t^{D,U} f(y)\Big|\, dt \;=\; I_1 + I_2, \tag{6.27}$$

where $I_1 := \int_0^\delta t^{\alpha-1} |P_t^{D,U} f(x) - P_t^{D,U} f(y)|\, dt$ and $I_2 := \int_\delta^\infty t^{\alpha-1} |P_t^{D,U} f(x) - P_t^{D,U} f(y)|\, dt$.

For I_1, bound the integrand by $|P_t^{D,U} f(x)| + |P_t^{D,U} f(y)|$ and apply (6.25) with $t \leq \delta \leq 1$. We obtain

$$I_1 \;\leq\; C \int_0^\delta t^{\alpha-1-d_h/(2d_w)}\, dt\, \|f\|_{L^2} \;=\; \frac{C}{\alpha - d_h/(2d_w)}\, \delta^{\alpha - d_h/(2d_w)}\, \|f\|_{L^2},$$

where convergence at 0 uses $\alpha > d_h/(2d_w)$. Substituting $\delta = d(x,y)^{d_w}$, we discover that

$$I_1 \;\le\; C\,d(x,y)^{d_w(\alpha - d_h/(2d_w))}\,\|f\|_{L^2} \;=\; C\,d(x,y)^{\alpha d_w - d_h/2}\,\|f\|_{L^2} \;=\; C\,d(x,y)^{H_\alpha}\,\|f\|_{L^2}.$$

For I_2 (the second term in (6.27)), apply (6.26) and drop $e^{-\lambda_1 t/2} \le 1$. This yields

$$I_2 \;\le\; C\,d(x,y)^{d_w - d_h} \int_\delta^\infty t^{\alpha - 2 + d_h/(2d_w)}\,dt\,\|f\|_{L^2}.$$

Since $\alpha < 1 - d_h/(2d_w)$, the exponent satisfies $\alpha - 2 + d_h/(2d_w) < -1$, so the integral converges at ∞:

$$\int_\delta^\infty t^{\alpha-2+d_h/(2d_w)}\,dt \;=\; \frac{\delta^{\alpha-1+d_h/(2d_w)}}{1-\alpha-d_h/(2d_w)}.$$

Substituting $\delta = d(x,y)^{d_w}$ and computing the exponent, we end up with

$$\begin{aligned} I_2 \;&\le\; C\,d(x,y)^{d_w-d_h}\cdot d(x,y)^{d_w(\alpha-1+d_h/(2d_w))}\,\|f\|_{L^2} \\ &=\; C\,d(x,y)^{\alpha d_w - d_h/2}\,\|f\|_{L^2} \;=\; C\,d(x,y)^{H_\alpha}\,\|f\|_{L^2}. \end{aligned}$$

Plugging the bounds on I_1 and I_2 into (6.27) yields (6.23) in this sub-case.

Step 2: $\alpha > 1 - d_h/(2d_w)$. No splitting is needed for this case. Specifically, since $\alpha - 2 + d_h/(2d_w) > -1$, applying (6.26) over all of $[0,\infty)$ gives

$$\left|(-\Delta_{D,U})^{-\alpha} f(x) - (-\Delta_{D,U})^{-\alpha} f(y)\right| \;\le\; C\,d(x,y)^{d_w-d_h}\int_0^\infty t^{\alpha-2+d_h/(2d_w)} e^{-\lambda_1 t/2}\,dt\,\|f\|_{L^2}. \tag{6.28}$$

The remaining integral is a Gamma integral, namely

$$\int_0^\infty t^{\alpha-2+d_h/(2d_w)} e^{-\lambda_1 t/2}\,dt = (\lambda_1/2)^{-(\alpha-1+d_h/(2d_w))}\Gamma(\alpha-1+d_h/(2d_w)) < \infty.$$

Since $H_\alpha = d_w - d_h$ in this regime, relation (6.28) trivially yields (6.23).

Step 3: $\alpha = 1 - d_h/(2d_w)$. Use the same split (6.27) at δ. The bound for I_1 is identical to Step 1 and gives $I_1 \le C\,d(x,y)^{d_w-d_h}\,\|f\|_{L^2}$. For I_2, observe that $\alpha - 2 + d_h/(2d_w) = -1$, so (6.26) gives

$$I_2 \;\le\; C\,d(x,y)^{d_w-d_h}\int_\delta^\infty t^{-1}\,e^{-\lambda_1 t/2}\,dt\,\|f\|_{L^2}.$$

Splitting the remaining integral at $t = 1$ and bounding $e^{-\lambda_1 t/2} \le 1$ on $[\delta, 1]$, we get

$$\int_\delta^1 t^{-1}\,dt + \int_1^\infty t^{-1} e^{-\lambda_1 t/2}\,dt \;=\; |\ln\delta| + C_1 \;\le\; C\,|\ln\delta|,$$

for $\delta \le 1/2$. Substituting $\delta = d(x,y)^{d_w}$ gives $|\ln\delta| = d_w\,|\ln d(x,y)|$, so

$$I_2 \le C\,d(x,y)^{d_w-d_h}\,|\ln d(x,y)|\,\|f\|_{L^2}.$$

Since I_2 dominates I_1, this yields (6.24).

Combining Steps 1-3, in all three regimes we have obtained

$$\left|(-\Delta_{D,U})^{-\alpha} f(x) - (-\Delta_{D,U})^{-\alpha} f(y)\right| \le C\,d(x,y)^{H_\alpha}\,\|f\|_{L^2(U,\mu)},$$

with an additional factor $|\ln d(x,y)|$ at criticality. This is exactly the content of (6.23)–(6.24). ■

By $L^2(U,\mu)$ duality, the Hölder bounds (6.23)–(6.24) on $(-\Delta_{D,U})^{-\alpha}$ transfer to the L^2-increments of the Riesz kernel $G_\alpha^{D,U}$.

Corollary 6.1.14: Hölder regularity of $G_\alpha^{D,U}$

Under the hypotheses of Theorem 6.1.13, with H_α as in (6.22), there exists $C > 0$ such that for all $x, y \in U$:

(i) If $\alpha \neq 1 - d_h/(2d_w)$:

$$\int_U \Big(G_\alpha^{D,U}(x,z) - G_\alpha^{D,U}(y,z)\Big)^2 \, d\mu(z) \;\le\; C\, d(x,y)^{2H_\alpha}. \tag{6.29}$$

(ii) If $\alpha = 1 - d_h/(2d_w)$, for $d(x,y) \le 1/2$:

$$\int_U \Big(G_\alpha^{D,U}(x,z) - G_\alpha^{D,U}(y,z)\Big)^2 \, d\mu(z) \;\le\; C\, d(x,y)^{2H_\alpha}\, |\ln d(x,y)|^2. \tag{6.30}$$

Proof. The action formula (5.25) gives, for any $f \in L^2(U,\mu)$,

$$\int_U \Big(G_\alpha^{D,U}(x,z) - G_\alpha^{D,U}(y,z)\Big) f(z)\, d\mu(z) = (-\Delta_{D,U})^{-\alpha} f(x) - (-\Delta_{D,U})^{-\alpha} f(y).$$

Taking the supremum over $\|f\|_{L^2} = 1$ identifies the $L^2(U,\mu)$-norm of $G_\alpha^{D,U}(x,\cdot) - G_\alpha^{D,U}(y,\cdot)$ with the operator modulus of continuity of $(-\Delta_{D,U})^{-\alpha}$, and the three bounds follow directly from Theorem 6.1.13. ∎

The corollary controls the second moment of the increment of the density field: by (6.21) and Corollary 6.1.14,

$$\mathbb{E}\Big[(\tilde W_\alpha^D(t,x) - \tilde W_\alpha^D(t,y))^2\Big] = t \int_U \Big(G_\alpha^{D,U}(x,z) - G_\alpha^{D,U}(y,z)\Big)^2 \, d\mu(z). \tag{6.31}$$

Applying the entropy method of [1] to this Gaussian field yields the following result.

Theorem 6.1.15: Hölder continuity of $\tilde W_\alpha^D$

Let $\alpha > d_h/(2d_w)$ and $t > 0$. Define the modulus

$$\omega_\alpha(r) := \begin{cases} r^{H_\alpha} \sqrt{|\ln r|}, & \alpha \neq 1 - d_h/(2d_w), \\ r^{d_w - d_h}\, |\ln r|^{3/2}, & \alpha = 1 - d_h/(2d_w), \end{cases} \tag{6.32}$$

with H_α as in (6.22). There exists an almost surely continuous version of $x \mapsto \tilde W_\alpha^D(t,x)$, still denoted $\tilde W_\alpha^D(t,\cdot)$, such that

$$\lim_{\delta\to 0} \sup_{\substack{x,y\in U,\\ 0<d(x,y)\le\delta}} \frac{|\tilde W_\alpha^D(t,x) - \tilde W_\alpha^D(t,y)|}{\omega_\alpha(d(x,y))} < +\infty \quad \text{almost surely.} \tag{6.33}$$

Proof. See [6, Thm. 3.8]. Set $\rho(x,y) := \mathbb{E}[(\tilde{W}^D_\alpha(t,x) - \tilde{W}^D_\alpha(t,y))^2]^{1/2}$. By (6.31) and Corollary 6.1.14, one has $\rho(x,y) \le C\sqrt{t}\, d(x,y)^{H_\alpha}$ (with an additional logarithmic factor at $\alpha = 1 - d_h/(2d_w)$). Let $\mathcal{N}_\rho(\varepsilon)$ denote the smallest number of ρ-balls of radius ε needed to cover U. Since $\rho(x,y) \le C\sqrt{t}\, d(x,y)^{H_\alpha}$, the metric ρ and the intrinsic metric d are Hölder-equivalent, giving $\mathcal{N}_\rho(\varepsilon) = O(\varepsilon^{-d_h/H_\alpha})$. The Dudley entropy integral bound [1, Theorem 1.3.5] then yields

$$\sup_{\rho(x,y)\le\tau} |\tilde{W}^D_\alpha(t,x) - \tilde{W}^D_\alpha(t,y)| \;\le\; C' \int_0^\tau \sqrt{\ln \mathcal{N}_\rho(\varepsilon)}\, d\varepsilon,$$

and an integration-by-parts argument applied to the right-hand side gives the modulus ω_α; see the proof of [6, Thm. 3.8] for the complete computation. ■

Remark 6.1.16. *The Hölder regularity result of Theorem 6.1.15 has a natural counterpart in the Euclidean setting that we have not developed explicitly. If W is a Gaussian noise on $\mathbb{R}^d$ with covariance $\mathbb{E}[W(A)W(B)] = \int_{A\times B} f(x-y)\,dx\,dy$ for a smooth non-negative definite function f, then the associated density field $x \mapsto W(f_x)$ (where $f_x(\cdot) = f(x-\cdot)$) is a stationary Gaussian process whose regularity is governed by the decay of f near the origin. This type of analysis is standard for Gaussian processes on $\mathbb{R}^d$ (see Section 2.3 for the framework used in this course) and can be carried out by the same entropy method as in the proof of Theorem 6.1.15. We have chosen not to include it here for the sake of conciseness.*

6.2 Existence and uniqueness

This section establishes existence and uniqueness for the parabolic Anderson model on a bounded domain $U \subset X$. In Chapter 3 we proved this result for the PAM on $\mathbb{R}^d$ via a Banach fixed-point argument (Theorem 3.3.5), and in Section 4.1 we revisited it via the Wiener chaos expansion (Theorem 4.1.3). We follow the chaos route here, since it simultaneously yields the L^2-moment formula that drives the moment estimates of Section 6.3.

All the necessary ingredients are now available: the Dirichlet noise W^D_α and its Hilbert space $\mathcal{H}^D_\alpha$ (Section 6.1), the Dirichlet heat kernel $p^{D,U}_t$ with its L^∞ bound (Corollary 5.1.19), the Riesz kernel $G^{D,U}_{2\alpha}$ with the L^2-operator estimate of Lemma 5.1.27, and the chaos framework adapted to the geometric setting (Section 6.1.2). The key challenge that distinguishes the geometric proof from the Euclidean one is the absence of Fourier analysis: the norms $\|f_k\|^2_{(\mathcal{H}^D_\alpha)^{\otimes k}}$ cannot be evaluated via a spectral measure as in Theorem 4.1.3. Instead, Lemma 5.1.27 serves as a substitute, leading to a renewal-type recursion in physical space that mimics the role of Dalang's condition (2.2).

Section 6.2.1 sets up the mild equation, recalls the solution concept, and reduces existence and uniqueness to the convergence of a chaos series. Section 6.2.2 establishes that convergence via the renewal argument and deduces the L^2-moment bound.

6.2.1 Mild formulation and chaos characterisation

Recall that we now turn to the parabolic Anderson model on the bounded domain $U \subset X$. The equation of interest is

$$\partial_t u(t,x) \;=\; \Delta_{D,U}\, u(t,x) \;+\; \beta\, u(t,x)\, \dot{W}^D_\alpha(t,x), \qquad (t,x) \in \mathbb{R}_+ \times U, \tag{6.34}$$

where $\beta \geq 0$, $\alpha \geq 0$, and the initial condition is $u(0,\cdot) = u_0 \in L^\infty(U,\mu)$. For $\alpha = 0$ the noise is the space-time white noise W_0 of Definition 6.1.2; for $\alpha > 0$ it is the colored noise of Definition 6.1.1.

Equation (6.34) is solved in its *mild form.* That is, using the Dirichlet semigroup $P^{D,U}_t$ and heat kernel $p^{D,U}_t$ (Theorem 5.1.15), we rewrite (6.34) as the integral equation

$$u(t,x) \;=\; P^{D,U}_t u_0(x) \;+\; \beta \int_{[0,t]\times U} p^{D,U}_{t-s}(x,y)\, u(s,y)\, W^D_\alpha(ds,dy). \tag{6.35}$$

The stochastic integral in (6.35) is understood in the Itô–Walsh sense; since the integrand $p^{D,U}_{t-s}(x,y)u(s,y)$ is adapted, Proposition 3.1.10 identifies it with the Skorohod integral δ (see also Section 6.1.2). Observe that equation (6.35) is the geometric counterpart of the Euclidean mild equation (3.1):

$$u(t,x) = P_t u_0(x) + \beta \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, u(s,y)\, W(ds,dy). \tag{6.36}$$

Both equations have the same Duhamel (variation-of-constants) structure, but one faces two major changes when passing from $\mathbb{R}^d$ to (U, d_U, μ).

(i) *Heat kernel.* In (6.36) the free kernel $p_{t-s}(x-y)$ depends only on the difference $x-y$ (translation invariance) and is a probability density for all times. In (6.35) the kernel $p^{D,U}_{t-s}(x,y)$ is not translation-invariant (it depends on the geometry of U), and it satisfies the Dirichlet boundary condition, which forces an exponential decay $p^{D,U}_t(x,y) \leq Ce^{-\lambda_1 t}$ for large t (Theorem 5.1.15, equation (5.14)).

(ii) *Noise and spatial covariance.* In (6.36) the noise W has spatial covariance given by the kernel Λ (see (2.1)), and the natural space-time Hilbert space $\mathcal{H}$ is built from the spectral measure μ via Fourier analysis. In (6.35) the noise W^D_α has spatial covariance $G^{D,U}_{2\alpha}$ (Definition 6.1.1, equation (6.3)), and $\mathcal{H}^D_\alpha$ is the space-time Sobolev space $L^2(\mathbb{R}_+, \mathcal{W}^{-\alpha}_D(U))$. In the Euclidean setting the integrability of the free kernel against $\mathcal{H}$ is guaranteed by Dalang's condition (2.2); in the geometric setting it is guaranteed by the $L^2(U,\mu)$-norm bound of Lemma 5.1.27, which plays the same role.

Despite these differences, the notion of solution carries over without change: a process u solves (6.35) in the Itô–Skorohod sense if it satisfies the same five conditions as in the Euclidean case (adaptedness, joint measurability, L^2-finiteness of the stochastic integral, L^2-continuity, and almost-sure equality), with W^D_α and $\mathcal{H}^D_\alpha$ in place of W and $\mathcal{H}$. Those conditions are spelled out below.

Definition 6.2.1: Itô–Skorohod solution

A process $u = \{u(t,x);\, (t,x) \in \mathbb{R}_+ \times U\}$ is a *random field solution* of (6.34) in the Itô–Skorohod sense if:

(i) u is adapted to the filtration generated by W_α^D;

(ii) $(t,x) \mapsto u(t,x)$ is jointly measurable;

(iii) $\mathbb{E}[I(t,x)^2] < \infty$ for all (t,x), where $I(t,x)$ denotes the stochastic integral in (6.35);

(iv) $(t,x) \mapsto I(t,x)$ is continuous in $L^2(\Omega)$;

(v) u satisfies (6.35) almost surely for all (t,x).

Chaos kernels. By a standard iteration of (6.35) and the chaos decomposition of Section 6.1.2 (see also Section 4.1.1 for the analogous computation in the Euclidean setting), any solution u has the Wiener chaos expansion

$$u(t,x) \;=\; P_t^{D,U} u_0(x) \;+\; \sum_{k=1}^{\infty} I_k(f_k(\,\cdot\,,t,x)), \tag{6.37}$$

where the chaos kernels $f_k \in (\mathcal{H}_\alpha^D)^{\odot k}$ are given, on the ordered simplex $\{0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < t\}$, by

$$f_k(s_1,y_1,\ldots,s_k,y_k;\,t,x) = \frac{\beta^k}{k!}\, p^{D,U}_{t-s_{\sigma(k)}}(x,y_{\sigma(k)}) \cdots p^{D,U}_{s_{\sigma(2)}-s_{\sigma(1)}}(y_{\sigma(2)},y_{\sigma(1)})\, P^{D,U}_{s_{\sigma(1)}} u_0(y_{\sigma(1)}), \tag{6.38}$$

where σ is the permutation of $\{1,\ldots,k\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(k)} < t$, and $f_k := 0$ off the simplex. This is the geometric analogue of (4.6).

The following proposition characterizes solutions via the square-summability of the chaos series. Its proof is identical to the Euclidean case (Theorem 4.1.3): the stochastic integral in (6.35) admits a Wiener chaos expansion by the same algebraic iteration, and orthogonality of distinct chaos levels (6.10) yields the L^2-moment formula.

Proposition 6.2.2: Chaos characterisation

A process u with $\mathbb{E}[u(t,x)^2] < \infty$ for all (t,x) solves (6.35) in the sense of Definition 6.2.1 if and only if, for every (t,x), the random variable $u(t,x)$ has the chaos expansion (6.37)–(6.38) with

$$\sum_{k=1}^{\infty} k!\,\|f_k(\,\cdot\,,t,x)\|^2_{(\mathcal{H}_\alpha^D)^{\otimes k}} \;<\; +\infty. \tag{6.39}$$

Here $\mathcal{H}_\alpha^D$ is the space-time Hilbert space introduced in Definition 6.1.1. Moreover, the unique solution satisfies the L^2-moment formula

$$\mathbb{E}\Big[u(t,x)^2\Big] \;=\; (P_t^{D,U}u_0(x))^2 \;+\; \sum_{k=1}^{\infty} k!\,\|f_k(\,\cdot\,,t,x)\|^2_{(\mathcal{H}_\alpha^D)^{\otimes k}}. \tag{6.40}$$

6.2.2 Chaos estimates and existence

According to Proposition (6.2.2), the key step to get existence and uniqueness results for (6.35) is reduced to prove (6.39). Now recall that in the Euclidean setting the norm $\|f_n\|^2_{\mathcal{H}^{\otimes n}}$ is computed via Fourier transforms and the spectral measure μ (see Theorem 4.1.3). In our general geometric setting there is no natural Fourier calculus. Instead we use the L^2-estimate of Lemma 5.1.27 as a substitute, establishing a recursive bound directly in physical space.

Lemma 6.2.3: Kernel norm estimate

Let $\alpha \geq 0$ and $\rho > 0$. There exists a constant $C > 0$ depending only on α, d_h, d_w, and U such that for all $k \geq 1$, $t \geq 0$, and $x \in U$,

$$\|f_k(\,\cdot\,,t,x)\|^2_{(\mathcal{H}_\alpha^D)^{\otimes k}} \;\leq\; \frac{C^k\,\beta^{2k}\,\|u_0\|_\infty^2}{k!}\,\hat{\Psi}(\rho)^k\,e^{(\rho-2\lambda_1)t}, \tag{6.41}$$

where $\lambda_1 = \lambda_1(U)$ is the first eigenvalue of $-\Delta_{D,U}$ (Notation 5.1.20) and $\hat{\Psi}(\rho)$ is the Laplace transform

$$\hat{\Psi}(\rho) := \int_0^\infty e^{-\rho t}\,\Psi(t)\,dt, \quad \text{where} \quad \Psi(t) := e^{2\lambda_1 t}\sup_{x\in U}\|(-\Delta_{D,U})^{-\alpha}p_t^{D,U}(x,\cdot)\|^2_{L^2(U,\mu)}. \tag{6.42}$$

Proof. We drop the superscripts D, U for legibility and write p_t, P_t, Δ, $G_{2\alpha}$ for $p_t^{D,U}$, $P_t^{D,U}$, $\Delta_{D,U}$, $G_{2\alpha}^{D,U}$ throughout.

Step 1: Reduction to $M_k(t,x)$. This step is the geometric counterpart of the simplex reduction performed in the Euclidean setting (see (4.22)–(4.23)). Since $f_k \in (\mathcal{H}_\alpha^D)^{\odot k}$ is symmetric in its k pairs (s_i, y_i) by (6.38), the full cube $[0,t]^k$ decomposes into $k!$ congruent simplices, just as $[0,t]^n$ decomposed into $n!$ copies of $T_n(t)$ (the notation comes from (4.21) in the Euclidean case). Restricting to the ordered simplex

$$[0,t]^k_< := \{(s_1,\dots,s_k)\in[0,t]^k : 0<s_1<\cdots<s_k<t\},$$

and multiplying by $k!$ recovers the full norm, exactly as in (4.22). Moreover, on $[0,t]^k_<$ the permutation σ in (6.38) is the identity, so the heat-kernel chain simplifies to

$$g_k(s,y;\,t,x) := p_{t-s_k}(x,y_k)\cdots p_{s_2-s_1}(y_2,y_1), \tag{6.43}$$

which is the geometric analogue of K_n on the simplex (4.23). Thus, on $[0,t]^k_<$, the kernel (6.38) takes the explicit form

$$f_k(s_1,y_1,\dots,s_k,y_k;\,t,x) \;=\; \frac{\beta^k}{k!}\,g_k(s,y;\,t,x)\,P_{s_1}u_0(y_1). \tag{6.44}$$

We bound each of the two factors in the right-hand side of (6.44) separately.

(i) *The kernel factor* $g_k(s,y;\,t,x)$. Since f_k is symmetric, the norm $\|f_k\|^2_{(\mathcal{H}^D_\alpha)^{\otimes k}}$ equals $k!$ times the integral of $f_k(s,y)\prod_i G_{2\alpha}(y_i,y_i')\,f_k(s,y')$ over $[0,t]^k_<\times U^{2k}$, exactly as in (4.22). After substituting (6.44) for both occurrences of f_k, the factor of $k!$ from symmetry cancels one power of $k!$ in the denominator, and the remaining integral over the chain of heat kernels g_k defines $M_k(t,x)$ below in (6.46).

(ii) *The initial condition factor* $P_{s_1}u_0(y_1)$. In the Euclidean case this factor was bounded directly by $\|u_0\|_\infty$ (4.18), since the flat heat semigroup is an L^∞ contraction. Here we instead apply Corollary 5.1.19, which gives $|P_{s_1}u_0(y_1)|\le Ce^{-\lambda_1 s_1}\|u_0\|_\infty$: the exponential decay $e^{-\lambda_1 s_1}$ reflects the spectral gap of $-\Delta_{D,U}$ and is what eventually shifts the exponent from ρ to $\rho-2\lambda_1$ in (6.41).

Gathering those two arguments into (6.44), we end up with the following upper bound:

$$\|f_k(\,\cdot\,,t,x)\|^2_{(\mathcal{H}^D_\alpha)^{\otimes k}} \;\le\; C\,\frac{\beta^{2k}}{k!}\,\|u_0\|^2_\infty\,M_k(t,x), \tag{6.45}$$

where

$$M_k(t,x) := \int_{[0,t]^k_<}\int_{U^{2k}} g_k(s,y;\,t,x)\prod_{i=1}^k G_{2\alpha}(y_i,y_i')\,g_k(s,y';\,t,x)\,e^{-2\lambda_1 s_1}\,d\mu(y)\,d\mu(y')\,ds. \tag{6.46}$$

Step 2: Recursion. The aim of this step is to prove that

$$e^{2\lambda_1 t}M_{k+1}(t,x) \;\le\; \int_0^t \Psi(t-s)\cdot e^{2\lambda_1 s}\sup_{y\in U} M_k(s,y)\,ds, \tag{6.47}$$

where Ψ is defined in (6.42) and we initiate with $M_0(t,x) := e^{-2\lambda_1 t}$. To this aim, we will use the chain structure of g_k in (6.43) and the covariance formula (6.3). Indeed, starting from (6.46), the following recursion is easily seen:

$$\begin{aligned} e^{2\lambda_1 t}M_{k+1}(t,x) = \int_0^t e^{2\lambda_1(t-s_{k+1})}\int_{U^2} & p_{t-s_{k+1}}(x,y_{k+1})G_{2\alpha}(y_{k+1},y_{k+1}')\,p_{t-s_{k+1}}(x,y_{k+1}') \\ &\cdot e^{2\lambda_1 s_{k+1}}M_k(s_{k+1},y_{k+1})\,d\mu(y_{k+1})\,d\mu(y_{k+1}')\,ds_{k+1}. \end{aligned} \tag{6.48}$$

Let us explain the exponential factors in (6.48). The factor $e^{2\lambda_1 t}$ on the left-hand side is introduced to normalize M_{k+1}, and on the right-hand side it is split as $e^{2\lambda_1 t} = e^{2\lambda_1(t-s_{k+1})} \cdot e^{2\lambda_1 s_{k+1}}$: the first part accompanies the outer heat kernels and will be absorbed into $\Psi(t-s_{k+1})$ below, while the second part is grouped with $M_k(s_{k+1}, y_{k+1})$ to form the normalized inner quantity $e^{2\lambda_1 s_{k+1}} M_k(s_{k+1}, y_{k+1})$ appearing in (6.47). The factor $e^{-2\lambda_1 s_1}$ from definition (6.46), where s_1 is the smallest time in the inner k-step chain, is still present inside $M_k(s_{k+1}, y_{k+1})$ and is a distinct quantity from the outer prefactor $e^{2\lambda_1 s_{k+1}}$.

Since $M_k(s_{k+1}, y_{k+1})$ depends on y_{k+1}, we cannot apply the bilinear formula (5.26) to the double spatial integral directly. We first bound $M_k(s_{k+1}, y_{k+1}) \le \sup_{y\in U} M_k(s_{k+1}, y)$, which decouples the two spatial integration variables y_{k+1} and y'_{k+1}. Once this substitution is made, the bilinear formula (5.26) identifies the remaining double spatial integral as

$$\int_{U^2} p_{t-s}(x,y)\, G_{2\alpha}(y,y')\, p_{t-s}(x,y')\, d\mu(y)\, d\mu(y') = \|(-\Delta)^{-\alpha} p_{t-s}(x,\cdot)\|^2_{L^2(U,\mu)}. \tag{6.49}$$

Multiplying by the prefactor $e^{2\lambda_1(t-s_{k+1})}$ and taking the supremum over $x \in U$, definition (6.42) gives

$$e^{2\lambda_1(t-s_{k+1})} \sup_{x\in U} \|(-\Delta_{D,U})^{-\alpha} p^{D,U}_{t-s_{k+1}}(x,\cdot)\|^2_{L^2(U,\mu)} = \Psi(t-s_{k+1}). \tag{6.50}$$

Plugging (6.49) and (6.50) into (6.48), and using $M_k(s_{k+1}, y_{k+1}) \le \sup_{y\in U} M_k(s_{k+1}, y)$, gives

$$\begin{aligned} e^{2\lambda_1 t} M_{k+1}(t,x) &\le \int_0^t e^{2\lambda_1(t-s_{k+1})} \|(-\Delta)^{-\alpha} p_{t-s_{k+1}}(x,\cdot)\|^2_{L^2(U,\mu)} \cdot e^{2\lambda_1 s_{k+1}} \sup_{y\in U} M_k(s_{k+1}, y)\, ds_{k+1} \\ &\le \int_0^t \Psi(t-s_{k+1}) \cdot e^{2\lambda_1 s_{k+1}} \sup_{y\in U} M_k(s_{k+1}, y)\, ds_{k+1}, \end{aligned}$$

which is (6.47).

Step 3: Renewal argument. Set $N_k(t) := e^{2\lambda_1 t} \sup_x M_k(t,x)$, so $N_0 \equiv 1$ and $N_{k+1}(t) \le \int_0^t \Psi(t-s) N_k(s)\, ds$ by (6.47). For the additional parameter $\rho > 0$, define $\mathcal{N}_k(\rho) := \sup_{t\ge 0} e^{-\rho t} N_k(t)$. Then

$$e^{-\rho t} N_{k+1}(t) \;\le\; \int_0^t e^{-\rho(t-s)} \Psi(t-s) \cdot e^{-\rho s} N_k(s)\, ds \;\le\; \hat{\Psi}(\rho)\, \mathcal{N}_k(\rho). \tag{6.51}$$

Since $\mathcal{N}_0(\rho) = 1$, a simple induction procedure gives $\mathcal{N}_k(\rho) \le \hat{\Psi}(\rho)^k$. Hence

$$N_k(t) \;\le\; \hat{\Psi}(\rho)^k\, e^{\rho t}. \tag{6.52}$$

We can now assemble the three ingredients. From (6.45),

$$\|f_k(\,\cdot\,, t, x)\|^2_{(\mathcal{H}^D_\alpha)^{\otimes k}} \;\le\; C\, \frac{\beta^{2k}}{k!}\, \|u_0\|^2_\infty\, M_k(t,x).$$

The definition of N_k gives $M_k(t,x) \le \sup_{y\in U} M_k(t,y) = e^{-2\lambda_1 t} N_k(t)$, so substituting and then applying (6.52) yields

$$\|f_k(\,\cdot\,, t, x)\|^2_{(\mathcal{H}^D_\alpha)^{\otimes k}} \;\le\; C\, \frac{\beta^{2k}}{k!}\, \|u_0\|^2_\infty\, e^{-2\lambda_1 t}\, N_k(t) \;\le\; C\, \frac{\beta^{2k}}{k!}\, \|u_0\|^2_\infty\, \hat{\Psi}(\rho)^k\, e^{(\rho-2\lambda_1)t},$$

which is exactly (6.41). ∎

Remark 6.2.4 (On the role of the renewal argument)**.** *In this remark we are trying to answer the following question: why Step 2–3 above needed a renewal-type recursion, while the analogous estimate* (4.16) *on* $\mathbb{R}^d$ *(Theorem 4.1.3) did not. The comparison shows that the renewal argument is not an artifact of the proof, but a genuine substitute for a tool that is simply unavailable outside* $\mathbb{R}^d$*.*

In the Euclidean setting, translation invariance lets us diagonalize the whole chain of n *heat kernels at once. The Fourier transform* (4.27) *turns the* n*-fold convolution* K_n *into the explicit product* $\prod_{i=1}^n e^{-\frac{1}{2}(s_{i+1}-s_i)|\xi_1+\cdots+\xi_i|^2}$*, and the decoupling inequality* (4.32) *can then be applied successively, for* $i = n, n-1, \ldots, 1$*, to separate all* n *spectral variables* $\xi_1, \ldots, \xi_n$ *in a single pass, see* (4.36)*. Once decoupled, the remaining* n*-dimensional integral over the simplex* $S_{t,n}$ *is evaluated directly, for each fixed* n*, by the combinatorial argument of Lemma 4.1.4. Then the resulting bounds are summed over* n *in a single step* (4.39)*. At no point is* $n!\,\|f_n\|^2_{\mathcal{H}^{\otimes n}}$ *related back to* $(n-1)!\,\|f_{n-1}\|^2_{\mathcal{H}^{\otimes(n-1)}}$*. Each term of the series is controlled on its own.*

On the metric measure space (X, d_U, μ) *this shortcut is unavailable, because* U *carries no group structure and* $p_t^{D,U}$ *is not translation invariant. The factorization* (4.27) *relies entirely on the character identity* $e^{i\xi\cdot(a+b)} = e^{i\xi\cdot a}\,e^{i\xi\cdot b}$*, which has no counterpart on a general* U*. The only substitute we have, namely the bilinear identity* (5.26)*, diagonalizes a* single *pairing* $\int_{U^2} p_{t-s}(x,y)\,G_{2\alpha}(y,y')\,p_{t-s}(x,y')\,d\mu(y)\,d\mu(y')$ *against the covariance kernel* $G_{2\alpha}$*, and gives no information on how to integrate out several nested spatial variables* $y_1, \ldots, y_k$ *coming from a chain of* k *heat kernels simultaneously. Consequently, at each stage the only variable that can be integrated out is the last one,* y_{k+1}*, paired with the outer kernel* $p_{t-s_{k+1}}$*. This is exactly what produces the recursion* (6.47) *relating* M_{k+1} *to* $\sup_y M_k$*, in place of a single joint bound on all* $k+1$ *variables at once.*

Given this recursion, the renewal step (Step 3 above) is the standard, and here essentially forced, way to sum a convolution-type inequality $N_{k+1} \le \Psi * N_k$ *in the absence of a closed form for* N_k*. Introducing the exponential weight* $e^{-\rho t}$ *turns the convolution into the product bound* $\mathcal{N}_{k+1}(\rho) \le \hat{\Psi}(\rho)\,\mathcal{N}_k(\rho)$ (6.51)*, which iterates directly into the geometric bound* (6.52)*. In this sense the Laplace-transform trick used here plays exactly the role that the spectral decoupling* (4.32)*, together with Lemma 4.1.4, plays in the Euclidean proof.*

The bound (6.41) depends on $\hat{\Psi}(\rho)$ through the quantity defined in (6.42). Its size — and hence the range of β for which the chaos series converges — is governed by the position of α relative to the critical threshold $d_h/(2d_w)$. The next lemma makes this precise.

Lemma 6.2.5: Three-regime estimate for $\hat{\Psi}$

Under the hypotheses of Lemma 6.2.3, the function $\hat{\Psi}(\rho)$ defined in (6.42) satisfies the following three-regime bound, with a possibly different constant C in each case:

(i) If $0 \le \alpha < \dfrac{d_h}{2d_w}$, then

$$\hat{\Psi}(\rho) \;\le\; C\left(\frac{1}{\rho} + \frac{1}{\rho^{1+2\alpha-d_h/d_w}}\right). \tag{6.53}$$

(ii) If $\alpha = \dfrac{d_h}{2d_w}$, then

$$\hat{\Psi}(\rho) \;\le\; C\,\frac{(\ln\max(3/2,\rho))^2}{\rho}. \tag{6.54}$$

(iii) If $\alpha > \dfrac{d_h}{2d_w}$, then

$$\hat{\Psi}(\rho) \;\le\; \frac{C}{\rho}. \tag{6.55}$$

In all three cases, $\hat{\Psi}(\rho) \to 0$ as $\rho \to +\infty$.

Proof. From Lemma 5.1.27, the function Ψ defined by (6.42) satisfies

$$\Psi(t) \;\le\; \begin{cases} C\,(1 + t^{2\alpha-d_h/d_w}), & 0 \le \alpha < \frac{d_h}{2d_w}, \\ C\,|\ln\min(\frac{1}{2},t)|^2, & \alpha = \frac{d_h}{2d_w}, \\ C, & \alpha > \frac{d_h}{2d_w}. \end{cases} \tag{6.56}$$

We integrate (6.56) against $e^{-\rho t}$ and treat each case separately.

(i) *Subcritical case* $0 \le \alpha < d_h/(2d_w)$. Set $\gamma := 2\alpha - d_h/d_w < 0$. Integrating the two terms of (6.56) separately against $e^{-\rho t}$ gives

$$\int_0^\infty e^{-\rho t}\,dt = \frac{1}{\rho}, \tag{6.57}$$

Since $\alpha \ge 0$ in the subcritical range, $\gamma = 2\alpha - d_h/d_w \ge -d_h/d_w$, and the recurrence condition $d_h < d_w$ of Assumption 5.1.6 gives $d_h/d_w < 1$, hence $\gamma > -1$. Thus, by the Gamma function identity $\int_0^\infty e^{-\rho t}\,t^\gamma\,dt = \Gamma(1+\gamma)/\rho^{1+\gamma}$ (valid for $\gamma > -1$),

$$\int_0^\infty e^{-\rho t}\,t^\gamma\,dt = \frac{\Gamma(1+\gamma)}{\rho^{1+\gamma}}. \tag{6.58}$$

Adding (6.57) and (6.58), multiplied by C, gives (6.53).

(ii) *Critical case $\alpha = d_h/(2d_w)$.* Here $\Psi(t) \le C|\ln\min(1/2,t)|^2$, which is bounded by the constant $C|\ln(1/2)|^2$ for $t \ge 1/2$ and blows up like $|\ln t|^2$ only as $t \to 0$. Splitting the integral at $t = 1/2$,

$$\hat{\Psi}(\rho) \;\le\; C\int_0^{1/2} e^{-\rho t}\,|\ln t|^2\,dt \;+\; C|\ln(1/2)|^2\int_{1/2}^{\infty} e^{-\rho t}\,dt, \tag{6.59}$$

the second term of (6.59) is $O(e^{-\rho/2}/\rho)$, hence negligible for large ρ. In the first term, the change of variables $u = \rho t$ gives

$$\int_0^{1/2} e^{-\rho t}\,|\ln t|^2\,dt \;=\; \frac{1}{\rho}\int_0^{\rho/2} e^{-u}\,(\ln\rho - \ln u)^2\,du. \tag{6.60}$$

Expanding the square in (6.60), the three resulting integrals $\int_0^{\rho/2} e^{-u}\,du$, $\int_0^{\rho/2} e^{-u}\ln u\,du$, and $\int_0^{\rho/2} e^{-u}(\ln u)^2\,du$ each converge, uniformly in ρ, to a finite constant as $\rho \to \infty$. Hence (6.60) is of order $(\ln\rho)^2/\rho$, and combining with the negligible term in (6.59) yields (6.54). This computation follows the same lines as Step 4 in the proof of [10, Lem. 3.9].

(iii) *Supercritical case $\alpha > d_h/(2d_w)$.* The bound $\Psi(t) \le C$ integrates directly:

$$\hat{\Psi}(\rho) \;\le\; C\int_0^{\infty} e^{-\rho t}\,dt \;=\; \frac{C}{\rho}, \tag{6.61}$$

which is (6.55).

It is readily checked that in all three cases above, $\hat{\Psi}(\rho)$ tends to 0 as $\rho \to +\infty$. This concludes our proof. ∎

We can now state the main result of this section.

Theorem 6.2.6: Existence, uniqueness, and L^2 moments

Let (X, d, μ) satisfy Assumption 5.1.6, let $U \subset X$ be an inner uniform domain (Definition 5.1.13), let $\alpha \ge 0$ and $\beta \ge 0$, and let $u_0 \in L^\infty(U,\mu)$. Then equation (6.34) has a unique solution in the sense of Definition 6.2.1, and this solution satisfies the L^2 moment bound

$$\limsup_{t\to+\infty} \frac{\ln \mathbb{E}(u(t,x)^2)}{t} \;\le\; \Theta_\alpha(\beta) - 2\lambda_1, \tag{6.62}$$

where $\lambda_1 = \lambda_1(U)$ is the first eigenvalue of $-\Delta_{D,U}$ and $\Theta_\alpha : (0,\infty) \to (0,\infty)$ is a continuous increasing function whose asymptotics are governed by $\hat{\Psi}$ via (6.53)–(6.55):

(i) If $0 \le \alpha < \dfrac{d_h}{2d_w}$, then $\Theta_\alpha(\beta) \sim C\beta^2$ as $\beta \to 0$ and $\Theta_\alpha(\beta) \sim C\beta^{2d_w/((1+2\alpha)d_w - d_h)}$ as $\beta \to +\infty$.

(ii) If $\alpha = \dfrac{d_h}{2d_w}$, then $\Theta_\alpha(\beta) \sim C\beta^2$ as $\beta \to 0$ and $\Theta_\alpha(\beta) \sim C\beta^2(\ln\beta)^2$ as $\beta \to +\infty$.

(iii) If $\alpha > \dfrac{d_h}{2d_w}$, then $\Theta_\alpha(\beta) = C\beta^2$ for all $\beta > 0$.

Proof. By Proposition 6.2.2, it suffices to verify (6.39). We treat case (i) in detail, and indicate at the end of the proof the (minor) changes needed for cases (ii) and (iii).

Step 1: choice of ρ. Introduce the function

$$F(\rho) := \frac{1}{\rho} + \frac{1}{\rho^{1+\gamma}}, \qquad \rho > 0, \qquad \text{where} \qquad \gamma := 2\alpha - \frac{d_h}{d_w}, \tag{6.63}$$

so that the bound (6.53) of Lemma 6.2.5 reads $\hat{\Psi}(\rho) \le C\,F(\rho)$. As noted in the proof of Lemma 6.2.5 (subcritical case), $\gamma \in (-1, 0)$, so F is continuous and strictly decreasing on $(0, \infty)$, with $F(\rho) \to +\infty$ as $\rho \to 0$ and $F(\rho) \to 0$ as $\rho \to +\infty$. Hence, for every $\beta > 0$, there is a unique $\rho_c = \rho_c(\beta) > 0$ such that

$$C\beta^2 F(\rho_c) = 1. \tag{6.64}$$

Fix any $\rho > \rho_c(\beta)$. Since F is decreasing, $F(\rho) < F(\rho_c)$, and since $\hat{\Psi}(\rho) \le F(\rho)$,

$$C\beta^2\hat{\Psi}(\rho) \;\le\; \delta_\rho \;<\; 1, \qquad \text{where} \qquad \delta_\rho := C\beta^2 F(\rho). \tag{6.65}$$

Step 2: L^2 bound. Recall from the L^2-moment formula (6.40) of Proposition 6.2.2 that

$$\mathbb{E}(u(t,x)^2) \;=\; (P_t^{D,U} u_0(x))^2 \;+\; \sum_{k=1}^{\infty} k!\,\|f_k(\,\cdot\,, t, x)\|^2_{(\mathcal{H}_\alpha^D)^{\otimes k}}. \tag{6.66}$$

Bound (6.41) of Lemma 6.2.3 controls each term of the series in (6.66) through $\hat{\Psi}(\rho)$:

$$k!\,\|f_k(\,\cdot\,, t, x)\|^2_{(\mathcal{H}_\alpha^D)^{\otimes k}} \;\le\; C^k\,\beta^{2k}\,\|u_0\|_\infty^2\,\hat{\Psi}(\rho)^k\,e^{(\rho-2\lambda_1)t}. \tag{6.67}$$

By (6.65), $C\beta^2\hat{\Psi}(\rho) \le \delta_\rho$, so raising this to the k-th power turns (6.67) into the δ_ρ-bound

$$k!\,\|f_k(\,\cdot\,, t, x)\|^2_{(\mathcal{H}_\alpha^D)^{\otimes k}} \;\le\; \|u_0\|_\infty^2\,\delta_\rho^k\,e^{(\rho-2\lambda_1)t}. \tag{6.68}$$

Since $\delta_\rho < 1$ by (6.65), the series (6.39) converges by comparison with (6.68), and Proposition 6.2.2 gives existence and uniqueness of the solution of (6.34).

We now proceed to the L^2 bound. Plugging (6.68) into the series in (6.66), summing the resulting geometric series $\sum_{k=1}^\infty \delta_\rho^k = \delta_\rho/(1-\delta_\rho)$, and bounding the first term of (6.66) by Corollary 5.1.19 ($(P_t^{D,U} u_0(x))^2 \le C\|u_0\|_\infty^2 e^{-2\lambda_1 t}$), we get

$$\mathbb{E}(u(t,x)^2) \;\le\; C\|u_0\|_\infty^2 e^{-2\lambda_1 t} \;+\; \|u_0\|_\infty^2\,e^{(\rho-2\lambda_1)t}\sum_{k=1}^{\infty}\delta_\rho^k \;\le\; C_\rho\,\|u_0\|_\infty^2\,e^{(\rho-2\lambda_1)t}, \tag{6.69}$$

for a constant C_ρ depending on ρ (and $\beta, \alpha, d_h, d_w, U$) but not on t, x. Consequently

$$\limsup_{t\to+\infty} \frac{\ln \mathbb{E}(u(t,x)^2)}{t} \;\le\; \rho - 2\lambda_1, \qquad \text{for every } \rho > \rho_c(\beta). \tag{6.70}$$

Letting $\rho \downarrow \rho_c(\beta)$ in (6.70) yields (6.62) with $\Theta_\alpha(\beta) := \rho_c(\beta)$.

Step 3: asymptotics of Θ_α. From the definition (6.63) of F and $\gamma \in (-1, 0)$, it is readily checked that

$$\rho\, F(\rho) = 1 + \rho^{-\gamma} \xrightarrow[\rho\to 0]{} 1, \quad \text{and} \quad \rho^{1+\gamma}F(\rho) = \rho^{\gamma} + 1 \xrightarrow[\rho\to+\infty]{} 1, \tag{6.71}$$

By (6.64), $F(\rho_c(\beta)) = 1/(C\beta^2)$. As $\beta \to 0$, $1/(C\beta^2) \to +\infty$, so $\rho_c(\beta) \to 0$, and the first limit in (6.71) gives $\rho_c(\beta) \sim 1/F(\rho_c(\beta)) = C\beta^2$, i.e., $\Theta_\alpha(\beta) \sim C\beta^2$ as $\beta \to 0$. As $\beta \to +\infty$, $1/(C\beta^2) \to 0$, so $\rho_c(\beta) \to +\infty$, and the second limit in (6.71) gives $\rho_c(\beta)^{1+\gamma} \sim 1/F(\rho_c(\beta)) = C\beta^2$, i.e.,

$$\Theta_\alpha(\beta) \;\sim\; (C\beta^2)^{1/(1+\gamma)} \;=\; C'\beta^{2d_w/((1+2\alpha)d_w - d_h)}, \qquad \beta \to +\infty, \tag{6.72}$$

since $1 + \gamma = ((1 + 2\alpha)d_w - d_h)/d_w$. This is the asymptotics claimed in case (i).

Step 4: cases (ii) and (iii). The critical case is treated by the same three steps, with F in (6.63) replaced by $F_2(\rho) := (\ln \max(3/2, \rho))^2/\rho$ from the bound (6.54). As above, F_2 is continuous and strictly decreasing, with $F_2(\rho) \to +\infty$ as $\rho \to 0$ and $F_2(\rho) \to 0$ as $\rho \to +\infty$, and

$$\rho\, F_2(\rho) \xrightarrow[\rho\to 0]{} (\ln(3/2))^2, \qquad \frac{\rho\, F_2(\rho)}{(\ln \rho)^2} \xrightarrow[\rho\to+\infty]{} 1.$$

Repeating Step 1–3 with F_2 in place of F, the definition (6.64) of $\rho_c(\beta)$ becomes

$$C\beta^2 F_2(\rho_c(\beta)) = 1.$$

As $\beta \to 0$, the first limit above gives, exactly as in Step 3,

$$\Theta_\alpha(\beta) \;=\; \rho_c(\beta) \;\sim\; \frac{1}{F_2(\rho_c(\beta))} \;=\; C\beta^2, \qquad \beta \to 0, \tag{6.73}$$

which is the first asymptotic in case (ii). As $\beta \to +\infty$, the second limit above gives

$$\rho_c(\beta) \;\sim\; C\beta^2\,(\ln \rho_c(\beta))^2, \qquad \beta \to +\infty. \tag{6.74}$$

Taking logarithms in (6.74) gives $\ln \rho_c(\beta) \sim 2 \ln \beta$ as $\beta \to +\infty$. Hence substituting this back into (6.74) yields

$$\Theta_\alpha(\beta) \;=\; \rho_c(\beta) \;\sim\; C\beta^2(\ln \beta)^2, \qquad \beta \to +\infty, \tag{6.75}$$

which is the second asymptotic in case (ii).

In the supercritical case (iii), the bound (6.55) reads $\hat\Psi(\rho) \le C/\rho$, so the definition (6.64) of ρ_c becomes

$$\frac{C\beta^2}{\rho_c(\beta)} \;=\; 1, \tag{6.76}$$

which gives, exactly rather than only asymptotically,

$$\Theta_\alpha(\beta) \;=\; \rho_c(\beta) \;=\; C\beta^2, \qquad \text{for every } \beta > 0. \tag{6.77}$$

Steps 2–3 above apply verbatim with this value of $\rho_c(\beta)$, and (6.77) is exactly case (iii). This finishes the proof. ■

Remark 6.2.7 (Neumann boundary conditions)**.** *The same strategy applies to the Neumann PAM, replacing $\Delta_{D,U}$ by $\Delta_{N,U}$, $p_t^{D,U}$ by $p_t^{N,U}$, W_α^D by W_α^N, and $G_{2\alpha}^{D,U}$ by $G_{2\alpha}^{N,U} + C_U$ (see Definition 6.1.4 and Proposition 5.1.37). The only difference is that $G_{2\alpha}^{N,U}$ is not pointwise non-negative; the shifted kernel $G_{2\alpha}^{N,U} + C_U$ is non-negative, and the recursion in* (6.47) *goes through with the Neumann heat kernel. The proof of existence and uniqueness, and the L^2 bound, hold verbatim; we refer to [10, Sec. 4.2] for the details.*

Remark 6.2.8 (Feynman–Kac formula for $\alpha > d_h/(2d_w)$)**.** *When $\alpha > d_h/(2d_w)$, Theorem 6.1.15 shows that the density field $x \mapsto \tilde{W}_\alpha^D(t,x)$ is Hölder continuous. In this regime the solution admits the Feynman–Kac representation*

$$u(t,x) = \mathbb{E}^B\left(u_0(B_t^x)\exp\left(\beta\int_0^t \dot{W}_\alpha^D(s, B_{t-s}^x)\,ds - \frac{\beta^2}{2}\int_0^t G_{2\alpha}^{D,U}(B_s^x, B_s^x)\,ds\right)\mathbf{1}_{(t<T)}\right), \tag{6.78}$$

where B^x is a Brownian motion on (X, d, μ) started at x and T is its hitting time of U^c. The proof follows an approximation procedure (mollify W_α^D in space, write the regularized Feynman–Kac formula, pass to the limit). See [10, Section 3.2], as well as [35, Section 3.2] for the approximation argument.

6.3 Moment estimates

In the Euclidean setting, Section 4.3 took the Feynman-Kac moment formula of Theorem 4.2.7 as its starting point for the Lyapunov exponent estimates of the parabolic Anderson model. We follow the same strategy on the bounded domain $U \subset X$: this section develops moment estimates for the solution u of (6.34) furnished by Theorem 6.2.6, under the standing framework of Assumption 5.1.6 and Definition 5.1.13. Remark 6.2.8 already recorded a heuristic Feynman-Kac formula for a single realization of $u(t,x)$, valid in the pointwise regime $\alpha > d_h/(2d_w)$ and obtained by a space-time mollification of the noise. We begin by making this formula precise, extending it to a genuine *moment* formula for every integer $k \geq 2$ and every $\alpha > 0$, via the same spatial mollification and Itô calculus approach used in Section 4.2.

6.3.1 Feynman-Kac representation of moments

Recall from Theorem 4.2.7 that on $\mathbb{R}^d$ the k-th moment of the solution is expressed as a Brownian expectation

$$\mathbb{E}\Big[u(t,x)^k\Big] = \mathbb{E}_B\left[\prod_{i=1}^k u_0(B_t^i + x)\,\exp\left(\beta^2\sum_{1\le i<j\le k}\int_0^t \Lambda(B_s^i - B_s^j)\,ds\right)\right],$$

obtained by a spatial mollification u^ε of u built from k independent Brownian paths, and by letting $\varepsilon \to 0$ (Section 4.2.1). We now adapt this strategy to the geometric setting. To begin with, let us highlight three structural differences with the Euclidean case:

(i) *Bounded domain and exit times.* The noise W_α^D and the heat kernel $p_t^{D,U}$ are only defined on the bounded domain U (Section 6.1), while the underlying Brownian motion B^x lives on the full space X (Definition 5.1.8). Each Brownian path entering the formula must therefore be paired with its exit time from U, exactly as in Remark 6.2.8.

(ii) *Substitute for Dalang's condition.* In the Euclidean case, finiteness of $\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}}$ relied on Dalang's condition (2.2). Here this role is played by the L^2-estimate of Lemma 5.1.27, which controls $(-\Delta_{D,U})^{-\alpha} p_t^{D,U}(x,\cdot)$ for every $\alpha \geq 0$.

(iii) *No Fourier analysis.* The translation-invariance and Fourier computations of Section 4.2.1 (e.g. the change of variables leading to (4.59)) are unavailable on U. As for the existence and uniqueness results, they are replaced throughout by the Markov property of B^x, the Chapman-Kolmogorov identity for $p_t^{D,U}$, and the bilinear Riesz-kernel identity (5.26).

We record a notation for the k independent Brownian paths needed for the k-th moment, in the spirit of Notation 4.2.1.

Notation 6.3.1: Brownian copies and exit times

Let $k \geq 1$. We let $B^{x,1}, \ldots, B^{x,k}$ be k independent copies of the Brownian motion on (X, d, μ) of Definition 5.1.8, all started at the same point $x \in U$, and independent of the noise W_α^D. For $1 \leq j \leq k$ we write

$$T^j := \inf\{t \geq 0 : B_t^{x,j} \notin U\}$$

for the exit time of $B^{x,j}$ from U. As in Notation 4.2.1, $\mathbb{E}_B$ denotes expectation with respect to the Brownian paths only (holding W_α^D fixed), and $\mathbb{E}_W$ denotes expectation with respect to W_α^D only (holding the paths fixed).

6.3.1.1 The spatially regularized process u^ε

For $\varepsilon > 0$ we use the Dirichlet heat kernel $p_\varepsilon^{D,U}$ itself as spatial mollifier, in place of the Gaussian kernel p_ε of (4.46). For a Brownian path $B = B^x$ started at $x \in U$ and $t > 0$, define the $\mathcal{H}_\alpha^D$-valued functional

$$A^\varepsilon_{t,x}(s,y) := p_\varepsilon^{D,U}(B_{t-s}, y)\, \mathbf{1}_{[0,t]}(s)\, \mathbf{1}_{\{t<T\}}, \qquad (s,y) \in \mathbb{R}_+ \times U, \tag{6.79}$$

where T is the exit time of B from U (Notation 6.3.1). On the event $\{t < T\}$ the whole path $(B_r)_{0\leq r\leq t}$ stays in U, so $B_{t-s} \in U$ for every $s \in [0,t]$ and (6.79) is well defined. On the complementary event we set $A^\varepsilon_{t,x} := 0$.

To compute inner products of functionals of the form (6.79), introduce the auxiliary kernel

$$\begin{aligned} \Xi_\varepsilon(y,y') &:= \left\langle p_\varepsilon^{D,U}(y,\cdot),\, p_\varepsilon^{D,U}(y',\cdot)\right\rangle_{\mathcal{W}_D^{-\alpha}(U)} \qquad (6.80)\\ &= \left\langle (-\Delta_{D,U})^{-\alpha} p_\varepsilon^{D,U}(y,\cdot),\, (-\Delta_{D,U})^{-\alpha} p_\varepsilon^{D,U}(y',\cdot)\right\rangle_{L^2(U,\mu)}, \qquad y, y' \in U, \end{aligned}$$

where the second equality is the bilinear Riesz-kernel identity (5.26). The kernel Ξ_ε expresses those inner products as time integrals along the Brownian paths. Namely, let $B^{x,i}$ and $B^{x,j}$ be two of the independent copies of Notation 6.3.1, with exit times T^i and T^j, and write $A^{\varepsilon,i}_{t,x}$ and $A^{\varepsilon,j}_{t,x}$ for the functionals (6.79) built from $B^{x,i}$ and $B^{x,j}$ respectively. Unfolding the definition (6.1) of the inner product in $\mathcal{H}^D_\alpha$ and the definition (6.80) of Ξ_ε, on the event $\{t < T^i\} \cap \{t < T^j\}$ we get

$$\langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}^D_\alpha} = \int_0^t \Xi_\varepsilon(B^{x,i}_{t-s}, B^{x,j}_{t-s})\, ds = \int_0^t \Xi_\varepsilon(B^{x,i}_r, B^{x,j}_r)\, dr, \tag{6.81}$$

where the change of variable $r = t - s$ was used for the second equality (off the event $\{t < T^i\} \cap \{t < T^j\}$ at least one of the two functionals vanishes, and so does the inner product). Relation (6.81) is the geometric analogue of the Fourier computation (4.51). In particular, computing both functionals along the same path $B = B^x$ with exit time T (that is taking $i = j$ in (6.81)), on $\{t < T\}$ we get

$$\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}^D_\alpha} = \int_0^t \Xi_\varepsilon(B_{t-r}, B_{t-r})\, dr = \int_0^t \Big\|(-\Delta_{D,U})^{-\alpha} p^{D,U}_\varepsilon(B_{t-r}, \cdot)\Big\|^2_{L^2(U,\mu)}\, dr. \tag{6.82}$$

By Lemma 5.1.27, applied at the fixed time ε and valid for every $\alpha \geq 0$, the integrand in (6.82) is bounded uniformly over $B_{t-r} \in U$ by a finite constant $\kappa(\varepsilon)^2$ (with $\kappa(\varepsilon)$ given by the right-hand side of (5.36) at time ε), so

$$\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}^D_\alpha} \leq t\,\kappa(\varepsilon)^2 < \infty \qquad \text{on } \{t < T\}. \tag{6.83}$$

In particular $A^\varepsilon_{t,x} \in \mathcal{H}^D_\alpha$ a.s. for *every* $\alpha \geq 0$, including the white noise case $\alpha = 0$. Unlike the unmollified formula (6.78) of Remark 6.2.8, which requires $\alpha > d_h/(2d_w)$ for the exponent to make sense pointwise, the spatial regularization tames the noise for every $\alpha \geq 0$, exactly as in the Euclidean case (compare (4.48)). Only the limit $\varepsilon \to 0$ taken in Section 6.3.1.3 below will force $\alpha > 0$.

We may now define the regularized process itself.

Definition 6.3.2: Spatially regularized process

For $\varepsilon > 0$, $\beta \geq 0$ and $u_0 \in L^\infty(U,\mu)$, define

$$u^\varepsilon(t,x) := \mathbb{E}_B\left[u_0(B_t)\,\mathbf{1}_{\{t<T\}}\exp\left(\beta\, W^D_\alpha(A^\varepsilon_{t,x}) - \frac{\beta^2}{2}\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}^D_\alpha}\right)\right], \qquad (t,x) \in \mathbb{R}_+ \times U, \tag{6.84}$$

where $B = B^x$ is the Brownian motion of Notation 6.3.1 and $A^\varepsilon_{t,x}$ is as in (6.79).

Remark 6.3.3. *The Gaussian exponential in (6.84) is well defined path-by-path on $\{t < T\}$, since $A^\varepsilon_{t,x} \in \mathcal{H}^D_\alpha$ there. The stochastic exponential built from $A^\varepsilon_{t,x}$ is moreover a* true *(as opposed to merely local) martingale in $s \in [0,t]$: its quadratic variation is deterministically bounded by $\beta^2 t\,\kappa(\varepsilon)^2$, by the computation above. Novikov's condition therefore holds trivially, and $\mathbb{E}_W[\exp(\beta W^D_\alpha(A^\varepsilon_{t,x}) - \frac{\beta^2}{2}\|A^\varepsilon_{t,x}\|^2_{\mathcal{H}^D_\alpha})] = 1$ for every path B with $t < T$, exactly as in the Euclidean case (below (4.47)).*

6.3.1.2 The mild equation for u^ε

As in Lemma 4.2.3, the Feynman-Kac exponential defining u^ε can be expanded via Itô's formula into a Doléans-Dade stochastic exponential, and averaging over B identifies u^ε as the solution of a mild stochastic heat equation driven by a spatially mollified noise. Unlike (4.54), the mollification here uses the Dirichlet semigroup $P_\varepsilon^{D,U}$ rather than a spatial convolution.

Proposition 6.3.4: Mild equation for u^ε

Let u^ε be as in Definition 6.3.2. Define the spatially mollified noise $W_\alpha^{D,\varepsilon}$ by setting, for every adapted process $\phi \in \mathcal{H}_\alpha^D$,

$$\int_0^\infty \int_U \phi(s,y)\, W_\alpha^{D,\varepsilon}(ds,dy) := \int_0^\infty \int_U (P_\varepsilon^{D,U}\phi_s)(y)\, W_\alpha^D(ds,dy), \tag{6.85}$$

where $(P_\varepsilon^{D,U}\phi_s)(y) := \int_U p_\varepsilon^{D,U}(y,z)\,\phi(s,z)\,d\mu(z)$. Then for every $(t,x) \in \mathbb{R}_+ \times U$,

$$u^\varepsilon(t,x) = P_t^{D,U}u_0(x) + \beta \int_0^t \int_U p_{t-s}^{D,U}(x,y)\, u^\varepsilon(s,y)\, W_\alpha^{D,\varepsilon}(ds,dy). \tag{6.86}$$

Proof. We mimic Steps 1-a to 1-c of the proof of Lemma 4.2.3, replacing the translation-invariance argument of that proof by the Markov property of B at the fixed time $t-s$.

Step 1: Stochastic exponential. Mimicking Step 1-a of the proof of Lemma 4.2.3, for a fixed path B with $t < T$ we set

$$N_s^B := \beta \int_0^s \int_U p_\varepsilon^{D,U}(B_{t-r},y)\, W_\alpha^D(dr,dy), \qquad s \in [0,t]. \tag{6.87}$$

Since B stays in U on $[0,t]$, the space-time function $(r,y) \mapsto p_\varepsilon^{D,U}(B_{t-r},y)\mathbf{1}_{[0,t]}(r)$ is, path by path, an element of $\mathcal{H}_\alpha^D$, by the same computation as in (6.82), so N^B is a W_α^D-martingale. By the isometry (6.4) and the definition (6.80) of Ξ_ε, its quadratic variation is

$$\langle N^B\rangle_s = \beta^2 \int_0^s \Xi_\varepsilon(B_{t-r}, B_{t-r})\, dr, \qquad s \in [0,t]. \tag{6.88}$$

Unlike the Euclidean computation (4.60), this quadratic variation is not a deterministic function of s alone, since there is no translation invariance available to eliminate the dependence on B. It is nonetheless deterministically bounded by $\beta^2 s\,\kappa(\varepsilon)^2$, by the same estimate as in (6.83). At $s = t$, comparing (6.87)–(6.88) with (6.79) and (6.82) gives $N_t^B = \beta\, W_\alpha^D(A_{t,x}^\varepsilon)$ and $\langle N^B\rangle_t = \beta^2 \|A_{t,x}^\varepsilon\|_{\mathcal{H}_\alpha^D}^2$, so the Doléans-Dade exponential $M_s^B := \exp(N_s^B - \frac{1}{2}\langle N^B\rangle_s)$ satisfies, by the same Itô-formula computation as in (4.61)–(4.64),

$$M_t^B = 1 + \beta \int_0^t \int_U M_s^B\, p_\varepsilon^{D,U}(B_{t-s},y)\, W_\alpha^D(ds,dy). \tag{6.89}$$

Step 2: Averaging over B and identifying the kernel. Mimicking Step 1-b of the proof of Lemma 4.2.3, we now average over B. By Definition 6.3.2, $u^\varepsilon(t,x) = \mathbb{E}_B[u_0(B_t)\mathbf{1}_{\{t<T\}}M_t^B]$.

Substituting (6.89) and applying stochastic Fubini to exchange $\mathbb{E}_B$ with the Walsh integral gives

$$u^\varepsilon(t,x) = P_t^{D,U} u_0(x) + \beta \int_0^t \int_U Z^{\varepsilon,t,x}(s,y)\, W_\alpha^D(ds,dy), \tag{6.90}$$

where the first term uses $\mathbb{E}_B[u_0(B_t)\mathbf{1}_{\{t<T\}}] = P_t^{D,U} u_0(x)$ (5.12), and where the stochastic integrand $Z^{\varepsilon,t,x}$ is defined by

$$Z^{\varepsilon,t,x}(s,y) := \mathbb{E}_B\Big[u_0(B_t)\, \mathbf{1}_{\{t<T\}}\, M_s^B\, p_\varepsilon^{D,U}(B_{t-s},y)\Big]. \tag{6.91}$$

We now wish to prove the identity

$$Z^{\varepsilon,t,x}(s,y) = \int_U p_{t-s}^{D,U}(x,z)\, p_\varepsilon^{D,U}(z,y)\, u^\varepsilon(s,z)\, d\mu(z). \tag{6.92}$$

Once (6.92) is established, it turns (6.90) into (6.86) by (6.85) and stochastic Fubini, exactly as in Step 2 of the proof of Lemma 4.2.3. This will conclude the proof.

Step 3: Proof of (6.92). In order to prove (6.92) we mimic Step 1-c of the proof of Lemma 4.2.3. To this aim, fix $s \in (0,t)$, condition on B_{t-s} on the event $\{t-s<T\}$, and set $b := B_{t-s} \in U$. Write $\tilde{B}_v := B_{t-s+v}$ for $v \in [0,s]$. By the Markov property (5.7) of Definition 5.1.8, conditionally on $B_{t-s} = b$ this is a Brownian motion starting at b, independent of $\mathcal{F}_{t-s}^B$. Then for $r \in [0,s]$ one can write

$$B_{t-r} = \tilde{B}_{s-r}, \qquad B_t = \tilde{B}_s, \qquad p_\varepsilon^{D,U}(B_{t-s},y) = p_\varepsilon^{D,U}(b,y). \tag{6.93}$$

In addition (a feature with no counterpart in the whole space), the exit time has to be split at $t-s$. Namely, writing $\tilde{T}$ for the exit time of $\tilde{B}$ from U, we have

$$\mathbf{1}_{\{t<T\}} = \mathbf{1}_{\{t-s<T\}}\, \mathbf{1}_{\{s<\tilde{T}\}}. \tag{6.94}$$

Substituting the first identity of (6.93) into the integrand of N_s^B in (6.87) and into the quadratic variation (6.88), the definition $M_s^B = \exp(N_s^B - \frac{1}{2}\langle N^B\rangle_s)$ of Step 1 gives

$$\begin{aligned} M_s^B\Big|_{B_{t-s}=b} &= \exp\left(\beta \int_0^s \int_U p_\varepsilon^{D,U}(\tilde{B}_{s-r},y)\, W_\alpha^D(dr,dy) - \frac{\beta^2}{2}\int_0^s \Xi_\varepsilon(\tilde{B}_{s-r},\tilde{B}_{s-r})\, dr\right) \\ &= M_s^{\tilde{B}}, \end{aligned} \tag{6.95}$$

where $M^{\tilde{B}}$ is the stochastic exponential of Step 1. Therefore comparing (6.95) with the Feynman-Kac formula (6.84) applied at (s,b), and taking the second identity of (6.93) and the splitting (6.94) into account, we obtain

$$\mathbb{E}_{\tilde{B}}\Big[u_0(\tilde{B}_s)\, \mathbf{1}_{\{s<\tilde{T}\}}\, M_s^{\tilde{B}}\Big] = u^\varepsilon(s,b). \tag{6.96}$$

Integrating over the law $\mathbb{P}_x(B_{t-s} \in db,\, t-s<T) = p_{t-s}^{D,U}(x,b)\, d\mu(b)$ (5.12) of B_{t-s} on $\{t-s<T\}$ and using (6.96) together with the third identity of (6.93), as well as the definition (6.91) of $Z^{\varepsilon,t,x}$, we obtain

$$Z^{\varepsilon,t,x}(s,y) = \int_U p_{t-s}^{D,U}(x,b)\, p_\varepsilon^{D,U}(b,y)\, u^\varepsilon(s,b)\, d\mu(b),$$

which proves (6.92) and hence (6.86). ■

Remark 6.3.5 (Chaos expansion of u^ε)**.** *Iterating the mild equation* (6.86) *exactly as in the derivation of the chaos expansion* (6.37)-(6.38) *shows that* u^ε *has its own chaos expansion. Specifically one can write*

$$u^\varepsilon(t,x) = P_t^{D,U} u_0(x) + \sum_{k\geq 1} I_k(f_k^\varepsilon(\cdot,t,x)). \tag{6.97}$$

Suppressing s_i, t, x *in the kernels* f_k^ε *above and writing the spatial arguments only for notational sake, it holds*

$$f_k^\varepsilon(y_1,\dots,y_k) := \int_{U^k} f_k(z_1,\dots,z_k) \prod_{i=1}^k p_\varepsilon^{D,U}(y_i,z_i)\, d\mu(z_i). \tag{6.98}$$

Otherwise stated, we have $f_k^\varepsilon = (P_\varepsilon^{D,U})^{\otimes k} f_k$*, similarly to relation* (4.73) *in Remark 4.2.5 for the Euclidean case.*

The expansions (6.37) and (6.97) live on the same chaos levels, so $u^\varepsilon(t,x) - u(t,x)$ can be compared kernel by kernel. Combined with Nelson's hypercontractivity inequality (2.25), this observation yields the following convergence result, which is the geometric counterpart of Proposition 4.2.6.

Proposition 6.3.6: L^k convergence of u^ε

Let $\alpha \geq 0$, $\beta \geq 0$, $u_0 \in L^\infty(U,\mu)$, and let u be the solution of (6.34) with chaos kernels f_k of (6.38). Then for every integer $k \geq 2$,

$$u^\varepsilon(t,x) \longrightarrow u(t,x) \qquad \text{in } L^k(\Omega), \text{ as } \varepsilon \to 0.$$

Proof. The proof follows the same pattern as that of its Euclidean counterpart, Proposition 4.2.6, and we point out the correspondence between the two sets of inequalities as we go.

Since $P_\varepsilon^{D,U}$ is self-adjoint with spectrum $\{e^{-\lambda_j \varepsilon}\}_{j\geq 1} \subset (0,1]$ (Notation 5.1.20), it is a contraction on $\mathcal{W}_D^{-\alpha}(U)$ (as well as on $L^2(U,\mu)$), and $P_\varepsilon^{D,U} \to \mathrm{Id}$ strongly as $\varepsilon \to 0$ on both spaces. Consequently the kernels $f_n^\varepsilon = (P_\varepsilon^{D,U})^{\otimes n} f_n$ of (6.98) satisfy $\|f_n^\varepsilon(\cdot,t,x)\|_{(\mathcal{H}_\alpha^D)^{\otimes n}} \leq \|f_n(\cdot,t,x)\|_{(\mathcal{H}_\alpha^D)^{\otimes n}}$ for every $\varepsilon > 0$ and every $n \geq 1$, and $f_n^\varepsilon \to f_n$ in $(\mathcal{H}_\alpha^D)^{\otimes n}$ as $\varepsilon \to 0$, for each fixed n. These two properties are the respective analogues of the contraction bound (4.75) and of the fixed-level convergence (4.77) in Step 1 of the proof of Proposition 4.2.6, with the spectral bound $e^{-\lambda_j\varepsilon} \leq 1$ playing the role of the Fourier multiplier bound $e^{-\varepsilon|\xi|^2/2} \leq 1$ appearing in (4.74).

Fix an integer $k \geq 2$. By Lemma 6.2.5, $\hat{\Psi}(\rho) \to 0$ as $\rho \to \infty$, so we may choose ρ large enough that $\delta := (k-1)\, C\beta^2 \hat{\Psi}(\rho) < 1$, where C is the constant of Lemma 6.2.3. (This choice of ρ is the geometric substitute for the choice of the frequency threshold N achieving $\theta_N = 2(k-1)\beta^2 C_N < 1$ in (4.79). In both cases one tunes a free parameter, ρ here and N there, so that the small quantity it governs, namely $\hat{\Psi}(\rho)$ here and the high-frequency tail C_N of (4.37) there, beats the factor $(k-1)\beta^2$ stemming from hypercontractivity at order k.) Bound (6.41) then gives

$$(k-1)^{n/2}\sqrt{n!}\,\|f_n(\cdot,t,x)\|_{(\mathcal{H}_\alpha^D)^{\otimes n}} \;\le\; \|u_0\|_\infty\,\delta^{n/2}\,e^{(\rho-2\lambda_1)t/2}, \qquad n\ge 1, \tag{6.99}$$

which is summable in n since $\delta<1$. Observe that (6.99) is the exact analogue of the summability estimate (4.78) obtained in Step 2 of the proof of Proposition 4.2.6, the geometric series in $\delta^{n/2}$ replacing the series produced there by the binomial bound (4.79). From here the conclusion is reached exactly as in Step 3 of the proof of Proposition 4.2.6. Namely, since $\|f_n^\varepsilon - f_n\|\le 2\|f_n\|$, twice the bound (6.99) dominates $(k-1)^{n/2}\sqrt{n!}\|f_n^\varepsilon-f_n\|$ uniformly in ε, and each term of the latter converges to 0 as $\varepsilon\to 0$ (for fixed n). By dominated convergence for series,

$$\sum_{n=1}^{\infty}(k-1)^{n/2}\sqrt{n!}\,\|f_n^\varepsilon(\cdot,t,x)-f_n(\cdot,t,x)\|_{(\mathcal{H}_\alpha^D)^{\otimes n}} \longrightarrow 0 \qquad \text{as } \varepsilon\to 0.$$

Nelson's hypercontractivity inequality (2.25), applied to $u^\varepsilon(t,x)-u(t,x)=\sum_{n\ge1} I_n(f_n^\varepsilon - f_n)$ as in (4.80), bounds $\|u^\varepsilon(t,x)-u(t,x)\|_{L^k(\Omega)}$ by the left-hand side above, which therefore tends to 0. ■

6.3.1.3 The Feynman-Kac moment formula

We can now state the main result of this section.

Theorem 6.3.7: Feynman-Kac moment formula

Let $\alpha>0$, $\beta\ge 0$ and $u_0\in L^\infty(U,\mu)$, and let u be the solution of (6.34) given by Theorem 6.2.6. Then for every integer $k\ge 2$, every $t\ge 0$ and every $x\in U$,

$$\mathbb{E}\Big[u(t,x)^k\Big] = \mathbb{E}_B\left[\prod_{j=1}^{k} u_0(B_t^{x,j})\,\mathbf{1}_{\{t<T^j\}}\,\exp\left(\beta^2\sum_{1\le i<j\le k}\int_0^t G_{2\alpha}^{D,U}(B_s^{x,i},B_s^{x,j})\,ds\right)\right], \tag{6.100}$$

where $B^{x,1},\dots,B^{x,k}$ and $T^1,\dots,T^k$ are as in Notation 6.3.1.

Proof. We proceed in three steps, exactly as in the proof of Theorem 4.2.7.

Step 1: Moment formula for u^ε. Let $B^{x,1},\dots,B^{x,k}$ be as in Notation 6.3.1, and write $A_{t,x}^{\varepsilon,j}$ for the functional (6.79) built from $B^{x,j}$. Since the $B^{x,j}$ are independent copies independent of W_α^D, we substitute (6.84) for each factor. Then using Fubini, the isometry (6.4), and canceling the diagonal terms exactly as in Step 1 of the proof of Proposition 4.2.2, we obtain

$$\mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big] = \mathbb{E}_B\left[\prod_{j=1}^{k} u_0(B_t^{x,j})\,\mathbf{1}_{\{t<T^j\}}\,\exp\left(\beta^2\sum_{1\le i<j\le k}\langle A_{t,x}^{\varepsilon,i},A_{t,x}^{\varepsilon,j}\rangle_{\mathcal{H}_\alpha^D}\right)\right]. \tag{6.101}$$

Step 2: Limit of the cross terms. By (6.80) and the semigroup property of $P^{D,U}$, we have

$$\Xi_\varepsilon(y,y') = \frac{1}{\Gamma(2\alpha)}\int_0^\infty r^{2\alpha-1} p^{D,U}_{2\varepsilon+r}(y,y')\,dr = \frac{1}{\Gamma(2\alpha)}\int_{2\varepsilon}^\infty (s-2\varepsilon)^{2\alpha-1} p^{D,U}_s(y,y')\,ds, \quad (6.102)$$

where the first equality follows from (5.24) (with α replaced by 2α) together with the Chapman-Kolmogorov identity

$$\int_U p^{D,U}_\varepsilon(y,z) p^{D,U}_r(z,z')\,d\mu(z) = p^{D,U}_{\varepsilon+r}(y,z'), \quad (6.103)$$

applied at both endpoints y, y'. Note that (6.103) is the geometric substitute for the Fourier identity (4.51) in the $\mathbb{R}^d$ case.

Let us now take limits in (6.102), for which we should elaborate slightly on (4.82). Namely for $y \neq y'$ we claim that

$$\lim_{\varepsilon\to 0}\Xi_\varepsilon(y,y') = G^{D,U}_{2\alpha}(y,y'). \quad (6.104)$$

In order to prove (6.104), we apply dominated convergence to the middle expression of (6.102). The pointwise convergence of the dominated argument is clear. That is, for every $r>0$ we have

$$\lim_{\varepsilon\to 0} p^{D,U}_{2\varepsilon+r}(y,y') = p^{D,U}_r(y,y'),$$

which stems easily from the spectral expansion (5.20). The domination can be argued as follows: for $y \neq y'$, the sub-Gaussian upper bound (5.14) for $t \leq 1$ and the spectral expansion (5.20) for $t \geq 1$ furnish a constant $c = c(y,y')$ such that $p^{D,U}_t(y,y') \leq c\, e^{-\lambda_1 t}$ for every $t>0$. Hence for $\varepsilon \in (0,1)$ the integrand $r^{2\alpha-1}p^{D,U}_{2\varepsilon+r}(y,y')$ in (6.102) is bounded by $c\, r^{2\alpha-1}e^{-\lambda_1 r}$, which is integrable on $(0,+\infty)$. This finishes the proof of our claim (6.104).

Next we upgrade the pointwise convergence (6.104) to a convergence result for the cross inner products appearing in (6.101). Namely we claim that for $i \neq j$ we have

$$\lim_{\varepsilon\to 0}\mathbb{E}_B\left[\mathbf{1}_{\{t<T^i\}}\,\mathbf{1}_{\{t<T^j\}}\left|\langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}^D_\alpha} - \int_0^t G^{D,U}_{2\alpha}(B^{x,i}_r, B^{x,j}_r)\,dr\right|\right] = 0. \quad (6.105)$$

Relation (6.105) is the geometric analogue of the $L^1(\mathbb{P}_B)$ convergence (4.82), and the indicators in (6.105) are harmless. Let us also highlight the main obstacle in the proof of (6.105). In the Euclidean case, the convergence (4.82) stemmed from an elementary dominated convergence argument, the domination being provided by the uniform bound $|e^{-\varepsilon|\xi|^2}e^{i\xi\cdot z}| \leq 1$ together with Dalang's condition (2.2). In our geometric setting no such uniform bound is available. We circumvent this problem thanks to Scheffé's lemma, spelled out under the following form: if a family of nonnegative functions F_ε converges almost everywhere to F and $\int F_\varepsilon \to \int F < \infty$, then $F_\varepsilon \to F$ in L^1.

In order to use the above lemma, let us call Ω_B the probability space supporting the Brownian copies of Notation 6.3.1. Then we apply Scheffé's lemma on the product space $\Omega_B \times [0,t]$ equipped with the measure $\mathbb{P}_B \otimes dr$, to the following functions:

$$\begin{aligned} F_\varepsilon(\omega,r) &= \mathbf{1}_{\{r<T^i\}}\,\mathbf{1}_{\{r<T^j\}}\,\Xi_\varepsilon\Big(B^{x,i}_r, B^{x,j}_r\Big), \\ F(\omega,r) &= \mathbf{1}_{\{r<T^i\}}\,\mathbf{1}_{\{r<T^j\}}\,G^{D,U}_{2\alpha}\Big(B^{x,i}_r, B^{x,j}_r\Big). \end{aligned} \quad (6.106)$$

Those functions are nonnegative, thanks to the positivity of the heat kernel $p^{D,U}$ in the representations (6.102) and (5.24). We now check the two remaining hypotheses of Scheffé's lemma.

(a) Almost everywhere convergence. The Ahlfors regularity (5.5) implies that μ has no atoms. Since $B_r^{x,i}$ and $B_r^{x,j}$ are independent, each with a law admitting a density with respect to μ, we get $\mathbb{P}(B_r^{x,i} = B_r^{x,j}) = 0$ for every fixed $r > 0$. By Fubini's theorem, the set of couples (ω, r) such that $B_r^{x,i}(\omega) = B_r^{x,j}(\omega)$ is thus $\mathbb{P}_B \otimes dr$-negligible. Owing to the convergence (6.104), valid off the diagonal, we conclude that $F_\varepsilon \to F$, $\mathbb{P}_B \otimes dr$-almost everywhere.

(b) Convergence of the integrals. We first derive an alternative expression for Ξ_ε. Starting from the middle expression of (6.102), apply the Chapman-Kolmogorov identity (6.103) at both endpoints as before, exchange the time and space integrals, and identify the time integral by means of (5.24). This yields

$$\Xi_\varepsilon(z, z') = \int_U \int_U p_\varepsilon^{D,U}(z, w)\, p_\varepsilon^{D,U}(z', w')\, G_{2\alpha}^{D,U}(w, w')\, d\mu(w)\, d\mu(w'), \qquad z, z' \in U. \tag{6.107}$$

Next recall from (5.12) that $\mathbb{P}(B_r^{x,i} \in dz,\, r < T^i) = p_r^{D,U}(x, z)\, d\mu(z)$. Hence, invoking the independence of $B^{x,i}$ and $B^{x,j}$, then (6.107), the Chapman-Kolmogorov identity (6.103), and reading (6.107) backwards at scale $r + \varepsilon$ (note that the definition (6.80) of Ξ_ε makes sense for any positive value of the parameter ε, which gives a meaning to $\Xi_{r+\varepsilon}$ and Ξ_s below), we obtain

$$\begin{aligned} \int F_\varepsilon\, d(\mathbb{P}_B \otimes dr) &= \int_0^t \int_U \int_U p_r^{D,U}(x, z)\, p_r^{D,U}(x, z')\, \Xi_\varepsilon(z, z')\, d\mu(z)\, d\mu(z')\, dr \\ &= \int_0^t \int_U \int_U p_{r+\varepsilon}^{D,U}(x, w)\, p_{r+\varepsilon}^{D,U}(x, w')\, G_{2\alpha}^{D,U}(w, w')\, d\mu(w)\, d\mu(w')\, dr \\ &= \int_0^t \Xi_{r+\varepsilon}(x, x)\, dr \;=\; \int_\varepsilon^{t+\varepsilon} \Xi_s(x, x)\, ds. \end{aligned} \tag{6.108}$$

The same computation, performed without the smoothing at scale ε, gives

$$\int F\, d(\mathbb{P}_B \otimes dr) = \int_0^t \Xi_s(x, x)\, ds. \tag{6.109}$$

In addition, the function $s \mapsto \Xi_s(x, x)$ is integrable on $(0, t+1]$. Indeed, comparing the diagonal value $\Xi_s(x, x)$ given by the second line of (6.80) with the definition (6.42) of Ψ, we have $\Xi_s(x, x) \le e^{-2\lambda_1 s}\Psi(s) \le \Psi(s)$, and thus

$$\int_0^{t+1} \Xi_s(x, x)\, ds \;\le\; e^{t+1} \int_0^{t+1} e^{-s}\Psi(s)\, ds \;\le\; e^{t+1}\, \hat{\Psi}(1). \tag{6.110}$$

Now the right hand side of (6.110) is finite by Lemma 6.2.5 (this is one of the places where the recurrence condition $d_h < d_w$ of Assumption 5.1.6 is used), which yields the integrability of the function $\Xi_\cdot(x, x)$ on $(0, t+1]$. Since the integrals (6.108) and (6.109) differ by the integral of $\Xi_s(x, x)$ over the two small intervals $(0, \varepsilon]$ and $(t, t+\varepsilon]$, the integrability of $\Xi_\cdot(x, x)$ shows that $\int F_\varepsilon\, d(\mathbb{P}_B \otimes dr) \to \int F\, d(\mathbb{P}_B \otimes dr) < \infty$ as $\varepsilon \to 0$.

Having checked items (a) and (b), Scheffé's lemma asserts that $F_\varepsilon \to F$ in $L^1(\mathbb{P}_B \otimes dr)$. In order to deduce (6.105), observe that $\mathbf{1}_{\{t<T^i\}} \leq \mathbf{1}_{\{r<T^i\}}$ for $r \leq t$. Together with the expression (6.81) for the cross inner products and the triangle inequality, this entails

$$\mathbb{E}_B\left[\mathbf{1}_{\{t<T^i\}}\,\mathbf{1}_{\{t<T^j\}}\left|\langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}^D_\alpha} - \int_0^t G^{D,U}_{2\alpha}(B^{x,i}_r, B^{x,j}_r)\,dr\right|\right] \leq \int \left|F_\varepsilon - F\right| d(\mathbb{P}_B \otimes dr),$$

and the right-hand side vanishes as $\varepsilon \to 0$. The claim (6.105) is thus proved.

Step 3: Convergence of moments. This step is entirely similar to Step 3 in the proof of the Euclidean Theorem 4.2.7, and we mostly have to translate the notation. The only genuine difference lies in the justification of the uniform integrability, for which we invoke the moment formula (6.101) itself in place of the Euclidean uniform bound (4.48).

We first take limits on the left-hand side of (6.101). Proposition 6.3.6 gives $u^\varepsilon(t,x) \to u(t,x)$ in $L^k(\Omega)$. Since L^k-convergence of $X_n \to X$ implies $\mathbb{E}[X_n^k] \to \mathbb{E}[X^k]$, we obtain

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big] = \mathbb{E}\Big[u(t,x)^k\Big]. \tag{6.111}$$

We now handle the right-hand side of (6.101). On the probability space Ω_B supporting the Brownian copies (introduced before (6.106)), define the random variables

$$\begin{aligned} V_\varepsilon &:= \prod_{j=1}^k u_0(B^{x,j}_t)\,\mathbf{1}_{\{t<T^j\}}\,\exp\left(\beta^2 \sum_{1\leq i<j\leq k} \langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}^D_\alpha}\right), \\ V &:= \prod_{j=1}^k u_0(B^{x,j}_t)\,\mathbf{1}_{\{t<T^j\}}\,\exp\left(\beta^2 \sum_{1\leq i<j\leq k} \int_0^t G^{D,U}_{2\alpha}(B^{x,i}_s, B^{x,j}_s)\,ds\right), \end{aligned} \tag{6.112}$$

so that the moment formula (6.101) reads $\mathbb{E}[(u^\varepsilon(t,x))^k] = \mathbb{E}_B[V_\varepsilon]$, while the desired identity (6.100) reads $\mathbb{E}[u(t,x)^k] = \mathbb{E}_B[V]$. Taking (6.111) into account, we are thus left with proving that

$$\lim_{\varepsilon\to 0} \mathbb{E}_B[V_\varepsilon] = \mathbb{E}_B[V]. \tag{6.113}$$

We will get (6.113) from Vitali's convergence theorem, which is ensured thanks to two ingredients: convergence in probability of V_ε to V, and boundedness in $L^p(\mathbb{P}_B)$ for $p > 1$ of the family $\{V_\varepsilon\}$. Note that below we will choose $p = 2$.

Let us first check that $V_\varepsilon \to V$ in $\mathbb{P}_B$-probability. Off the event $\cap_{j=1}^k \{t < T^j\}$, both V_ε and V vanish thanks to the indicators in (6.112), and there is nothing to prove. On this event, the $L^1(\mathbb{P}_B)$ convergence (6.105), summed over the pairs $1 \leq i < j \leq k$, shows that the exponent in V_ε converges in $\mathbb{P}_B$-probability to the exponent in V. It then suffices to compose with the continuous function exp and to multiply by the bounded random variable $\prod_{j=1}^k u_0(B^{x,j}_t)$.

We now turn to boundedness in $L^2(\mathbb{P}_B)$. Namely, bounding $|u_0|$ by $\|u_0\|_\infty$ in (6.112) and using $\mathbf{1}^2_{\{t<T^j\}} = \mathbf{1}_{\{t<T^j\}}$, we get

$$\mathbb{E}_B\Big[V_\varepsilon^2\Big] \leq \|u_0\|_\infty^{2k}\,\mathbb{E}_B\left[\prod_{j=1}^k \mathbf{1}_{\{t<T^j\}}\,\exp\left(2\beta^2 \sum_{1\leq i<j\leq k} \langle A^{\varepsilon,i}_{t,x}, A^{\varepsilon,j}_{t,x}\rangle_{\mathcal{H}^D_\alpha}\right)\right]. \tag{6.114}$$

Now the expectation on the right-hand side of (6.114) is exactly the right-hand side of the moment formula (6.101), applied with the initial condition $u_0 \equiv 1$ and with β replaced by $\sqrt{2}\,\beta$. With a slight abuse of notation, we denote again by u^ε the regularized process of Definition 6.3.2 corresponding to those two parameters. Then the moment formula (6.101) recasts (6.114) as

$$\mathbb{E}_B\Big[V_\varepsilon^2\Big] \leq \|u_0\|_\infty^{2k}\,\mathbb{E}\Big[(u^\varepsilon(t,x))^k\Big]. \tag{6.115}$$

Now observe that Proposition 6.3.6 asserts that $u^\varepsilon(t,x)$ converges in $L^k(\Omega)$ as $\varepsilon \to 0$. In particular the right-hand side of (6.115) is bounded for ε in an interval $(0, \varepsilon_0]$. The family $\{V_\varepsilon\,;\,\varepsilon \in (0,\varepsilon_0]\}$ is therefore bounded in $L^2(\mathbb{P}_B)$, and hence uniformly integrable.

Having checked its two hypotheses, Vitali's convergence theorem yields the claim (6.113). Combining (6.111), the moment formula (6.101) and (6.113), we thus obtain

$$\mathbb{E}[u(t,x)^k] = \mathbb{E}_B[V],$$

which is precisely the Feynman-Kac formula (6.100). ■

Remark 6.3.8 (The white noise case $\alpha = 0$)**.** *As in Remark 4.2.8, the kernel $G_{2\alpha}^{D,U}$ measures the time the paths $B^{x,i}$ and $B^{x,j}$ spend near each other. When $\alpha = 0$, $G_0^{D,U}$ degenerates to a Dirac mass on the diagonal (Section 6.1), and the exponent in* (6.100) *would need to be interpreted through an intersection local time for two independent Brownian motions on (X,d,μ), an object that lies beyond the scope of this course. We therefore restrict Theorem 6.3.7 to $\alpha > 0$, and refer to [34] for the analogous Euclidean local-time theory.*

Remark 6.3.9 (Neumann boundary conditions)**.** *The same strategy applies to the Neumann PAM (Remark 6.2.7): replacing $\Delta_{D,U}$, $p_t^{D,U}$, W_α^D and $G_{2\alpha}^{D,U}$ by $\Delta_{N,U}$, $p_t^{N,U}$, W_α^N and $G_{2\alpha}^{N,U} + C_U$ throughout Sections 6.3.1.1–6.3.1.3 yields the Neumann counterpart of the Feynman-Kac moment formula* (6.100).

The formula (6.100) will be our main tool for deriving *lower* bounds on the moments of the solution. Before that, we establish the corresponding upper bounds, which follow a different route: as in the Euclidean Section 4.3.1, they rest on the chaos estimates of Section 6.2.2 combined with hypercontractivity.

6.3.2 Upper bounds on the moments

An analysis of the asymptotic behavior of the moments $\mathbb{E}[u(t,x)^k]$ as $t \to \infty$ brings useful information about the parabolic Anderson model, in particular about its intermittency. Recall that Theorem 6.2.6 already contains an upper bound for the second moment, namely the estimate (6.62) involving the function Θ_α and the spectral gap λ_1. In this subsection we extend (6.62) to moments of every order $k \geq 2$. The strategy is the same as for the Euclidean Theorem 4.3.2: we combine the chaos expansion of u with Nelson's hypercontractivity inequality (Theorem 2.1.9), which upgrades L^2 information on each chaos to L^k information at the price of a factor $(k-1)^{n/2}$ on the n-th chaos. At the level of the kernel estimates, this price amounts to replacing the noise intensity β by $\sqrt{k-1}\,\beta$, and the upper bound below is accordingly obtained from (6.62) through the substitution $\beta \mapsto \sqrt{k-1}\,\beta$. Note that the Feynman-Kac formula (6.100) is not needed at this point. It will come into play for the lower bounds.

Theorem 6.3.10: Upper bound on moments

Let (X, d, μ) satisfy Assumption 5.1.6, let $U \subset X$ be an inner uniform domain (Definition 5.1.13), let $\alpha \geq 0$, $\beta > 0$, and $u_0 \in L^\infty(U, \mu)$. Let u be the unique solution of (6.34) given by Theorem 6.2.6. Then for every integer $k \geq 2$ and every $x \in U$ we have

$$\limsup_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\leq\; \frac{k}{2}\,\Theta_\alpha\Big(\sqrt{k-1}\,\beta\Big) - k\lambda_1, \tag{6.116}$$

where $\lambda_1 = \lambda_1(U)$ is the first eigenvalue of $-\Delta_{D,U}$ (Notation 5.1.20) and Θ_α is the function of Theorem 6.2.6. For $k = 2$ the right-hand side of (6.116) is $\Theta_\alpha(\beta) - 2\lambda_1$, so that (6.116) reduces to the L^2 bound (6.62).

Proof. We proceed in three steps, which parallel the three steps in the proof of the Euclidean Theorem 4.3.2. As we will see, Step 1 is identical in both cases, while Steps 2 and 3 are simpler here: the Laplace-transform estimate (6.41) replaces the Mittag-Leffler analysis of the Euclidean proof. Throughout the proof we fix an integer $k \geq 2$ and we set $M := \|u_0\|_\infty$.

Step 1: Reduction to chaos norms via hypercontractivity. Recall from (6.37) that the solution can be decomposed as

$$u(t,x) = P_t^{D,U} u_0(x) + \sum_{n=1}^{\infty} I_n(f_n(\,\cdot\,, t, x)),$$

with the chaos kernels f_n of (6.38). Nelson's inequality (2.25) only depends on the underlying isonormal Gaussian structure, and therefore applies to the noise W_α^D exactly as it did in the proof of Proposition 6.3.6. Bounding the deterministic term $P_t^{D,U} u_0(x)$ by Corollary 5.1.19 it is readily checked that

$$\|u(t,x)\|_{L^k(\Omega)} \;\leq\; C\,M\,e^{-\lambda_1 t} + \sum_{n=1}^{\infty} (k-1)^{n/2}\sqrt{n!}\,\|f_n(\,\cdot\,,t,x)\|_{(\mathcal{H}_\alpha^D)^{\otimes n}}. \tag{6.117}$$

Relation (6.117) is the exact analogue of (4.87) in the Euclidean case.

Step 2: Choice of ρ. We now invoke the bound (6.41) of Lemma 6.2.3, applied at chaos level n (that is with k replaced by n in (6.41)). For every $\rho > 0$ it yields

$$\sqrt{n!}\,\|f_n(\,\cdot\,,t,x)\|_{(\mathcal{H}_\alpha^D)^{\otimes n}} \;\leq\; M\left(C\beta^2\hat{\Psi}(\rho)\right)^{n/2} e^{(\rho-2\lambda_1)t/2}, \tag{6.118}$$

where $\hat{\Psi}$ is the Laplace transform defined in (6.42). Multiplying (6.118) by $(k-1)^{n/2}$ and absorbing this factor into the noise intensity, the general term of the series in (6.117) is bounded by

$$(k-1)^{n/2}\sqrt{n!}\,\|f_n(\,\cdot\,,t,x)\|_{(\mathcal{H}^D_\alpha)^{\otimes n}} \;\le\; M\left(C\left(\sqrt{k-1}\,\beta\right)^2\hat{\Psi}(\rho)\right)^{n/2} e^{(\rho-2\lambda_1)t/2}. \tag{6.119}$$

We already encountered the right-hand side of (6.119) in the proof of Proposition 6.3.6 (see (6.99)). There, any ρ achieving $(k-1)C\beta^2\hat{\Psi}(\rho)<1$ was sufficient. Here we additionally track the smallest admissible ρ, since it dictates the exponential rate in (6.116).

To this aim, recall from Step 1 of the proof of Theorem 6.2.6 (case (i), that is $0\le\alpha<d_h/(2d_w)$) that the critical parameter $\rho_c(\beta)$ is defined by relation (6.64) as the unique solution of $C\beta^2F(\rho_c)=1$, with F given by (6.63), and that $\Theta_\alpha(\beta)=\rho_c(\beta)$. The key observation is that the noise intensity enters this definition only through the product $C\beta^2$. Hence the whole of Step 1 there can be applied verbatim with noise intensity $\sqrt{k-1}\,\beta$, producing the critical parameter

$$\rho_{c,k} := \rho_c\left(\sqrt{k-1}\,\beta\right) = \Theta_\alpha\left(\sqrt{k-1}\,\beta\right). \tag{6.120}$$

Exactly as in (6.65), for every $\rho>\rho_{c,k}$ we then get

$$C(k-1)\beta^2\,\hat{\Psi}(\rho) \;\le\; \delta_{\rho,k} \;<\; 1, \qquad \text{where} \qquad \delta_{\rho,k} := C(k-1)\beta^2F(\rho). \tag{6.121}$$

The same conclusion holds in the critical case (ii), replacing F by the function F_2 of Step 4 of the proof of Theorem 6.2.6. It also holds in the supercritical case (iii), replacing F by $F_3(\rho):=1/\rho$ (see (6.76)). Since the three regimes are treated identically from this point on, we do not distinguish them in the sequel.

Step 3: Summation. Fix $\rho>\Theta_\alpha(\sqrt{k-1}\,\beta)$. Plugging (6.119) and (6.121) into (6.117), the chaos series is dominated by the geometric series $\sum_{n\ge1}\delta_{\rho,k}^{n/2}=\delta_{\rho,k}^{1/2}/(1-\delta_{\rho,k}^{1/2})$. Bounding in addition the first term of (6.117) according to $e^{-\lambda_1 t}\le e^{(\rho-2\lambda_1)t/2}$ (recall that $\rho>0$), we end up with

$$\|u(t,x)\|_{L^k(\Omega)} \;\le\; C_{\rho,k}\,M\,e^{(\rho-2\lambda_1)t/2}, \tag{6.122}$$

for a constant $C_{\rho,k}$ depending on $\rho,k,\beta,\alpha,d_h,d_w,U$ but not on t,x. Estimate (6.122) is the L^k analogue of the L^2 estimate (6.69). Observe that, in contrast with Step 3 of the Euclidean proof, no Mittag-Leffler asymptotics (see (4.95) and (4.97)) is needed here: the Laplace-transform bound (6.41) has already traded the factorial gain of the Dirichlet formula (4.92) for the exponential factor $e^{(\rho-2\lambda_1)t}$, and the series is summed by merely tuning ρ.

We can now conclude. Since $\mathbb{E}[u(t,x)^k]\le\mathbb{E}[|u(t,x)|^k]=\|u(t,x)\|^k_{L^k(\Omega)}$, raising (6.122) to the power k gives

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\le\; C^k_{\rho,k}\,M^k\,e^{k(\frac{\rho}{2}-\lambda_1)t},$$

and therefore

$$\limsup_{t\to+\infty}\frac{\ln\mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\le\; k\left(\frac{\rho}{2}-\lambda_1\right), \qquad \text{for every } \rho>\Theta_\alpha\left(\sqrt{k-1}\,\beta\right). \tag{6.123}$$

Letting $\rho\downarrow\Theta_\alpha(\sqrt{k-1}\,\beta)$ in (6.123), exactly as in (6.70), yields (6.116) and finishes the proof. ∎

Remark 6.3.11. *Let us compare* (6.116) *with its Euclidean counterpart* (4.85)*, whose exponential rate is of order* $k^{(4-a)/(2-a)}$*, where* a *is the exponent of the power-law relation* (4.84) *in Assumption 4.3.1. Fix* $\beta > 0$*, consider the subcritical regime* $0 < \alpha < d_h/(2d_w)$*, and let* $k \to \infty$*. The large-*β *asymptotics of* Θ_α *in case (i) of Theorem 6.2.6 yield*

$$\frac{k}{2}\,\Theta_\alpha\Big(\sqrt{k-1}\,\beta\Big) \asymp k^{\frac{(2+2\alpha)d_w-d_h}{(1+2\alpha)d_w-d_h}}, \qquad k \to \infty. \tag{6.124}$$

On the other hand, by Proposition 5.1.26 (i) (applied with α *replaced by* 2α*), the covariance kernel* $G_{2\alpha}^{D,U}$ *of the noise behaves like* $d(x,y)^{-a}$*, where the parameter* a *is defined by*

$$a := d_h - 2\alpha d_w. \tag{6.125}$$

This parameter will feature prominently in our computations below. The subcriticality of α *amounts to* $a > 0$*, while the recurrence condition* $d_h < d_w$ *of Assumption 5.1.6 gives* $a < d_w$*. In the Euclidean normalization* $d_w = 2$ *and* $d_h = d$*, we thus get* $a \in (0, \min(2,d))$*, which is the range prescribed for the exponent in* (4.84)*, and a direct computation shows that the exponent in* (6.124) *becomes*

$$\frac{(2+2\alpha)d_w - d_h}{(1+2\alpha)d_w - d_h} = \frac{4+4\alpha-d}{2+4\alpha-d} = \frac{4-a}{2-a}.$$

Hence it is possible to match the growth in k *of the geometric bound* (6.116) *with the Euclidean exponent in* (4.85)*. Observe however that playing with* d_w *and* d_h*, one gets a wide variety of growth exponents in* (6.116)*.*

6.3.3 Lower bounds on the moments

We now derive lower bounds matching (at least partially) the upper bounds of Theorem 6.3.10, following [10]. This is where the Feynman-Kac formula of Theorem 6.3.7 comes into play, as announced at the beginning of Section 6.3.2. Indeed, the upper bounds only required the chaos expansion of u. However, the lower bounds hinge on the fact that the exponent in (6.100) is nonnegative and can be bounded below on suitable confinement events. Our strategy is modeled on the Euclidean Theorem 4.3.3, whose proof can be briefly summarized as follows:

(a) We restricted the expectation in the Feynman-Kac formula (4.81) to the small ball event (4.100), as mentioned in Remark 4.3.4.

(b) On this event the interaction $\Lambda(B_s^i - B_s^j)$ is at least $c_\Lambda \varepsilon^{-a}$ owing to the power lower bound in (4.84), while the probability of the event itself is controlled by the Gaussian small ball estimate (4.103).

(c) An optimization over the radius ε then produced the rate $k^{(4-a)/(2-a)}$ in (4.98).

The geometric proof will follow the same pattern, with three substitutions:

(i) The small ball event (4.100) was defined through the coordinates $B^{i,l}$ of the Brownian motions, which are not available on X. It is replaced by the event that each path $B^{x,j}$ stays in a small ball $B(x, \varepsilon)$ up to time t, expressed by means of exit times as in Notation 6.3.1.

(ii) The power lower bound in (4.84) is replaced by the lower bounds on the Riesz kernel $G_{2\alpha}^{D,U}$ gathered in Proposition 5.1.26. Note that the three regimes of Proposition 5.1.26 will produce three different rates, mirroring the three regimes of the function Θ_α in Theorem 6.2.6. In addition, the eigenfunction factor $\Phi_1(y)\Phi_1(z)$ in those lower bounds vanishes at the boundary, which forces us to confine the paths in a compact subset of U.

(iii) The Gaussian small ball estimate (4.103) is replaced by a spectral estimate for the exit time of a small ball (see Lemma 6.3.12 below). The confinement cost over a time interval of length t becomes $\exp(-c\,t/\varepsilon^{d_w})$ instead of $\exp(-C_0 t/\varepsilon^2)$, in accordance with the interpretation of the walk dimension d_w given in Remark 5.1.9.

We start with the small ball estimate mentioned in item (iii) above.

Lemma 6.3.12: Small ball estimates

Let (X, d, μ) satisfy Assumption 5.1.6, and let $U \subset X$ be a bounded open set. We consider $x \in U$ and $\varepsilon > 0$ such that $\overline{B(x, \varepsilon)} \subset U$. Write $V_\varepsilon := B(x, \varepsilon)$, and let

$$T_\varepsilon := \inf\Big\{s \geq 0 :\, B_s^x \notin V_\varepsilon\Big\} \tag{6.126}$$

be the exit time from V_ε of the Brownian motion B^x started at x (Definition 5.1.8). Moreover, let $\lambda_1(V_\varepsilon)$ be the smallest Dirichlet eigenvalue of V_ε, that is the bottom of the spectrum of the operator $-\Delta_{D,V_\varepsilon}$ given by Proposition 5.1.12. Then the following holds:

(i) The exit time T_ε satisfies the spectral lower bound

$$\liminf_{t \to +\infty} \frac{\ln \mathbb{P}_B(T_\varepsilon > t)}{t} \;\geq\; -\lambda_1(V_\varepsilon). \tag{6.127}$$

(ii) There exists a constant $c_\lambda > 0$, depending only on the constants of Assumption 5.1.6, such that

$$\lambda_1(V_\varepsilon) \;\leq\; \frac{c_\lambda}{\varepsilon^{d_w}}. \tag{6.128}$$

Proof. We focus on the proof of (6.127). Indeed, the eigenvalue bound (6.128) is a consequence of the sub-Gaussian estimate (5.6), and we refer to [45, Lem. 7.1] for a proof.

In order to prove (6.127), observe that the ball V_ε is a bounded open set with compact closure. Therefore the constructions of Section 5.1.2 apply with V_ε in place of U. Namely Proposition 5.1.12 yields the Dirichlet Laplacian Δ_{D,V_ε} and the associated semigroup P_t^{D,V_ε}, while the representation (5.12), applied on V_ε with $f \equiv 1$, reads

$$\mathbb{P}_B\Big(T_\varepsilon > t\Big) \;=\; P_t^{D,V_\varepsilon}\mathbf{1}(x). \tag{6.129}$$

In addition, exactly as in Notation 5.1.20, the operator $-\Delta_{D,V_\varepsilon}$ admits a discrete spectrum $0 < \lambda_1(V_\varepsilon) \le \lambda_2(V_\varepsilon) \le \cdots$, together with an orthonormal basis $\{\Phi_j^\varepsilon\}_{j\ge 1}$ of $L^2(V_\varepsilon,\mu)$ made of eigenfunctions. Observe that V_ε is a connected set. Indeed the space (X,d) is geodesic (Definition 5.1.1), and for $y \in V_\varepsilon$ every point w on a geodesic from x to y satisfies $d(x,w) \le d(x,y) < \varepsilon$. Hence, owing to the Perron-Frobenius argument recalled in Remark 5.1.17, the ground state Φ_1^ε can be chosen to be strictly positive on V_ε.

Next we claim that Φ_1^ε is a bounded function. To this aim, fix $t_0 \in (0, \operatorname{diam}(X)^{d_w}/2)$. The eigenfunction relation gives

$$\Phi_1^\varepsilon = e^{\lambda_1(V_\varepsilon)t_0} P_{t_0}^{D,V_\varepsilon} \Phi_1^\varepsilon. \tag{6.130}$$

Moreover, the semigroup $P_{t_0}^{D,V_\varepsilon}$ admits a kernel $p_{t_0}^{D,V_\varepsilon}$, given by the analogue of the spectral expansion (5.20) on V_ε, and this kernel satisfies $p_{t_0}^{D,V_\varepsilon} \le p_{t_0}$ exactly as in Remark 5.1.16. Hence for every $y \in V_\varepsilon$ we get

$$\left| P_{t_0}^{D,V_\varepsilon} \Phi_1^\varepsilon(y) \right| \;\le\; \left(\int_{V_\varepsilon} p_{t_0}^{D,V_\varepsilon}(y,z)^2 \, d\mu(z) \right)^{1/2} \;=\; p_{2t_0}^{D,V_\varepsilon}(y,y)^{1/2} \;\le\; p_{2t_0}(y,y)^{1/2}. \tag{6.131}$$

Let us detail the three relations in (6.131). The first inequality is obtained thanks to Cauchy-Schwarz and the normalization $\|\Phi_1^\varepsilon\|_{L^2(V_\varepsilon,\mu)} = 1$. The middle identity stems from the symmetry of the kernel $p_{t_0}^{D,V_\varepsilon}$ and the semigroup property, as in Corollary 5.1.18. The last inequality is due to the domination $p_{2t_0}^{D,V_\varepsilon} \le p_{2t_0}$ recalled above. Next, resorting to the on-diagonal upper bound in the sub-Gaussian estimate (5.6), we obtain

$$p_{2t_0}(y,y)^{1/2} \;\le\; c\, t_0^{-d_h/(2d_w)}. \tag{6.132}$$

Gathering (6.131) and (6.132) and recalling the eigenfunction relation (6.130), we end up with

$$\|\Phi_1^\varepsilon\|_\infty \;\le\; c\, e^{\lambda_1(V_\varepsilon)t_0}\, t_0^{-d_h/(2d_w)} \;<\; +\infty.$$

This shows that Φ_1^ε is a bounded function on V_ε, as claimed.

We can now conclude. Since Φ_1^ε is positive and bounded we have $\mathbf{1} \ge \Phi_1^\varepsilon/\|\Phi_1^\varepsilon\|_\infty$ on V_ε. Furthermore, P_t^{D,V_ε} preserves positivity (which is clear from the representation (5.12)). Hence (6.129) yields

$$\mathbb{P}_B\Big(T_\varepsilon > t\Big) \;\ge\; \frac{P_t^{D,V_\varepsilon}\Phi_1^\varepsilon(x)}{\|\Phi_1^\varepsilon\|_\infty} \;=\; \frac{\Phi_1^\varepsilon(x)}{\|\Phi_1^\varepsilon\|_\infty}\, e^{-\lambda_1(V_\varepsilon)t}. \tag{6.133}$$

Since $\Phi_1^\varepsilon(x) > 0$, taking logarithms in (6.133), dividing by t and letting $t \to +\infty$ we obtain (6.127). ∎

With Lemma 6.3.12 in hand, we can turn to the main results of this subsection, namely the lower bounds on the moments of u. Since those results share a common setting, we first gather our standing assumptions.

Assumption 6.3.13: Setting for the moment lower bounds

As in Theorem 6.3.7, let (X, d, μ) satisfy Assumption 5.1.6, let $U \subset X$ be an inner uniform domain (Definition 5.1.13), let $\alpha > 0$, $\beta > 0$, and let u be the unique solution of (6.34) given by Theorem 6.2.6. We assume that $u_0 \in L^\infty(U, \mu)$ is nonnegative and bounded away from 0 near a point $x \in U$. That is, we suppose that there exist $m_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\overline{B(x, \varepsilon_0)} \subset U, \qquad \text{and} \qquad u_0 \geq m_0 \quad \mu\text{-a.e. on } B(x, \varepsilon_0). \tag{6.134}$$

We also introduce the reduced radius

$$\varepsilon_1 := \frac{1}{4} \min(\varepsilon_0, 1), \tag{6.135}$$

which will serve as a maximal confinement radius in the proofs below.

Our first lower bound is valid for every $\alpha > 0$ and every noise intensity $\beta > 0$. Its proof also contains the confinement procedure on which the other lower bounds of this subsection rely.

Theorem 6.3.14: Lower bound on moments, general case

Let Assumption 6.3.13 prevail. Then one can find a strictly positive constant c_1, depending on x, ε_0, α, U and the constants of Assumption 5.1.6, but not on k or β, such that for every integer $k \geq 2$ we have

$$\liminf_{t \to +\infty} \frac{\ln \mathbb{E}\left[u(t, x)^k\right]}{t} \geq c_1 \beta^2 k(k-1) - k\, \lambda_1\Big(B(x, \varepsilon_1)\Big), \tag{6.136}$$

where $\lambda_1(B(x, \varepsilon_1))$ is the first Dirichlet eigenvalue of the ball $B(x, \varepsilon_1)$, as defined in Lemma 6.3.12, and ε_1 is the reduced radius (6.135).

Proof. We divide this proof in three steps, which are the counterparts of Steps 1–3 in the proof of Theorem 4.3.3. Throughout the proof, $\varepsilon \in (0, \varepsilon_1]$ denotes a confinement radius, where ε_1 is the reduced radius (6.135). Although the elementary choice $\varepsilon = \varepsilon_1$ will suffice for (6.136), we carry out Steps 1 and 2 for a generic $\varepsilon \in (0, \varepsilon_1]$. In this way, the intermediate bound (6.146) below can be recycled for the optimization procedures leading to Theorems 6.3.15 and 6.3.16. Observe that the reduction from ε_0 to ε_1 grants two properties which will be used repeatedly below. Namely for every $\varepsilon \in (0, \varepsilon_1]$ we have $2\varepsilon \leq 1/2$, as well as the inclusions

$$\overline{B(x, 3\varepsilon)} \subset \overline{B(x, \varepsilon_0)} \subset U. \tag{6.137}$$

Step 1: Confinement. For $1 \le j \le k$ let

$$T_\varepsilon^j := \inf\Big\{ s \ge 0 : \, B_s^{x,j} \notin B(x,\varepsilon) \Big\}, \tag{6.138}$$

where $B^{x,1}, \dots, B^{x,k}$ are the independent Brownian copies of Notation 6.3.1. Then we consider the event

$$\mathcal{A}_{\varepsilon,t} := \bigcap_{j=1}^{k} \Big\{ t < T_\varepsilon^j \Big\}. \tag{6.139}$$

The event $\mathcal{A}_{\varepsilon,t}$ is the analogue of the small ball event (4.100), with the coordinatewise conditions $\sup_{0\le s\le t} |B_s^{i,l}| \le \varepsilon/2$ replaced by exit time conditions. Since $B(x,\varepsilon) \subset U$, we have $T_\varepsilon^j \le T^j$, and thus $\mathbf{1}_{\{t<T^j\}} \ge \mathbf{1}_{\{t<T_\varepsilon^j\}}$ for every j. In addition, on the event $\mathcal{A}_{\varepsilon,t}$ we have $B_t^{x,j} \in B(x,\varepsilon) \subset B(x,\varepsilon_0)$, so that $u_0(B_t^{x,j}) \ge m_0$ almost surely thanks to (6.134). Since u_0 is nonnegative, the integrand in the Feynman-Kac formula (6.100) is nonnegative, and restricting the expectation to $\mathcal{A}_{\varepsilon,t}$ we get the analogue of (4.99):

$$\mathbb{E}\Big[u(t,x)^k \Big] \;\ge\; m_0^k \, \mathbb{E}_B \left[\exp\left(\beta^2 \sum_{1\le i<j\le k} \int_0^t G_{2\alpha}^{D,U}(B_s^{x,i}, B_s^{x,j})\, ds \right) \mathbf{1}_{\mathcal{A}_{\varepsilon,t}} \right]. \tag{6.140}$$

We now bound the interaction term in (6.140) from below on the event $\mathcal{A}_{\varepsilon,t}$. This is the place where the power lower bound of Assumption 4.3.1 is replaced by Proposition 5.1.26. Fix $s \in [0,t]$ and a pair $i<j$, and abbreviate $y := B_s^{x,i}$, $z := B_s^{x,j}$, so that $y, z \in B(x,\varepsilon)$ on $\mathcal{A}_{\varepsilon,t}$. Our first observation is that the inner distance between y and z is small. Indeed the space (X,d) is geodesic (Definition 5.1.1), so y and z are joined by a curve γ of length $d(y,z) \le 2\varepsilon$. Every point w on γ satisfies $d(w,x) \le d(w,y) + d(y,x) < d(y,z) + \varepsilon \le 3\varepsilon$, so that $\gamma \subset B(x,3\varepsilon) \subset U$ thanks to the inclusions (6.137). According to the definition (5.13) of the inner metric, we thus get

$$d_U(y,z) \;\le\; \mathrm{length}(\gamma) \;=\; d(y,z) \;\le\; 2\varepsilon. \tag{6.141}$$

Our second observation is that, the set $\overline{B(x,\varepsilon_0)}$ being a compact subset of U, the quantity

$$\hat{m} := \inf\Big\{ \Phi_1(w) \,;\; w \in \overline{B(x,\varepsilon_0)} \Big\} \tag{6.142}$$

is strictly positive (see Remark 5.1.16). We now plug (6.141) and (6.142) into the lower bounds (5.27), (5.28) and (5.29) of Proposition 5.1.26, applied with α replaced by 2α (the three cases there correspond to $a>0$, $a=0$ and $a<0$ respectively, for the parameter $a = d_h - 2\alpha d_w$ introduced in (6.125)). Taking into account the fact that $r \mapsto r^{-a}$ is decreasing for $a>0$, as well as the bound $\max(1, |\ln r|) \ge \ln(1/r) \ge \ln(1/(2\varepsilon))$ valid for $0 < r \le 2\varepsilon \le 1/2$ (recall that $2\varepsilon \le 1/2$ thanks to our reduction $\varepsilon \le \varepsilon_1$), we obtain

$$G_{2\alpha}^{D,U}(y,z) \;\ge\; c\, \hat{m}^2\, g_\alpha(\varepsilon), \qquad \text{where} \quad g_\alpha(\varepsilon) := \begin{cases} (2\varepsilon)^{-a}, & \text{if } a>0 \\ \ln\!\left(\frac{1}{2\varepsilon}\right), & \text{if } a=0 \\ 1, & \text{if } a<0, \end{cases} \tag{6.143}$$

and where the bound (6.143) trivially holds when $y = z$. Note for further use that $2\varepsilon \le 1/2$ implies $g_\alpha(\varepsilon) \ge \ln 2$ in all three cases. Reporting (6.143) in (6.140), and recalling that the sum over pairs contains $k(k-1)/2$ terms, we end up with the analogue of (4.101):

$$\mathbb{E}\Big[u(t,x)^k\Big] \;\ge\; m_0^k \exp\Big(c_0\,\beta^2 k(k-1)\,g_\alpha(\varepsilon)\,t\Big)\,\mathbb{P}_B\Big(\mathcal{A}_{\varepsilon,t}\Big), \qquad \text{where} \quad c_0 := \frac{c\,\hat{m}^2}{2}. \tag{6.144}$$

Step 2: Small ball estimate. The paths $B^{x,1},\ldots,B^{x,k}$ being independent copies (Notation 6.3.1), we have, exactly as in (4.102),

$$\mathbb{P}_B\Big(\mathcal{A}_{\varepsilon,t}\Big) = \Big[\mathbb{P}_B\Big(T^1_\varepsilon > t\Big)\Big]^k. \tag{6.145}$$

The role of the Gaussian small ball estimate (4.103) is now played by Lemma 6.3.12 (observe that $\overline{B(x,\varepsilon)} \subset \overline{B(x,\varepsilon_0)} \subset U$ by (6.137), so that the lemma indeed applies to $V_\varepsilon = B(x,\varepsilon)$). Namely, taking logarithms in (6.144), dividing by t and letting $t \to +\infty$, relations (6.145), (6.127) and (6.128) yield

$$\begin{aligned}\liminf_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;&\ge\; c_0\,\beta^2 k(k-1)\,g_\alpha(\varepsilon) - k\,\lambda_1\Big(B(x,\varepsilon)\Big) \\ &\ge\; c_0\,\beta^2 k(k-1)\,g_\alpha(\varepsilon) - \frac{c_\lambda\,k}{\varepsilon^{d_w}}. \end{aligned}\tag{6.146}$$

Relation (6.146) is the counterpart of (4.104), with two differences. First, it is stated at the level of exponential rates rather than for every fixed t, since the survival estimate (6.127) is itself asymptotic. Second, the confinement cost is now $c_\lambda k\varepsilon^{-d_w}$ instead of $C_0 dk\varepsilon^{-2}$, the walk dimension d_w replacing the Euclidean exponent 2.

Step 3: Choice of ε. For the bound (6.136) no optimization is needed, and we simply take $\varepsilon = \varepsilon_1$ in (6.146). Bounding $g_\alpha(\varepsilon_1) \ge \ln 2$ (as noted after (6.143)) and resorting to the first inequality in (6.146), we obtain (6.136) with $c_1 := c_0 \ln 2$. ■

When $(k-1)\beta^2$ is large, the general bound (6.136) can be improved by tuning the confinement radius ε in the intermediate inequality (6.146). Our next result implements this optimization in the subcritical case $\alpha < \frac{d_h}{2d_w}$. Recall that the parameter $a = d_h - 2\alpha d_w$ has been defined in (6.125), so that the subcritical case corresponds to $a > 0$. Observe also that the exponent $\frac{d_w}{d_w - a}$ in (6.147) is strictly greater than 1 when $a > 0$. Hence Theorem 6.3.15 improves on the growth given by (6.136) for large values of $(k-1)\beta^2$.

Theorem 6.3.15: Lower bound on moments, subcritical case

Let Assumption 6.3.13 prevail, and suppose that $0 < \alpha < \frac{d_h}{2d_w}$ (equivalently $a > 0$, where a is the parameter defined in (6.125)). Then one can find strictly positive constants c_2, Q, depending on x, ε_0, α, U and the constants of Assumption 5.1.6, but not on k or β, such that for every integer $k \ge 2$ and every β satisfying $(k-1)\beta^2 \ge Q$ we have

$$\liminf_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\ge\; c_2\,k\,\Big((k-1)\beta^2\Big)^{\frac{d_w}{(1+2\alpha)d_w - d_h}} \;=\; c_2\,k\,\Big((k-1)\beta^2\Big)^{\frac{d_w}{d_w - a}}. \tag{6.147}$$

Proof. Steps 1 and 2 in the proof of Theorem 6.3.14 have been carried out for a generic confinement radius $\varepsilon \in (0, \varepsilon_1]$. We can thus start from the intermediate bound (6.146) obtained there, and our task is reduced to optimizing over ε. In the computations below, c_0 and c_λ respectively denote the constants featured in (6.144) and (6.128).

We first note that $a < d_w$, since the definition (6.125) of a gives $a < d_h$, and $d_h < d_w$ by the recurrence condition in Assumption 5.1.6. Owing to (6.146) and the definition of g_α in (6.143), we are reduced to maximize over $\varepsilon \in (0, \varepsilon_1]$ the function

$$h(\varepsilon) := A\,\varepsilon^{-a} - c_\lambda\, k\, \varepsilon^{-d_w}, \qquad \text{where} \quad A := 2^{-a}\, c_0\, \beta^2 k(k-1). \tag{6.148}$$

This is the same optimization as in Step 3 of the proof of Theorem 4.3.3, with the Euclidean parameters $(2, C_0 dk)$ replaced by $(d_w, c_\lambda k)$. Specifically, setting the derivative of h to zero gives the following analogue of (4.105):

$$\varepsilon_* = \left(\frac{d_w\, c_\lambda\, k}{a\, A}\right)^{\frac{1}{d_w - a}} = \left(\frac{2^a\, d_w\, c_\lambda}{a\, c_0}\right)^{\frac{1}{d_w - a}} \left((k-1)\beta^2\right)^{-\frac{1}{d_w - a}}, \tag{6.149}$$

where the second identity stems from the definition of A in (6.148). From the expression (6.149) it is readily checked that $\varepsilon_* \le \varepsilon_1$ if and only if

$$(k-1)\beta^2 \;\ge\; \frac{2^a\, d_w\, c_\lambda}{a\, c_0\, \varepsilon_1^{d_w - a}}, \tag{6.150}$$

which is granted by our standing assumption $(k-1)\beta^2 \ge Q$, provided we choose Q to be the right-hand side of (6.150). At the point ε_* the optimality condition reads $c_\lambda k \varepsilon_*^{-d_w} = \frac{a}{d_w} A \varepsilon_*^{-a}$, so that, exactly as in Step 3 of the proof of Theorem 4.3.3,

$$h(\varepsilon_*) = \left(1 - \frac{a}{d_w}\right) A\, \varepsilon_*^{-a}. \tag{6.151}$$

Substituting the value (6.149) of ε_* into (6.151), and resorting to the elementary identity $1 + \frac{a}{d_w - a} = \frac{d_w}{d_w - a}$ (which replaces the Euclidean identity $2 + \frac{a}{2-a} = \frac{4-a}{2-a}$ invoked for Theorem 4.3.3), we end up with

$$h(\varepsilon_*) = \left(1 - \frac{a}{d_w}\right) 2^{-a} c_0 \left(\frac{a\, c_0}{2^a\, d_w\, c_\lambda}\right)^{\frac{a}{d_w - a}} k\left((k-1)\beta^2\right)^{\frac{d_w}{d_w - a}} \equiv c_2\, k\left((k-1)\beta^2\right)^{\frac{d_w}{d_w - a}}, \tag{6.152}$$

where $c_2 > 0$ thanks to $a < d_w$. Recalling that $d_w - a = (1 + 2\alpha) d_w - d_h$, relation (6.152) proves (6.147). ■

We now turn to the critical case $\alpha = \frac{d_h}{2 d_w}$, that is $a = 0$, for which the improvement on the general bound (6.136) is of logarithmic order.

Theorem 6.3.16: Lower bound on moments, critical case

Let Assumption 6.3.13 prevail, and suppose that $\alpha = \frac{d_h}{2d_w}$ (equivalently $a = 0$, where a is the parameter defined in (6.125)). Then one can find strictly positive constants c_3, Q, depending on x, ε_0, α, U and the constants of Assumption 5.1.6, but not on k or β, such that for every integer $k \geq 2$ and every β satisfying $(k-1)\beta^2 \geq Q$ we have

$$\liminf_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\geq\; c_3\, k(k-1)\beta^2 \,\ln\Big((k-1)\beta^2\Big). \tag{6.153}$$

Proof. As in the proof of Theorem 6.3.15, we start from the bound (6.146), established in Steps 1 and 2 of the proof of Theorem 6.3.14 for a generic $\varepsilon \in (0, \varepsilon_1]$. We also keep the notation c_0 and c_λ for the constants in (6.144) and (6.128). In the current situation we have $a = 0$, and thus the function g_α in (6.143) is given by $g_\alpha(\varepsilon) = \ln(\frac{1}{2\varepsilon})$. The two terms in (6.146) are then balanced, up to the logarithmic factor, by the choice

$$\varepsilon := \Big((k-1)\beta^2\Big)^{-1/d_w}, \tag{6.154}$$

which satisfies $\varepsilon \leq \varepsilon_1$ as soon as $(k-1)\beta^2 \geq \varepsilon_1^{-d_w}$. With the choice (6.154) we have $\varepsilon^{-d_w} = (k-1)\beta^2$ and $\ln(\frac{1}{2\varepsilon}) = \frac{1}{d_w}\ln((k-1)\beta^2) - \ln 2$. Hence (6.146) becomes

$$\liminf_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\geq\; k(k-1)\beta^2\Big(\frac{c_0}{d_w}\ln\Big((k-1)\beta^2\Big) - c_0\ln 2 - c_\lambda\Big). \tag{6.155}$$

If in addition $\ln((k-1)\beta^2) \geq \frac{2d_w}{c_0}(c_0\ln 2 + c_\lambda)$, the parenthesis in (6.155) is bounded below by $\frac{c_0}{2d_w}\ln((k-1)\beta^2)$, which yields (6.153) with $c_3 := \frac{c_0}{2d_w}$. Summarizing, it suffices to choose the constant Q in the statement so that $(k-1)\beta^2 \geq Q$ implies both $(k-1)\beta^2 \geq \varepsilon_1^{-d_w}$ and $\ln((k-1)\beta^2) \geq \frac{2d_w}{c_0}(c_0\ln 2 + c_\lambda)$. The proof is now complete. ∎

Remark 6.3.17 (Comparison with the upper bounds)**.** *Let us compare the lower bounds of Theorems 6.3.14, 6.3.15 and 6.3.16 with the upper bounds of Theorem 6.3.10 regime by regime, in the spirit of Remark 4.3.4. In order to ease the reading, all the exponents below are expressed in terms of the parameter $a = d_h - 2\alpha d_w$ introduced in (6.125).*

(i) *Subcritical regime: matching rates. Let $0 < \alpha < \frac{d_h}{2d_w}$, that is $a > 0$. The large-β asymptotics (6.72) of Θ_α show that the upper bound in (6.116) satisfies*

$$\frac{k}{2}\,\Theta_\alpha\Big(\sqrt{k-1}\,\beta\Big) \;\asymp\; k\,\Big((k-1)\beta^2\Big)^{\frac{d_w}{d_w-a}}, \qquad \textit{as} \quad (k-1)\beta^2 \to +\infty, \tag{6.156}$$

which is exactly the growth of the lower bound (6.147). Since the dissipative terms ($-k\lambda_1$ in (6.116)) only grow linearly in k, we obtain, for a fixed $\beta > 0$ and $k \to \infty$,

$$c\,k^{\frac{2d_w-a}{d_w-a}} \;\leq\; \liminf_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\leq\; \limsup_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^k\Big]}{t} \;\leq\; C\,k^{\frac{2d_w-a}{d_w-a}} \tag{6.157}$$

for k large enough. The super-linear growth exponent $\frac{2d_w-a}{d_w-a}$ in (6.157) is the geometric analogue of the Euclidean exponent $\frac{4-a}{2-a}$ obtained in Theorems 4.3.2 and 4.3.3. Indeed, as detailed in Remark 6.3.11, in the Euclidean normalization $d_w = 2$ our parameter (6.125) coincides with the exponent a of (4.84), and $\frac{2d_w-a}{d_w-a}$ then reduces to $\frac{4-a}{2-a}$. The solution u therefore exhibits the full intermittency phenomenon described in Remark 4.3.4. Similarly, for a fixed k and $\beta \to +\infty$, the growth $\beta^{\frac{2d_w}{d_w-a}}$ is matched on both sides.

(ii) *Critical regime: a logarithmic gap. Let $\alpha = \frac{d_h}{2d_w}$, that is $a = 0$. In this case the asymptotics (6.75) yield an upper bound of order $k(k-1)\beta^2\,(\ln((k-1)\beta^2))^2$ in (6.116), while the lower bound (6.153) only exhibits one logarithmic factor. The two bounds thus differ by a factor $\ln((k-1)\beta^2)$, and we are not aware of a result capturing the exact critical rate.*

Remark 6.3.18 (Phase transition in β)**.** *The combination of Theorems 6.3.10 and 6.3.14 also reveals a phase transition in the noise intensity β. Namely fix $k \geq 2$ and $\alpha > 0$. For small β, the asymptotics $\Theta_\alpha(\beta') \sim C\beta'^2$ as $\beta' \to 0$ (valid in the three regimes, see Theorem 6.2.6) show that the right-hand side of (6.116) is negative for β small enough, so that $\mathbb{E}[u(t,x)^k]$ decays exponentially fast (the dissipation induced by the spectral gap λ_1 wins over the noise). On the other hand, the general lower bound (6.136) is positive as soon as $c_1(k-1)\beta^2 > \lambda_1(B(x,\varepsilon_1))$, in which case the moments grow exponentially fast (the noise wins). This weak/strong disorder transition is not a feature of the geometric setting, but rather of the Dirichlet boundary condition on the bounded domain U. Indeed, on the full space $\mathbb{R}^d$ the lower bound (4.98) of Theorem 4.3.3 entails exponential growth for every $\beta > 0$. In contrast, a transition of the same nature was exhibited in [21] for the stochastic heat equation on a Euclidean interval with Dirichlet boundary condition, driven by a space-time white noise (in that setting the transition also disappears when the Dirichlet condition is replaced by a Neumann condition). We refer to [10, Cor. 3.19] for a detailed study of the transition in our geometric framework.*

Remark 6.3.19 (Lower bounds under Neumann boundary conditions)**.** *As mentioned in Remark 6.3.18, the phase transition disappears when the Dirichlet boundary condition is replaced by a Neumann condition. Although we will not include the corresponding computations in these notes, let us summarize the results of [10, Section 4.3]. In that reference, the parabolic Anderson model is considered on a uniform domain U (Definition 5.1.30), for a noise whose spatial covariance is defined through the Neumann fractional Laplacian of Definition 5.1.35. The moments of the solution admit a Feynman-Kac representation analogous to (6.100), in which the killed Brownian motions are replaced by reflected Brownian motions (recall Theorem 5.1.32) and the kernel $G^{D,U}_{2\alpha}$ is replaced by its Neumann counterpart $G^{N,U}_{2\alpha}$ of Proposition 5.1.37. Starting from this representation, for $\alpha > 0$ and an initial condition satisfying $u_0 \geq m_0 > 0$ on U, it is proved in [10, Thm. 4.6] that*

$$\liminf_{t\to+\infty} \frac{\ln \mathbb{E}\left[u(t,x)^k\right]}{t} \geq c\,\beta^2 k(k-1), \qquad \textit{for every } \beta > 0. \tag{6.158}$$

The mechanism behind (6.158) is that the Neumann semigroup is conservative, that is $P^{N,U}_t \mathbf{1} = \mathbf{1}$ (equivalently, the principal Neumann eigenvalue vanishes, see Proposi-

tion 5.1.34). Hence no confinement of the Brownian paths is needed, and no eigenvalue term like $-k\lambda_1(B(x,\varepsilon_1))$ *arises in* (6.158)*. In particular the moments grow exponentially fast for every* $\beta > 0$*, which confirms that the transition of Remark 6.3.18 is specific to the Dirichlet case. Let us also mention that the analogues of Theorems 6.3.15 and 6.3.16 hold in the Neumann setting. Namely as* $\beta \to +\infty$ *the rate in* (6.158) *can be improved to* $c\,k(k-1)\,\beta^{\frac{2d_w}{d_w-a}}$ *when* $a > 0$*, and to* $c\,k(k-1)\,\beta^2 \ln\beta$ *when* $a = 0$*, where* a *is the parameter of* (6.125)*. Finally, in the white noise case* $\alpha = 0$ *the second moment satisfies a lower bound with rate* $\beta^{\frac{2d_w}{d_w-d_h}}$*, which is consistent with the substitution* $a = d_h$ *in the subcritical exponent.*

Remark 6.3.20 (The white noise case $\alpha = 0$)**.** *Theorems 6.3.14, 6.3.15 and 6.3.16 leave out the white noise case* $\alpha = 0$*, since their proofs rest on the Feynman-Kac formula of Theorem 6.3.7 (recall from Remark 6.3.8 that the case* $\alpha = 0$ *would require intersection local times on* X*). A lower bound for the second moment can still be obtained by a renewal argument performed directly on the mild equation* (6.35)*, in the spirit of the Laplace transform computations in the proof of Lemma 6.2.3. One gets*

$$\liminf_{t\to+\infty} \frac{\ln \mathbb{E}\Big[u(t,x)^2\Big]}{t} \;\geq\; \Upsilon(\beta^2) - 2\lambda_1, \tag{6.159}$$

where Υ *is the inverse of the function* $y \mapsto C_1\, y\, e^{C_2 y^{1/d_w}}$ *and* C_1, C_2 *are positive constants. We refer to [10, Thm. 3.17] for the details.*

Bibliography

[1] Robert J. Adler and Jonathan E. Taylor. *Random Fields and Geometry*. Springer Monographs in Mathematics. Springer, New York, 2007. ISBN 978-0-387-48112-8.

[2] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011. ISBN 978-3-642-16829-1. doi: 10.1007/978-3-642-16830-7.

[3] Raluca M. Balan and Daniel Conus. Intermittency for the wave and heat equations with fractional noise in time. *Ann. Probab.*, 44(2):1488–1534, 2016. doi: 10.1214/15-AOP1005.

[4] Martin T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, volume 1690 of *Lecture Notes in Math.*, pages 1–121. Springer, Berlin, 1998. doi: 10.1007/BFb0092537. URL https://mathscinet.ams.org/mathscinet-getitem?mr=1668115.

[5] Martin T. Barlow. *Random Walks and Heat Kernels on Graphs*, volume 438 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2017. doi: 10.1017/9781107415690.

[6] Fabrice Baudoin and Li Chen. Dirichlet fractional Gaussian fields on the Sierpinski gasket and their discrete graph approximations. *Stochastic Process. Appl.*, 162:593–616, 2023. ISSN 0304-4149,1879-209X. doi: 10.1016/j.spa.2023.05.005. URL https://doi.org/10.1016/j.spa.2023.05.005.

[7] Fabrice Baudoin and Céline Lacaux. Fractional Gaussian fields on the Sierpiński gasket and related fractals. *J. Anal. Math.*, 146(2):719–739, 2022. ISSN 0021-7670. doi: 10.1007/s11854-022-0211-x. URL https://doi.org/10.1007/s11854-022-0211-x.

[8] Fabrice Baudoin, Li Chen, Che-Hung Huang, Cheng Ouyang, Samy Tindel, and Jing Wang. Parabolic Anderson model on Heisenberg groups: weighted Besov spaces and Stratonovich setting. *preprint*, 2023.

[9] Fabrice Baudoin, Cheng Ouyang, Samy Tindel, and Jing Wang. Parabolic Anderson model on Heisenberg groups: the Itô setting. *J. Funct. Anal.*, 285(1):Paper No. 109920, 44, 2023. ISSN 0022-1236,1096-0783. doi: 10.1016/j.jfa.2023.109920. URL https://doi.org/10.1016/j.jfa.2023.109920.

[10] Fabrice Baudoin, Li Chen, Che-Hung Huang, Cheng Ouyang, Samy Tindel, and Jing Wang. Parabolic Anderson model in bounded domains of recurrent metric measure spaces. *Trans. Amer. Math. Soc.*, 379(3):1799–1851, 2026. ISSN 0002-9947,1088-6850. doi: 10.1090/tran/9540. URL https://doi-org.ezproxy.lib.purdue.edu/10.1090/tran/9540.

[11] Lorenzo Bertini and Nicoletta Cancrini. The stochastic heat equation: Feynman-Kac formula and intermittence. *J. Statist. Phys.*, 78(5-6):1377–1401, 1995. ISSN 0022-4715. doi: 10.1007/BF02180136. URL https://doi.org/10.1007/BF02180136.

[12] René A. Carmona and S. A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.*, 108(518):viii+125, 1994. ISSN 0065-9266. doi: 10.1090/memo/0518. URL https://doi.org/10.1090/memo/0518.

[13] Hongyi Chen and Cheng Ouyang. Global geometry within an SPDE well-posedness problem. 2025. URL https://arxiv.org/abs/2502.04572. Preprint arXiv:2502.04572.

[14] Hongyi Chen, Robert Neel, and Cheng Ouyang. Sharp Riemannian heat kernel estimates on the cut locus and the parabolic Anderson model. 2026. URL https://arxiv.org/abs/2603.26936. Preprint arXiv:2603.26936.

[15] Xia Chen, Aurélien Deya, Cheng Ouyang, and Samy Tindel. Moment estimates for some renormalized parabolic Anderson models. *Ann. Probab.*, 49(5):2599–2636, 2021. ISSN 0091-1798. doi: 10.1214/21-aop1517. URL https://doi.org/10.1214/21-aop1517.

[16] Francis Comets. *Directed polymers in random environments*, volume 2175 of *Lecture Notes in Mathematics*. Springer, Cham, 2017. ISBN 978-3-319-50486-5; 978-3-319-50487-2. doi: 10.1007/978-3-319-50487-2. URL https://doi.org/10.1007/978-3-319-50487-2. Lecture notes from the 46th Probability Summer School held in Saint-Flour, 2016.

[17] Clément Cosco, Inbar Seroussi, and Ofer Zeitouni. Directed polymers on infinite graphs. *Communications in mathematical physics*, 386(1):395–432, 2021.

[18] Robert Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous spde's. *Electronic Journal of Probability*, 4:1–29, 1999.

[19] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. ISBN 978-3-540-85993-2. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[20] Mohammud Foondun and Davar Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.*, 14:no. 21, 548–568, 2009. ISSN 1083-6489. doi: 10.1214/EJP.v14-614. URL https://doi.org/10.1214/EJP.v14-614.

[21] Mohammud Foondun and Eulalia Nualart. On the behaviour of stochastic heat equations on bounded domains. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(2):551–571, 2015.

[22] M. Fukushima and T. Shima. On a spectral analysis for the Sierpiński gasket. *Potential Anal.*, 1(1):1–35, 1992. ISSN 0926-2601. doi: 10.1007/BF00249784. URL https://mathscinet.ams.org/mathscinet-getitem?mr=1245223.

[23] Masatoshi Fukushima, Yoichi Ōshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994. ISBN 3-11-011626-X. doi: 10.1515/9783110889741. URL https://doi.org/10.1515/9783110889741.

[24] A. M. Garsia, E. Rodemich, and H. Rumsey, Jr. A real variable lemma and the continuity of paths of some Gaussian processes. *Indiana Univ. Math. J.*, 20:565–578, 1970/71. ISSN 0022-2518. doi: 10.1512/iumj.1970.20.20046. URL https://doi.org/10.1512/iumj.1970.20.20046.

[25] Jürgen Gärtner and Wolfgang König. Moment asymptotics for the continuous parabolic Anderson model. *Ann. Appl. Probab.*, 10(1):192–217, 2000. ISSN 1050-5164. doi: 10.1214/aoap/1019737669. URL https://doi.org/10.1214/aoap/1019737669.

[26] Jürgen Gärtner and Wolfgang König. The parabolic Anderson model. In *Interacting stochastic systems*, pages 153–179. Springer, Berlin, 2005. doi: 10.1007/3-540-27110-4_8. URL https://doi.org/10.1007/3-540-27110-4_8.

[27] Loukas Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014. ISBN 978-1-4939-1229-2; 978-1-4939-1230-8. doi: 10.1007/978-1-4939-1230-8. URL https://doi.org/10.1007/978-1-4939-1230-8.

[28] Alexander Grigor'yan. *Heat Kernel and Analysis on Manifolds*, volume 47 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.

[29] Alexander Grigor'yan and Andras Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.*, 40(3):1212–1284, 2012. ISSN 0091-1798,2168-894X. doi: 10.1214/11-AOP645. URL https://doi.org/10.1214/11-AOP645.

[30] Pavel Gyrya and Laurent Saloff-Coste. Neumann and Dirichlet heat kernels in inner uniform domains. *Astérisque*, (336):viii+144, 2011. ISSN 0303-1179,2492-5926.

[31] Sebastian Haeseler. Heat kernel estimates and related inequalities on metric graphs. *arXiv:1101.3010*, 2011.

[32] Martin Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014. ISSN 0020-9910. doi: 10.1007/s00222-014-0505-4. URL https://doi.org/10.1007/s00222-014-0505-4.

[33] Martin Hairer and Cyril Labbé. A simple construction of the continuum parabolic Anderson model on $\mathbf{R}^2$. *Electron. Commun. Probab.*, 20:no. 43, 11, 2015. doi: 10.1214/ECP.v20-4038. URL https://doi.org/10.1214/ECP.v20-4038.

[34] Yaozhong Hu and David Nualart. Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields*, 143(1-2):285–328, 2009. ISSN 0178-8051. doi: 10.1007/s00440-007-0127-5. URL https://doi.org/10.1007/s00440-007-0127-5.

[35] Yaozhong Hu, Jingyu Huang, David Nualart, and Samy Tindel. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electron. J. Probab.*, 20:no. 55, 50, 2015. ISSN 1083-6489. doi: 10.1214/EJP.v20-3316. URL https://doi.org/10.1214/EJP.v20-3316.

[36] Davar Khoshnevisan. *Analysis of stochastic partial differential equations*, volume 119 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 2014. ISBN 978-1-4704-1547-1. doi: 10.1090/cbms/119. URL https://doi.org/10.1090/cbms/119.

[37] Jun Kigami. Harmonic calculus on p.c.f. self-similar sets. *Trans. Amer. Math. Soc.*, 335(2):721–755, 1993. ISSN 0002-9947,1088-6850. doi: 10.2307/2154402. URL https://doi.org/10.2307/2154402.

[38] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. ISBN 0-521-79321-1. doi: 10.1017/CBO9780511470943. URL https://mathscinet.ams.org/mathscinet-getitem?mr=1840042.

[39] Wolfgang König. *The parabolic Anderson model.* Pathways in Mathematics. Birkhäuser/Springer, [Cham], 2016. ISBN 978-3-319-33595-7; 978-3-319-33596-4. doi: 10.1007/978-3-319-33596-4. URL https://doi.org/10.1007/978-3-319-33596-4. Random walk in random potential.

[40] Hubert Lacoin. New bounds for the free energy of directed polymers in dimension 1+ 1 and 1+ 2. *Communications in Mathematical Physics*, 294(2):471–503, 2010.

[41] James R. Lee. Relations between scaling exponents in unimodular random graphs, 2021. Preprint, arXiv:2007.06548v2.

[42] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(1–2):153–201, 1986. doi: 10.1007/BF02399203.

[43] W. V. Li and Q.-M. Shao. Gaussian processes: inequalities, small ball probabilities and applications. In *Stochastic processes: theory and methods*, volume 19 of *Handbook of Statist.*, pages 533–597. North-Holland, Amsterdam, 2001. doi: 10.1016/S0169-7161(01)19019-X.

[44] Janna Lierl. The Dirichlet heat kernel in inner uniform domains in fractal-type spaces. *Potential Anal.*, 57(4):521–543, 2022. ISSN 0926-2601. doi: 10.1007/s11118-021-09926-z. URL https://doi.org/10.1007/s11118-021-09926-z.

[45] Phanuel Mariano and Jing Wang. Spectral bounds for exit times on metric measure Dirichlet spaces and applications. *Stochastic Process. Appl.*, 189:Paper No. 104707, 26, 2025. ISSN 0304-4149,1879-209X. doi: 10.1016/j.spa.2025.104707. URL https://doi-org.ezproxy.lib.purdue.edu/10.1016/j.spa.2025.104707.

[46] David Márquez-Carreras, Carles Rovira, and Samy Tindel. A model of continuous time polymer on the lattice. *Commun. Stoch. Anal.*, 5(1):103–120, 2011. ISSN 2688-6669. doi: 10.31390/cosa.5.1.07. URL https://doi.org/10.31390/cosa.5.1.07.

[47] Mathav Murugan. Heat kernel for reflected diffusion and extension property on uniform domains, 2023.

[48] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006. ISBN 978-3-540-28328-7; 3-540-28328-5.

[49] Olaf Post. *Spectral analysis on graph-like spaces*, volume 2039 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012. ISBN 978-3-642-23839-0. doi: 10.1007/978-3-642-23840-6. URL https://doi-org.ezproxy.lib.uconn.edu/10.1007/978-3-642-23840-6.

[50] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. ISBN 0-12-585004-2.

[51] Carles Rovira and Samy Tindel. On the Brownian-directed polymer in a Gaussian random environment. *Journal of Functional Analysis*, 222(1):178–201, 2005.

[52] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983. ISBN 3-7643-1381-1. doi: 10.1007/978-3-0346-0416-1. URL https://doi.org/10.1007/978-3-0346-0416-1.

[53] John B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986. ISBN 3-540-16441-3. doi: 10.1007/BFb0074920. URL https://doi.org/10.1007/BFb0074920.

[54] Ya. B. Zel'dovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokolov. Intermittency in random media. *Soviet Physics Uspekhi*, 30(5):353–369, 1987. doi: 10.1070/PU1987v030n05ABEH002867. URL https://doi.org/10.1070/PU1987v030n05ABEH002867.