Exercise Solutions

February 2023

1

let $f : X \to \mathbb{P}^n_R$. Let $\mathcal{S} = \{\mathcal{O}(i)[-i] : 0 \leq i \leq n\}$. Then, we know from the first talk that $\mathcal{O}(l) \in \langle \mathcal{S} \rangle_{n+1}$ for all $l \leq 0$.
Note that as $\mathcal{O}(l)$ is flat for any l, we get that $f^*(\mathcal{O}(l)) \cong Lf^*(\mathcal{O}(l))$.
Now, $f^*(\mathcal{O}(l)) \cong Lf^*(\mathcal{O}(l)) \in Lf^*(\langle \mathcal{S} \rangle_{n+1}) \subseteq \langle Lf^* \mathcal{S} \rangle_{n+1}$
So, if $f^*(\mathcal{S}) \subset \mathcal{A} \implies \langle Lf^* \mathcal{S} \rangle_{n+1} \subset \mathcal{A}$ and hence $f^*(\mathcal{O}(l)) \in \mathcal{A}$ for all $l \leq 0$

2

let $\{E_i, f_i : E_i \to E_{i+1}\}$ be a special Cauchy sequence in $\mathbf{D}^{\mathrm{perf}}(X)$. Then, by looking at the long exact sequence corresponding to the triangle defining the homotopy colimit of this Cauchy sequence, it is easy to see that it lies in $\mathbf{D}^{\mathrm{b}}_{\mathbf{coh}}(X)$. Let the homotopy colimit be E.
We note the following important property of these special Cauchy sequences :
By shifting if necessary, we can assume that $E \in \mathbf{D}^{\mathrm{b}}_{\mathbf{coh}}(X)^{\geq 0}$. For large enough i, $H^n(E_i) \to H^n(E_{n+1})$ is an isomorphism for all $n \geq 0$. Then, $E \cong E_i^{\geq 0}$ for large i (First show that there exists a map $E_j \to E_i^{\geq 0}$ for a fixed large i, and j large. This gives a map $E \to E_i^{\geq 0}$. Then, show that it is a quasi - isomorphism.).
Now, suppose we have two special Cauchy sequences $\{D_i\}$ and $\{E_i\}$, and maps between the two. We can complete each of these to a distinguished triangle to get a special Cauchy sequence $\{F_i\}$. By shifting if necessary, we can assume that the homotopy colimits, D and E lie in $\mathbf{D}^{\mathrm{b}}_{\mathbf{coh}}(X)^{\geq 1} \subset \mathbf{D}^{\mathrm{b}}_{\mathbf{coh}}(X)^{\geq 0}$. This tells us that the homotopy limit of the third sequence, F, also lies in $\mathbf{D}^{\mathrm{b}}_{\mathbf{coh}}(X)^{\geq 0}$. We have a commutative diagram for large n :

$$\begin{tikzcd} D_n \arrow[r] \arrow[d] & D_n^{\geq 0} \arrow[r, "\cong"] \arrow[d] & D \arrow[d] \\ E_n \arrow[r] & E_n^{\geq 0} \arrow[r, "\cong"] & E \end{tikzcd}$$

We can complete these to triangles to get,

$$\begin{tikzcd}
D_n \arrow[r] \arrow[d] & D_n^{\geq 0} \arrow[r, "\cong"] \arrow[d] & D \arrow[d] \\
E_n \arrow[r] \arrow[d] & E_n^{\geq 0} \arrow[r, "\cong"] & E \arrow[d] \\
F_n \arrow[rr] \arrow[d] & & F' \arrow[d] \\
{} & & {}
\end{tikzcd}$$

But, from the long exact sequences of cohomology, we get that $F' \cong F_n^{\geq 0} \cong F$, and hence, we get a triangle in $\mathbf{D}^{\mathrm{b}}_{\mathbf{coh}}(X)$

3

(a)

We first state a small lemma.

Lemma 3.1. *Let $P \in \mathbf{K}^-(R - proj)$. For any $n \in \mathbb{Z}$ such that $H^i(P) = 0$, for all $i > n$, there exists a complex P_n quasi - isomorphic (and in fact homotopy isomorphic) to P, such that $P_n^i = 0$ for all $i > n$.*

Let $\{E_i, f_i : E_i \to E_{i+1}\}$ be a Cauchy sequence in $\mathbf{K}^b(R - \text{proj})$. The homotopy colimit clearly lies in $\mathbf{D}^-_{\mathbf{qc}}(X)$. By shifting if necessary, we can assume that it lies in $\mathbf{D}_{\mathbf{qc}}(X)^{\leq 0}$.
Upto passing to a subsequence, we can assume that $H^n(E_i) \to H^n(E_{i+1})$ is an isomorphism for all $n \geq -i$. By the Lemma, we can assume that each of the complexes have non - zero terms only in non - positive degrees.
Then, we proceed in the following manner :
We will choose representatives for E_i for $i > 1$, such that $E_i^n = E_{i-1}^n$ for all $n > -i$. We do this by induction. So assume that $E_i^n = E_{i-1}^n$ for all $n > -i$ and $i \leq l$ for some $l \in \mathbb{Z}$.
Complete the map $E_l \to E_{l+1}$ to a triangle $E_l \to E_{l+1} \to F \to$. Note that $F \in \mathbf{K}^b(R - \text{proj})$, as it is the mapping cone on a morphism in $\mathbf{K}^b(R - \text{proj})$. Then, as $H^n(E_l) \to H^n(E_{l+1})$ is an isomorphism for all $n \geq -l$, we get that $H^n(F) = 0$, for all $n > -l$. So, by the lemma, we get that upto isomorphism, we can assume $F^n = 0$ for all $n > -l$. But, now from the triangle $F[-1] \to E_l \to E_{l+1} \to$, we get that E_{l+1} is quasi - isomorphic to the cone on $F[-1] \to E_l$. This gives us the required representative of E_{l+1}
Further, note that with this choice, the map f_i^n is the identity map for all $i \geq -n$.

Now, it is easy to see that the complex E with $E^i = E_i^i$, and the differential given similarly, is quasi - isomorphic to the homotopy colimit of E_i. As $E \in$

$\mathbf{K}^-(R-\text{proj})$, we are done.
Conversely, given any complex $P \in \mathbf{K}^b(R-\text{proj})$, it is the homotopy colimit of its brutal truncations $\tau^{\geq -n}P$.

(b)

This follows the same way as Question 2.

4

(a)

Suppose we are given a Cauchy sequence $\{E_i, f_i : E_{i+1} \to E_i\}$ in $\mathbf{D}^b(R-\text{mod})$. Let E be the homotopy limit of this sequence. We note that as the sequence $\{H^n(E_i)\}_{i=1}^{\infty}$ eventually stabilises, $\Pi_i H^n(E_i) \xrightarrow{1-\text{shift}} \Pi_i H^n(E_i)$ is a split epimorhism. Then, from the triangle, $E \to \Pi E_i \to \Pi E_i \to$, we get the short exact sequence, $0 \to H^n(E) \to \Pi H^n(E_i) \to \Pi H^n(E_i) \to 0$. This tells us that $\varprojlim H^n(E_i) = H^n(E)$, which immediately gives us that $E \in \mathbf{D}^-(R-\text{mod}) = \mathbf{K}^-(R-\text{proj})$

Conversely, any complex $P \in \mathbf{K}^b(R-\text{proj})$ is the homotopy limit of its canonical truncations $P^{\geq -n}$

(b)

Let E be the homotopy colimit of a special Cauchy sequence $\{E_i, f_i : E_{i+1} \to E_i\}$ in $\mathbf{D}^b(R-\text{mod})$. Consider the triangle $\tau^{>-m}E \to E \to \tau^{\leq -m}E \to$ given by the brutal truncations of E. Note that the homotopy colimit of $\{(\tau^{\leq -m}E)^{\geq -n}\}_{n \geq m}$ is $\tau^{\leq -m}E$. This gives us a triangle

$$\bigoplus_{n \geq m} (\tau^{\leq -m}E)^{\geq -n} \to \tau^{\leq -m}E \to \bigoplus_{n \geq m} (\tau^{\leq -m}E)^{\geq -n}[1] \to$$

. Note that, $Hom(E, \mathbf{D}^b(R-\text{mod})^{\leq -m} = 0)$, and hence $(E \to \tau^{\leq -m}E \to \bigoplus_{n \geq m} (\tau^{\leq -m}E)^{\geq -n}[1]) = 0$. So, $E \to \tau^{\leq -m}E$ factors through a map $E \to \bigoplus_{n \geq m} (\tau^{\leq -m}E)^{\geq -n}$, which is zero. So, we get that E is a direct summand of $\tau^{>-m}E \in \mathbf{K}^b(R-\text{proj})$, and hence must itself belong to $\mathbf{K}^b(R-\text{proj})$.
The converse is easy.

5

This question follows from unpacking the definition of $\mathfrak{S}(\mathcal{T})$ for the sepcific cases, and from Questions 3 and 4 above.

6

(a)

We already know that $\text{Hom}(G[i], G) = 0$ for all $i < 0$. Consider $\mathcal{S} = \{t \in \mathcal{T}_G^{\leq 0} : \text{Hom}(G[i], t) = 0,\ \forall\ i < 0\}$. Then $\{G[i] : i \geq 0\} \subseteq \mathcal{S}$, $\text{Add}(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathcal{S} * \mathcal{S} \subset \mathcal{S}$, which implies that $\mathcal{T}_G^{\leq 0} \subseteq \mathcal{S}$.
Conversely, let $t \in \mathcal{T}$ be such that $\text{Hom}(G[i], t) = 0,\ \forall\ i < 0$. Consider the t - structure triangle, $t^{\leq 0} \to t \to t^{>0} \to t^{\leq 0}[1]$. Now, $\text{Hom}(G[i], t^{\leq 0}[1]) = 0$ for all $i < 0$, by above. Hence, we get $\text{Hom}(G[i], t^{>0}) = 0$ for all $i < 0$. We already know $\text{Hom}(G[i], t^{>0}) = 0$ for all $i \geq 0$, and so, as G is a generator, we get that $t^{>0} = 0$, and hence, $t \cong t^{\leq 0} \in \mathcal{T}_G^{\leq 0}$.

(b)

Let $t \in \mathcal{T}_G^{\leq 0}$. Consider the natural map $\bigoplus_{\text{Hom}(G,t)} G \to t$, and complete it to a triangle, $\bigoplus_{\text{Hom}(G,t)} G \to t \to s \to \bigoplus_{\text{Hom}(G,t)} G[1]$. Then, for any map $G[i] \to s$ with $i \leq 0$, we get that the composite $(G[i] \to s \to \bigoplus_{\text{Hom}(G,t)} G[1]) = 0$. Hence, it must factor through a map $G[i] \to t$. As $i \geq 0$, and $t \in \mathcal{T}_G^{\leq 0}$, for the map to be non - zero, i itself must be zero. Hence, this map must in turn must factor through $\bigoplus_{\text{Hom}(G,t)} G \to t$. So, we get that the original map $(G[i] \to s) = (G[i] \to \bigoplus_{\text{Hom}(G,t)} G \to t \to s) = 0$, as the last two maps are consecutive maps of a triangle. And so, by part (a), $s \in \mathcal{T}_G^{\leq -1}$, and we are done.