EXERCISES FOR
'TRIANGULATED CATEGORIES AND COMPLETIONS'

Exercise 1. Let A be a ring and M an A-module.

(a) Write M as a filtered colimit of finitely presented A-modules.
(b) Write M as a directed colimit of finitely presented A-modules. Hint: Use a presentation of M.

Exercise 2. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory. Show that the inclusion induces a fully faithful functor $\operatorname{Ind}(\mathcal{D}) \to \operatorname{Ind}(\mathcal{C})$.

Exercise 3 (Example 2.9)**.** Let $\mathcal{C}$ be an essentially small additive category with cokernels.

(a) Show that a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is left exact if and only if it is a filtered colimit of representable functors. Then $\operatorname{Ind}(\mathcal{C})$ identifies with the category of left exact functors $\operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.
(b) Assume $\mathcal{C}$ is abelian. Then the category of left exact functors is abelian. Show that the Yoneda functor $\mathcal{C} \to \operatorname{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is exact.

Exercise 4. View the rational numbers $(\mathbb{Q}, \leq)$ with the usual ordering as a category.

(a) Show that a Cauchy sequence $x \in \operatorname{Cau}(\mathbb{N}, \mathbb{Q})$ is an increasing sequence of rational numbers.
(b) Let $x \to y$ be an eventually invertible morphism of Cauchy sequences. Show that x and y are either both eventually constant or neither is eventually constant.
(c) Describe $\operatorname{Ind}_{\mathrm{Cau}}(\mathbb{Q})$.

Exercise 5. We consider the category $\mathcal{C} = \mathrm{fl}(\mathbb{Z})$.

(a) Let $X \in \operatorname{Fun}(\mathbb{N}, \mathcal{C})$ be given by $X_n = \bigoplus_{p \leqslant n \text{ prime}} \mathbb{Z}/p$ with the natural inclusion maps. Show that X is a Cauchy sequence in $\mathcal{C}$ and that X is not artinian.
(b) Find $\operatorname{Ind}_{\mathrm{Cau}}(\mathrm{fl}(\mathbb{Z}))$.

Exercise 6. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be triangulated categories.

(a) Assume that $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{E}$ are completions. Show that $g \circ f\colon \mathcal{C} \to \mathcal{E}$ is a completion. Under which conditions is $g \circ f$ sequential or Cauchy sequential?
(b) Show that the identity functor $\mathrm{id}\colon \mathcal{C} \to \mathcal{C}$ is a sequential Cauchy completion.

Exercise 7. Let $A = k[t]/(t^n)$ (or any self-injective k-algebra of finite representation type) and $\mathcal{C} = \operatorname{stmod}(A)$. Show that the restricted Yoneda functor $\operatorname{StMod}(A) \to \operatorname{Ind}(\operatorname{stmod}(A))$ is an equivalence. Hence for any essentially small triangulated subcategory $\mathcal{C}$ of $\operatorname{StMod}(A)$ containing the compacts $\operatorname{stmod}(A)$ the functor $\operatorname{stmod}(A) \to \mathcal{C}$ is a completion.