Exercises: *Triangulated categories via metric techniques.* Århus. March 2023

Question 1. In Talk 1 we proved that: if $X = \mathbb{P}^n_R$, and $(\mathcal{A}, \mathcal{B})$ is a t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$ such that $\mathcal{A}$ contains all of

$$\mathcal{O}, \quad \mathcal{O}(1)[-1], \quad \cdots, \quad \mathcal{O}(n-1)[-n+1], \quad \mathcal{O}(n)[-n],$$

then $\mathcal{A}$ must also contain $\mathcal{O}(-\ell)$ for all $\ell \geq 0$.

Prove that the same is true for all quasiprojective schemes over R, meaning schemes that admit a locally closed immersion $X \subset \mathbb{P}^n_R$.

Question 2. With (special) Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$ as in the "provisional" definition of Talk 2, prove that the homotopy colimit of a special Cauchy sequence of exact triangles in $\mathbf{D}^{\mathrm{perf}}(X)$ is an exact triangle in $\mathbf{D}^b_{\mathbf{coh}}(X)$.

Question 3. Let R be an associative ring. In the category $\mathbf{K}^b(R\text{–proj})$, a "provisional" Cauchy sequence is just as in $\mathbf{D}^{\mathrm{perf}}(X)$: it is a sequence of composable morphisms

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$$

such that, for any integer $m > 0$, there exists an integer $N > 0$ satisfying

- **•:** The morphisms $H^i(E_n) \longrightarrow H^i(E_{n+1})$ are isomorphisms for all $i \geq -m$ and all $n > N$.

The special Cauchy sequences are those Cauchy sequences where the limit has only finitely many nonzero cohomology groups.

(a) Prove that the objects in $\mathbf{K}^-(R\text{–proj})$ are precisely the homotopy colimits of Cauchy sequences.

(b) Deduce that an object in $\mathbf{K}^-(R\text{–proj})$ is the homotopy colimit of a special Cauchy sequence if and only if it has only finitely many nonzero cohomology groups.

Question 4. Let the notation be as in Question 3, but now assume further that the ring R is coherent. Let $\big(\mathbf{K}^-(R\text{–proj})^{\leq 0}, \mathbf{K}^-(R\text{–proj})^{\geq 0}\big)$ be the standard t-structure on $\mathbf{K}^-(R\text{–proj})$. Note that, for any object $A \in \mathbf{K}^-(R\text{–proj})$ and any integer $m < \infty$, the object $A^{\geq -m}$ belongs to $\mathbf{D}^b(R\text{–mod}) \subset \mathbf{K}^-(R\text{–proj})$, where the inclusion is by taking projective resolutions.

The (provisional) definition of Cauchy sequences in $\mathbf{D}^b(R\text{–mod})$ is that an inverse sequence

$$E_1 \longleftarrow E_2 \longleftarrow E_3 \longleftarrow \cdots$$

is Cauchy if, for any integer $m > 0$, there exists an integer $N > 0$ satisfying

- **•:** The morphisms $H^i(E_n) \longleftarrow H^i(E_{n+1})$ are isomorphisms for all $i \geq -m$ and all $n > N$.

The special Cauchy sequences are those Cauchy sequences where the homotopy inverse limit X satisfies $\mathrm{Hom}\big(X, \mathbf{D}^b(R\text{–mod})^{\leq -m}\big) = 0$ for some $m \gg 0$.

(a) Prove that the objects in $\mathbf{K}^-(R\text{–proj})$ are precisely the homotopy inverse colimits of Cauchy sequences.

(b) Deduce that an object in $\mathbf{K}^-(R\text{–proj})$ is the homotopy inverse limit of a special Cauchy sequence if and only if it belongs to $\mathbf{K}^b(R\text{–proj})$.

Question 5. Check that, in the special case where $X = \mathrm{Spec}(R)$, where $\mathcal{T} = \mathbf{D}_{\mathbf{qc}}(X)$, where $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is the standard t-structure, and where the metrics on $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^b_{\mathbf{coh}}(X)$ are given respectively by

$$B_n = \mathbf{D}^{\mathrm{perf}}(X) \cap \mathcal{T}^{\leq -n}, \qquad\qquad B_n = \mathbf{D}^b_{\mathbf{coh}}(X) \cap \mathcal{T}^{\leq -n}$$

then the equivalences

$$\mathfrak{S}\big[\mathbf{D}^b(R\text{–proj})\big] = \mathbf{D}^b(R\text{–mod})$$

and (assuming R is a coherent ring)

$$\mathfrak{S}\Big(\big[\mathbf{D}^b(R\text{–mod})\big]^{\mathrm{op}}\Big) = \big[\mathbf{D}^b(R\text{–proj})\big]^{\mathrm{op}}$$

agree with those of Questions 3 and 4.

Question 6. Let $\mathcal{T}$ be a triangulated category. Assume the object $G \in \mathcal{T}$ is a compact generator, and that $\operatorname{Hom}\big(G, G[i]\big) = 0$ for all $i > 0$. Prove that

(a) For the t-structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$, we have that $X \in \mathcal{T}_G^{\leq 0}$ is and only if $\operatorname{Hom}\big(G[i], X\big) = 0$ for all $i < 0$.

(b) Deduce that the category $\mathcal{T}$ must be approximable.