Triangulated categories via metric techniques, 1

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22 March 2023

Overview

1. t-structures: examples and formal definition
2. Ancient history
3. First application: a conjecture of Antieau, Gepner and Heller
4. Something about the proof

Example (the standard t–structure on $\mathbf{D}(\mathcal{A})$)

Let $\mathcal{A}$ be an abelian category. We define two full subcategories of $\mathbf{D}(\mathcal{A})$:

- $$\mathbf{D}(\mathcal{A})^{\leq 0} \quad = \quad \{A^* \in \mathbf{D}(\mathcal{A}) \mid H^i(A^*) = 0 \text{ for all } i > 0\}$$
- $$\mathbf{D}(\mathcal{A})^{\geq 0} \quad = \quad \{A^* \in \mathbf{D}(\mathcal{A}) \mid H^i(A^*) = 0 \text{ for all } i < 0\}$$

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Put $I = \mathrm{Im}(Y^{-1} \to Y^0)$, and $Q = Y^0/I$.

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For every $Y \in \mathbf{D}(\mathcal{A})$ we have produced

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with $X \in \mathbf{D}(\mathcal{A})^{\leq 0}[1]$ and with $Z \in \mathbf{D}(\mathcal{A})^{\geq 0}$.

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- $\mathrm{Hom}\Big(\mathcal{T}^{\leq 0}[1]\ ,\ \mathcal{T}^{\geq 0}\Big) = 0$
- For every object $B \in \mathcal{T}$ there exists a triangle $A \longrightarrow B \longrightarrow C \longrightarrow$ with $A \in \mathcal{T}^{\leq 0}[1]$ and $C \in \mathcal{T}^{\geq 0}$.

Given an object $B \in \mathcal{T}$, the third property of a t-structure says that there exists an exact triangle

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This triangle is often written

$$B^{\leq -1} \longrightarrow B \longrightarrow B^{\geq 0} \longrightarrow B^{\leq -1}[1]$$

Notation

For $n \in \mathbb{Z}$ we adopt the shorthand

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n] \ , \qquad\qquad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n] \ .$$

Definition (Bounded t-Structures)

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Definition (Bounded t-Structures)

A t-structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ is called bounded if, for every object $X \in \mathcal{T}$, there exists an integer $n > 0$ with

$$X[n] \in \mathcal{T}^{\leq 0} \qquad \text{and} \qquad X[-n] \in \mathcal{T}^{\geq 0} \ .$$

Let X be a coherent scheme and $Z \subset X$ a closed subset with quasicompact complement.

We define $\mathbf{D}^{-}_{\mathbf{coh},Z}(X)$ to be the category whose objects are cochain complexes of $\mathcal{O}_X$-modules, such that

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We define $\mathbf{D}^-_{\mathbf{coh},Z}(X)$ to be the category whose objects are cochain complexes of $\mathcal{O}_X$-modules, such that

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Take any $F \in \mathbf{D}^-_{\mathbf{coh},Z}(X)$.

Resolving F by vector bundles, we may represent it as a complex

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This gives an exact triangle

$$E \longrightarrow F \longrightarrow D[1],$$

with $E \in \mathbf{D}^{\mathrm{perf}}(X)$ and $D \in \mathbf{D}^{-}_{\mathbf{coh}}(X)^{\leq m}$.

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For an unconditional proof, one needs to use ideas from

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Joseph Lipman and Amnon Neeman, *Quasi-perfect scheme maps and boundedness of the twisted inverse image functor*, Illinois J. Math. **51** (2007), 209–236.

For a proof that works in the relative context, that is given $F \in \mathbf{D}^-_{\mathbf{coh},Z}(X)$ it produces a triangle

$$D \longrightarrow E \longrightarrow F \longrightarrow D[1],$$

with $E \in \mathbf{D}^{\mathrm{perf}}_Z(X)$ and $D \in \mathbf{D}^-_{\mathbf{coh},Z}(X)^{\leq m}$, see

Tag 36.14 in the Stacks Project.

Let $\mathcal{M}$ be a model category with homotopy category $\mathcal{T}$, and assume $\mathcal{T}$ has a bounded t–structure. Antieau, Gepner and Heller proved:

1. If the abelian category $\mathcal{T}^{\heartsuit}$ is noetherian, then $K_n(\mathcal{M}) = 0$ for $n < 0$.
2. Unconditionally we have $K_{-1}(\mathcal{M}) = 0$.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

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If $\mathcal{A}$ is an abelian category, and $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ with the usual model structure, the vanishing in negative K-theory is due to Schlichting.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Corollary

Let X be a finite-dimensional, noetherian scheme. Assume $K_{-1}(X)$ is nonzero. Then the category $\mathbf{D}^{\mathrm{perf}}(X)$ has no bounded t-structure.

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If $K_n(X)$ is nonzero for $n \leq -2$, then any bounded t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$ cannot have a noetherian heart.

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This can be found as Corollary 1.4 in

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Conjecture

Let X be a finite-dimensional, noetherian scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ has a bounded t-structure if and only if X is regular, in which case $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^{b}_{\mathbf{coh}}(X)$.

This can be found as Conjecture 1.5 in

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Theorem

Let X be a a scheme, and let $Z \subset X$ be a closed subset. Let $\mathbf{D}^{\mathrm{perf}}_Z(X)$ be the derived category, with objects the perfect complexes on X whose restriction to $X - Z$ is acyclic.

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For the proof see

Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, https://arxiv.org/abs/2202.08861.

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R is a regular local ring if and only if R/m is of finite projective dimension, if and only if every module is of finite projective dimension.

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Definition

Let $\mathcal{T}$ be a triangulated category. Two t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ are declared equivalent if there exists an integer $n > 0$ with

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We are given a bounded t-structure on $\mathbf{D}_Z^{\mathrm{perf}}(X)$, and we would like to compare it to the standard t-structure on $\mathbf{D}^b_{\mathbf{coh},Z}(X)$. For technical reasons this is easiest to do in $\mathbf{D}_{\mathbf{qc},Z}(X)$.

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We appeal to Theorem A.1 in

Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio, *Construction of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2523–2543 (electronic).

Theorem

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{A} \subset \mathcal{T}$ be a set of compact objects satisfying $\mathcal{A}[1] \subset \mathcal{A}$.

Let $\mathrm{Coprod}(\mathcal{A})$ be the smallest full subcategory of $\mathcal{T}$, containing $\mathcal{A}$ and closed under coproducts and extensions.

Then $\Big(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})[1]^{\perp}\Big)$ is a t-structure on $\mathcal{T}$.

This is Theorem A.1 in

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Lemma

Let $\left(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})[1]^{\perp}\right)$ be the induced t-structure on $\mathcal{T}$. If $E \in \mathcal{T}^c$ is an object, then $A = E^{\leq -1}$ and $B = E^{\geq 0}$ are the same, whether computed in $\mathcal{T}$ or in $\mathcal{T}^c$.

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Now we are assuming that we are given a bounded t-structure $(\mathcal{A},\mathcal{B})$ on the category $\mathbf{D}_Z^{\mathrm{perf}}(X)$, which is the category of compact objects in $\mathbf{D}_{\mathbf{qc},Z}(X)$.

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Suppose we could prove the inclusions

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Take $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)$. Without loss of generality assume $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)^{\geq 0}$. We want to show that $F \in \mathbf{D}_Z^{\mathrm{perf}}(X)$.

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The literature we explained gave us exact triangles

$$\begin{array}{ccccc} D & \longrightarrow & E & \longrightarrow & F \\ \cap & & \cap & & \cap \\ \mathbf{D}_{\mathbf{qc},Z}(X)^{\leq -m} & & \mathbf{D}_Z^{\mathrm{perf}}(X) & & \mathbf{D}_{\mathbf{qc},Z}(X)^{\geq 0} \end{array}$$

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\mathrm{Coprod}(\mathcal{A})[m-n] & & & & \Big(\mathrm{Coprod}(\mathcal{A})^{\perp}\Big)[n+1]
\end{array}$$

It suffices to show that the standard t-structure on $\mathbf{D}_{\mathbf{qc},Z}(X)$ is equivalent to the t-structure generated by $\mathcal{A}$, where $(\mathcal{A},\mathcal{B})$ is our bounded t-structure on $\mathbf{D}_Z^{\mathrm{perf}}(X)$.

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For simplicity we assume that X is projective and that $Z = X$.

Pick any object $F \in \mathbf{D_{qc}}(X)^{\leq 0}$. Resolving it, we may produce an isomorph

$$\cdots \longrightarrow \mathcal{V}^{m-1} \longrightarrow \mathcal{V}^{m} \longrightarrow \cdots \longrightarrow \mathcal{V}^{-1} \longrightarrow \mathcal{V}^{0} \longrightarrow 0 \longrightarrow \cdots$$

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Now $(\mathcal{A}, \mathcal{B})$ is a bounded t-structure on the category $\mathbf{D}^{\mathrm{perf}}(X)$.

Hence, given any integer $N > 0$, we can find an integer $M > 0$ such that

$$\mathcal{O}(-\ell) \in \mathcal{A}[-M] \qquad \text{for all } 0 \leq \ell \leq N.$$

Alexander A. Beĭlinson, *The derived category of coherent sheaves on* $\mathbf{P}^n$, Selecta Mathematica Sovietica, vol. 3, 1983/84, Selected translations, pp. 233–237.

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Dmitri O. Orlov, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.

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Tensoring together $n+1$ of these we deduce a quasi-isomorphism of R with the Koszul complex

$$\bigotimes_{i=0}^{n} \left(R[x_i] \xrightarrow{x_i} R[x_i] \right)$$

Applying Proj to this, we obtain a quasi-isomorphism of $\mathcal{O}(1)$ with a complex

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Hence the brutal truncation must be quasi-isomorphic to $\mathcal{O}(\ell) \oplus \mathcal{V}[n]$ for some vector bundle $\mathcal{V}$.

Applying the functor $(-)^\vee = \mathcal{RH}om(-, \mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell) \oplus \mathcal{V}^\vee[-n]$ with

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Thus if $\mathcal{A}[-M]$ contains

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But then

$$\mathbf{D_{qc}}(X)^{\leq 0} \quad \subset \quad \mathrm{Coprod}(\mathcal{A}[-M])\ .$$

Thank you!