Triangulated categories via metric techniques, 2

Amnon Neeman

Australian National University

amnon.neeman@anu.edu.au

23 March 2023

Overview

Theorem

Let X and Y be coherent schemes. Assume both X and Y are affine. Then following are equivalent:

1. *There is a triangle equivalence* $\mathbf{D}^b_{\mathbf{coh}}(X) \cong \mathbf{D}^b_{\mathbf{coh}}(Y)$.
2. *There is a triangle equivalence* $\mathbf{D}^{\mathrm{perf}}(X) \cong \mathbf{D}^{\mathrm{perf}}(Y)$.

Theorem

Let X and Y be coherent schemes. Assume both X and Y are affine. Then following are equivalent:

1. *There is a triangle equivalence $\mathbf{D}^b_{\mathbf{coh}}(X) \cong \mathbf{D}^b_{\mathbf{coh}}(Y)$.*
2. *There is a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(X) \cong \mathbf{D}^{\mathrm{perf}}(Y)$.*

In other notation:

$$\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod}) \quad \Longleftrightarrow \quad \mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj}) \ .$$

Theorem

Let X and Y be coherent schemes. Assume both X and Y are affine. Then following are equivalent:

1. *There is a triangle equivalence* $\mathbf{D}^b_{\mathbf{coh}}(X) \cong \mathbf{D}^b_{\mathbf{coh}}(Y)$.
2. *There is a triangle equivalence* $\mathbf{D}^{\mathrm{perf}}(X) \cong \mathbf{D}^{\mathrm{perf}}(Y)$.

In other notation:

$$\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod}) \quad \Longleftrightarrow \quad \mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj}) \ .$$

This can be found in Theorem 1.1 of:

Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure and Appl. Algebra **61** (1989), 303–317.

Theorem

Let X and Y be coherent schemes. Assume both X and Y are affine. Then following are equivalent:

1. *There is a triangle equivalence $\mathbf{D}^b_{\mathbf{coh}}(X) \cong \mathbf{D}^b_{\mathbf{coh}}(Y)$.*
2. *There is a triangle equivalence $\mathbf{D}^{\mathrm{perf}}(X) \cong \mathbf{D}^{\mathrm{perf}}(Y)$.*

In other notation:

$$\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod}) \quad \Longleftrightarrow \quad \mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj}) \ .$$

This can be found in Theorem 1.1 of:

Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure and Appl. Algebra **61** (1989), 303–317.

Question (Krause 2018): Is there an algorithm to produce $\mathbf{D}^b_{\mathbf{coh}}(X)$ out of $\mathbf{D}^{\mathrm{perf}}(X)$?

Take any $F \in \mathbf{D}^b_{\mathbf{coh}}(X)$.

It has a resolution by vector bundles, which we write

$$\cdots \longrightarrow \mathcal{V}^{m-1} \longrightarrow \mathcal{V}^{m} \longrightarrow \cdots \longrightarrow \mathcal{V}^{n-1} \longrightarrow \mathcal{V}^{n} \longrightarrow 0 \longrightarrow \cdots$$

Take any $F \in \mathbf{D}^b_{\mathbf{coh}}(X)$.

It has a resolution by vector bundles, which we write

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{V}^m & \longrightarrow & \cdots & \longrightarrow & \mathcal{V}^{n-1} & \longrightarrow & \mathcal{V}^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{V}^{m-1} & \longrightarrow & \mathcal{V}^m & \longrightarrow & \cdots & \longrightarrow & \mathcal{V}^{n-1} & \longrightarrow & \mathcal{V}^n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathcal{V}^{m-1} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

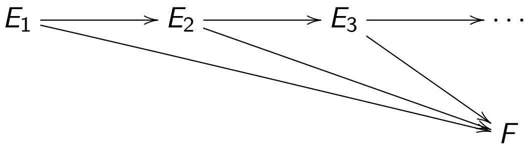
This gives an exact triangle, which we have met before

$$
\begin{array}{ccccc}
D_m & \longrightarrow & E_m & \longrightarrow & F \\
\pitchfork & & \pitchfork & & \pitchfork \\
\mathbf{D}^b_{\mathbf{coh}}(X)^{\leq m} & & \mathbf{D}^{\mathrm{perf}}(X) & & \mathbf{D}^b_{\mathbf{coh}}(X)
\end{array}
$$

This gives an exact triangle, which we have met before

$$\begin{array}{ccccc} D_m & \longrightarrow & E_m & \longrightarrow & F \\ \pitchfork & & \pitchfork & & \pitchfork \\ \mathbf{D}^b_{\mathbf{coh}}(X)^{\leq m} & & \mathbf{D}^{\mathrm{perf}}(X) & & \mathbf{D}^b_{\mathbf{coh}}(X) \end{array}$$

Next we want to observe that these brutal truncations fit into a sequence



This gives an exact triangle, which we have met before

$$\begin{array}{ccccc} D_m & \longrightarrow & E_m & \longrightarrow & F \\ \cap & & \cap & & \cap \\ \mathbf{D}^b_{\mathbf{coh}}(X)^{\leq m} & & \mathbf{D}^{\mathrm{perf}}(X) & & \mathbf{D}^b_{\mathbf{coh}}(X) \end{array}$$

Next we want to observe that these brutal truncations fit into a sequence

$$\begin{array}{ccccccc} E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & & & & & F \end{array}$$

where $\qquad F \cong \operatorname{Hocolim}_{\longrightarrow} E_m$

(Provisional) definition of special Cauchy sequences

Let

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow \cdots$$

be a sequence of composable morphisms in $\mathbf{D}^{\mathrm{perf}}(X)$.

(Provisional) definition of special Cauchy sequences

Let

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow \cdots$$

be a sequence of composable morphisms in $\mathbf{D}^{\mathrm{perf}}(X)$.

The sequence is Cauchy if

(Provisional) definition of special Cauchy sequences

Let

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow \cdots$$

be a sequence of composable morphisms in $\mathbf{D}^{\mathrm{perf}}(X)$.

The sequence is Cauchy if

1. For every integer $m > 0$, there exists an integer $N > 0$ such that the maps

$$\mathcal{H}^i(E_n) \longrightarrow \mathcal{H}^i(E_{n+1})$$

are isomorphisms for all $n > N$ and all $i > -m$.

(Provisional) definition of special Cauchy sequences

Let

$$E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow E_4 \longrightarrow \cdots$$

be a sequence of composable morphisms in $\mathbf{D}^{\mathrm{perf}}(X)$.

The sequence is Cauchy if

1. For every integer $m > 0$, there exists an integer $N > 0$ such that the maps

$$\mathcal{H}^i(E_n) \longrightarrow \mathcal{H}^i(E_{n+1})$$

 are isomorphisms for all $n > N$ and all $i > -m$.

2. A Cauchy sequence is special if, for all but finitely many $i \in \mathbb{Z}$, the limit of $\mathcal{H}^i(E_n)$ vanishes as $n \longrightarrow \infty$.

Hence it's natural to define $\mathbf{D}^b_{\mathbf{coh}}(X)$ as the completion of the special Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$. The objects are the special Cauchy sequences above, and the morphisms are the maps of Cauchy sequences with Ind-isomorphisms inverted.

Hence it's natural to define $\mathbf{D}^b_{\mathbf{coh}}(X)$ as the completion of the special Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$. The objects are the special Cauchy sequences above, and the morphisms are the maps of Cauchy sequences with Ind-isomorphisms inverted.

But what about the distinguished triangles?

Hence it's natural to define $\mathbf{D}^b_{\mathbf{coh}}(X)$ as the completion of the special Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$. The objects are the special Cauchy sequences above, and the morphisms are the maps of Cauchy sequences with Ind-isomorphisms inverted.

But what about the distinguished triangles?

A naive guess would be to define them to be special Cauchy sequences of distinguished triangles in $\mathbf{D}^{\mathrm{perf}}(X)$.

Hence it's natural to define $\mathbf{D}^b_{\mathbf{coh}}(X)$ as the completion of the special Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$. The objects are the special Cauchy sequences above, and the morphisms are the maps of Cauchy sequences with Ind-isomorphisms inverted.

But what about the distinguished triangles?

A naive guess would be to define them to be special Cauchy sequences of distinguished triangles in $\mathbf{D}^{\mathrm{perf}}(X)$.

Problem. We must be able to continue every morphism $A \longrightarrow B$ to a triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$, and

Hence it's natural to define $\mathbf{D}^b_{\mathbf{coh}}(X)$ as the completion of the special Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$. The objects are the special Cauchy sequences above, and the morphisms are the maps of Cauchy sequences with Ind-isomorphisms inverted.

But what about the distinguished triangles?

A naive guess would be to define them to be special Cauchy sequences of distinguished triangles in $\mathbf{D}^{\mathrm{perf}}(X)$.

Problem. We must be able to continue every morphism $A \longrightarrow B$ to a triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$, and

this triangle should be unique up to isomorphism!

To spell it out: let $f : A \longrightarrow B$ be a map of special Cauchy sequences, that is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(X)$

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots \end{array}$$

To spell it out: let $f : A \longrightarrow B$ be a map of special Cauchy sequences, that is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(X)$

Assume without loss that the colimits of A and B lie in $\mathbf{D}^{b}_{\mathbf{coh}}(X)^{\geq 0}$.

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots
\end{array}$$

To spell it out: let $f : A \longrightarrow B$ be a map of special Cauchy sequences, that is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(X)$
Assume without loss that the colimits of A and B lie in $\mathbf{D}^{b}_{\mathbf{coh}}(X)^{\geq 0}$.

$$
\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_1 & & C_2 & & C_3 & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_1[1] & \longrightarrow & A_2[1] & \longrightarrow & A_3[1] & \longrightarrow & \cdots
\end{array}
$$

To spell it out: let $f : A \longrightarrow B$ be a map of special Cauchy sequences, that is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(X)$

Assume without loss that the colimits of A and B lie in $\mathbf{D}^b_{\mathbf{coh}}(X)^{\geq 0}$.

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_1[1] & \longrightarrow & A_2[1] & \longrightarrow & A_3[1] & \longrightarrow & \cdots
\end{array}$$

To spell it out: let $f : A \longrightarrow B$ be a map of special Cauchy sequences, that is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(X)$

Assume without loss that the colimits of A and B lie in $\mathbf{D}^b_{\mathbf{coh}}(X)^{\geq 0}$.

$$\begin{array}{ccccccc}
A_{n+1}^{\geq -m} & \xrightarrow{\sim} & A_{n+2}^{\geq -m} & \xrightarrow{\sim} & A_{n+3}^{\geq -m} & \xrightarrow{\sim} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_{n+1}^{\geq -m} & \xrightarrow{\sim} & B_{n+2}^{\geq -m} & \xrightarrow{\sim} & B_{n+3}^{\geq -m} & \xrightarrow{\sim} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_{n+1}^{\geq -m} & \longrightarrow & C_{n+2}^{\geq -m} & \longrightarrow & C_{n+3}^{\geq -m} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_{n+1}[1]^{\geq -m} & \xrightarrow{\sim} & A_{n+2}[1]^{\geq -m} & \xrightarrow{\sim} & A_{n+3}[1]^{\geq -m} & \xrightarrow{\sim} & \cdots
\end{array}$$

To spell it out: let $f : A \longrightarrow B$ be a map of special Cauchy sequences, that is a commutative diagram in $\mathbf{D}^{\mathrm{perf}}(X)$

Assume without loss that the colimits of A and B lie in $\mathbf{D}^b_{\mathbf{coh}}(X)^{\geq 0}$.

$$\begin{array}{ccccccc}
A_{n+1}^{\geq -m} & \xrightarrow{\sim} & A_{n+2}^{\geq -m} & \xrightarrow{\sim} & A_{n+3}^{\geq -m} & \xrightarrow{\sim} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B_{n+1}^{\geq -m} & \xrightarrow{\sim} & B_{n+2}^{\geq -m} & \xrightarrow{\sim} & B_{n+3}^{\geq -m} & \xrightarrow{\sim} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_{n+1}^{\geq -m} & \xrightarrow{\sim} & C_{n+2}^{\geq -m} & \xrightarrow{\sim} & C_{n+3}^{\geq -m} & \xrightarrow{\sim} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_{n+1}[1]^{\geq -m} & \xrightarrow{\sim} & A_{n+2}[1]^{\geq -m} & \xrightarrow{\sim} & A_{n+3}[1]^{\geq -m} & \xrightarrow{\sim} & \cdots
\end{array}$$

Suppose we want to prove the octahedral axiom for such sequences. Let $X \longrightarrow Y \longrightarrow Z$ be two composable morphisms in $\mathbf{D}^b_{\mathbf{coh}}(X)$, which we want to complete to an octahedron using the Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$ converging to them.

Suppose we want to prove the octahedral axiom for such sequences. Let $X \longrightarrow Y \longrightarrow Z$ be two composable morphisms in $\mathbf{D}^b_{\mathbf{coh}}(X)$, which we want to complete to an octahedron using the Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$ converging to them. Assume without loss that X, Y and Z all belong to $\mathbf{D}^b_{\mathbf{coh}}(X)^{\geq 0}$.

Suppose we want to prove the octahedral axiom for such sequences. Let $X \longrightarrow Y \longrightarrow Z$ be two composable morphisms in $\mathbf{D}^b_{\mathbf{coh}}(X)$, which we want to complete to an octahedron using the Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$ converging to them. Assume without loss that X, Y and Z all belong to $\mathbf{D}^b_{\mathbf{coh}}(X)^{\geq 0}$.

Now choose excellent approximations of them, meaning a commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with X', Y' and Z' all in $\mathbf{D}^{\mathrm{perf}}(X)$ and with the vertical maps inducing isomorphisms in cohomology in degrees $> -1,000,000$.

Suppose we want to prove the octahedral axiom for such sequences. Let $X \longrightarrow Y \longrightarrow Z$ be two composable morphisms in $\mathbf{D}^b_{\mathbf{coh}}(X)$, which we want to complete to an octahedron using the Cauchy sequences in $\mathbf{D}^{\mathrm{perf}}(X)$ converging to them. Assume without loss that X, Y and Z all belong to $\mathbf{D}^b_{\mathbf{coh}}(X)^{\geq 0}$.

Now choose excellent approximations of them, meaning a commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with X', Y' and Z' all in $\mathbf{D}^{\mathrm{perf}}(X)$ and with the vertical maps inducing isomorphisms in cohomology in degrees $> -1,000,000$.

Now in the category $\mathbf{D}^{\mathrm{perf}}(X)$ we can complete $X' \longrightarrow Y' \longrightarrow Z'$ to an octahedron.

We obtain

$$
\begin{array}{ccccccc}
X' & \longrightarrow & Y' & \longrightarrow & U' & \longrightarrow & X'[1] \\
\| & & \downarrow & & \downarrow & & \| \\
X' & \longrightarrow & Z' & \longrightarrow & V' & \longrightarrow & X'[1] \\
 & & \downarrow & & \downarrow & & \\
 & & W' & = & W' & & \\
 & & \downarrow & & \downarrow & & \\
 & & Y'[1] & \longrightarrow & U'[1] & &
\end{array}
$$

A bunch of definitions

Reminder

Following a 1974 article of Lawvere, a metric on a category

1.

2.

A bunch of definitions

Reminder

Following a 1974 article of Lawvere, a metric on a category is a function that assigns a positive real number (length) to every morphism, satisfying:

1.
2.

A bunch of definitions

Reminder

Following a 1974 article of Lawvere, a metric on a category is a function that assigns a positive real number (length) to every morphism, satisfying:

1. For any identity map $\mathrm{id} : X \longrightarrow X$ we have

$$\text{Length}(\mathrm{id}) \quad = \quad 0\ ,$$

2.

A bunch of definitions

Reminder

Following a 1974 article of Lawvere, a metric on a category is a function that assigns a positive real number (length) to every morphism, satisfying:

1. For any identity map $\mathrm{id} : X \longrightarrow X$ we have
$$\text{Length}(\mathrm{id}) \quad = \quad 0 \ ,$$
2. and if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then
$$\text{Length}(gf) \quad \leq \quad \text{Length}(f) + \text{Length}(g) \ .$$

Definition (Equivalence of metrics)

We'd like to view two metrics on a category $\mathcal{C}$ as equivalent if the identity functor $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{C}$ is uniformly continuous in both directions.

equivalent

Definition (Equivalence of metrics)

We'd like to view two metrics on a category $\mathcal{C}$ as equivalent if the identity functor $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{C}$ is uniformly continuous in both directions.

More formally:

Let $\mathcal{C}$ be a category. Two metrics

$$\text{Length}_1 \qquad \text{and} \qquad \text{Length}_2$$

are declared equivalent if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\{\text{Length}_1(f) \ < \ \delta\} \qquad \Longrightarrow \qquad \{\text{Length}_2(f) \ < \ \varepsilon\}$$

and

$$\{\text{Length}_2(f) \ < \ \delta\} \qquad \Longrightarrow \qquad \{\text{Length}_1(f) \ < \ \varepsilon\}$$

Definition (Cauchy sequences)

Let $\mathcal{C}$ be a category with a metric. A Cauchy sequence in $\mathcal{C}$ is a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of composable morphisms such that, for any $\varepsilon > 0$, there exists an $M > 0$ such that the morphisms $E_i \longrightarrow E_j$ satisfy

$$\text{Length}(E_i \longrightarrow E_j) \quad < \quad \varepsilon$$

whenever $i, j > M$.

Definition (Cauchy sequences)

Let $\mathcal{C}$ be a category with a metric. A Cauchy sequence in $\mathcal{C}$ is a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of composable morphisms such that, for any $\varepsilon > 0$, there exists an $M > 0$ such that the morphisms $E_i \longrightarrow E_j$ satisfy

$$\text{Length}(E_i \longrightarrow E_j) \quad < \quad \varepsilon$$

whenever $i, j > M$.

We will assume the category $\mathcal{C}$ is $\mathbb{Z}$–linear. This means that $\mathrm{Hom}(a, b)$ is an abelian group for every pair of objects $a, b \in \mathcal{C}$, and that composition is bilinear.

Definition (The categories $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

1.

2.

Definition (The categories $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

1. Let $\mathfrak{L}(\mathcal{C})$ be the completion of $\mathcal{C}$, meaning the full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are the colimits in $\mathrm{Mod}\text{–}\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.

2.

Definition (The categories $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

1. Let $\mathfrak{L}(\mathcal{C})$ be the completion of $\mathcal{C}$, meaning the full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are the colimits in $\mathrm{Mod}\text{–}\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.
2. Define the full subcategory of $\mathfrak{S}(\mathcal{C}) \subset \mathfrak{L}(\mathcal{C})$ by the rule:

Definition (The categories $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

1. Let $\mathfrak{L}(\mathcal{C})$ be the completion of $\mathcal{C}$, meaning the full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are the colimits in $\mathrm{Mod}\text{–}\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.
2. Define the full subcategory of $\mathfrak{S}(\mathcal{C}) \subset \mathfrak{L}(\mathcal{C})$ by the rule:

   $F : \mathcal{C}^{\mathrm{op}} \longrightarrow Ab$ belongs to $\mathfrak{S}(\mathcal{C})$ if there exists an $\varepsilon > 0$ such that

$$\{\text{Length}(a \to b) < \varepsilon\} \quad \Longrightarrow$$

$$\{F(b) \longrightarrow F(a) \text{ is an isomorphism}\}.$$

Definition (The categories $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

1. Let $\mathfrak{L}(\mathcal{C})$ be the completion of $\mathcal{C}$, meaning the full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are the colimits in $\mathrm{Mod}\text{–}\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.
2. Define the full subcategory of $\mathfrak{S}(\mathcal{C}) \subset \mathfrak{L}(\mathcal{C})$ by the rule:

   $F : \mathcal{C}^{\mathrm{op}} \longrightarrow Ab$ belongs to $\mathfrak{S}(\mathcal{C})$ if there exists an $\varepsilon > 0$ such that

$$\{\text{Length}(a \to b) < \varepsilon\} \quad \Longrightarrow$$

$$\{F(b) \longrightarrow F(a) \text{ is an isomorphism}\}.$$

Equivalent metrics lead to identical $\mathfrak{L}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$.

Heuristic

We want to specialize the above to a situation in which we can actually prove something.

Let $\mathcal{S}$ be a triangulated category with a Lawvere metric.

Heuristic

We want to specialize the above to a situation in which we can actually prove something.

Let $\mathcal{S}$ be a triangulated category with a Lawvere metric.

We will only consider translation invariant metrics

Heuristic

We want to specialize the above to a situation in which we can actually prove something.

Let $\mathcal{S}$ be a triangulated category with a Lawvere metric.

We will only consider translation invariant metrics

which means that for any homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ c & \xrightarrow{g} & d \end{array}$$

we must have

$$\text{Length}(f) \quad = \quad \text{Length}(g)$$

Heuristic, continued

Given any $f : a \longrightarrow b$ we may form the homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{g} & x \end{array}$$

and our assumption tells us that

$$\mathrm{Length}(f) \quad = \quad \mathrm{Length}(g)$$

Hence it suffices to know the lengths of the morphisms

$$0 \longrightarrow x \ .$$

Heuristic, continued

We will soon be assuming that the metric is non-archimedean. Replacing the metric by an equivalent (if necessary), we may also assume our metric takes values in the set of rational numbers of the form

$$\{0, \infty\} \cup \{2^n \mid n \in \mathbb{Z}\} .$$

To know everything about the metric it therefore suffices to specify the balls

$$B_n \quad = \quad \left\{ x \in \mathcal{S} \;\middle|\; \text{the morphism } 0 \longrightarrow x \text{ has length } \leq \frac{1}{2^n} \right\}$$

Heuristic, continued

We will soon be assuming that the metric is non-archimedean. Replacing the metric by an equivalent (if necessary), we may also assume our metric takes values in the set of rational numbers of the form

$$\{0,\infty\}\cup\{2^n \mid n\in\mathbb{Z}\} .$$

To know everything about the metric it therefore suffices to specify the balls

$$B_n \quad = \quad \left\{ x\in\mathcal{S} \,\middle|\, \text{the morphism } 0\longrightarrow x \text{ has length } \leq \frac{1}{2^n} \right\}$$

If $f : x \longrightarrow y$ is any morphism, to compute its length you complete to a triangle $x \xrightarrow{f} y \longrightarrow z$, and then

$$\text{Length}(f) \quad = \quad \inf \left\{ \frac{1}{2^n} \,\middle|\, z\in B_n \right\}$$

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1.

2.

Example

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1. We want: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then $\mathrm{Length}(gf) \leq \max\big(\mathrm{Length}(f), \mathrm{Length}(g)\big)$.

2.

Example

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1. We want: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then $\text{Length}(gf) \leq \max\big(\text{Length}(f), \text{Length}(g)\big)$.

   This translates to $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

2.

Example

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1. We want: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then $\text{Length}(gf) \leq \max\big(\text{Length}(f), \text{Length}(g)\big)$.

   This translates to $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

2. $B_{n+1} \subset B_n$.

Example

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1. We want: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then $\text{Length}(gf) \leq \max\big(\text{Length}(f), \text{Length}(g)\big)$.

   This translates to $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

2. $B_{n+1}[-1] \cup B_{n+1} \cup B_{n+1}[1] \ \subset \ B_n$.

Example

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1. We want: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then $\text{Length}(gf) \leq \max\big(\text{Length}(f), \text{Length}(g)\big)$.

   This translates to $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

2. $B_{n+1}[-1] \cup B_{n+1} \cup B_{n+1}[1] \ \subset\ B_n$.

Example

Suppose $\mathcal{S}$ has a t-structure. The $B_n = \mathcal{S}^{\leq -n}$ works.

Theorem (1)

Let $\mathcal{S}$ be a category with a metric. Some slides ago we defined categories

$$\mathfrak{S}(\mathcal{S}) \quad \subset \quad \mathfrak{L}(\mathcal{S}) \ .$$

Theorem (1)

Let $\mathcal{S}$ be a triangulated category with a metric. Some slides ago we defined categories

$$\mathfrak{S}(\mathcal{S}) \quad \subset \quad \mathfrak{L}(\mathcal{S}) \ .$$

Now define the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

Theorem (1)

Let $\mathcal{S}$ be a triangulated category with a good metric. Some slides ago we defined categories

$$\mathfrak{S}(\mathcal{S}) \quad \subset \quad \mathfrak{L}(\mathcal{S}) \ .$$

Now define the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

With this definition of distinguished triangles, the category $\mathfrak{S}(\mathcal{S})$ is triangulated.

Example (the six triangulated categories to keep in mind)

Let R be an associative ring.

1. $\mathbf{D}(R)$ will be our shorthand for $\mathbf{D}(R\text{–}\mathrm{Mod})$; the objects are all cochain complexes of R-modules, no conditions.

2. $\mathbf{D}^b(R\text{–}\mathrm{proj})$ is the derived category of bounded complexes of finitely generated, projective R–modules.

3. Suppose the ring R is coherent. Then $\mathbf{D}^b(R\text{–}\mathrm{mod})$ is the bounded derived category of finitely presented R–modules.

Example (the six triangulated categories to keep in mind, continued)

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement.

4. $\mathbf{D_{qc}}_{,Z}(X)$ will be our shorthand for $\mathbf{D_{qc}}_{,Z}(\mathcal{O}_X\text{–Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the conditions are that (1) the cohomology must be quasicoherent, and (2) the restriction to $X - Z$ is acyclic.

5. The objects of $\mathbf{D}_Z^{\mathrm{perf}}(X) \subset \mathbf{D_{qc}}_{,Z}(X)$ are the perfect complexes. A complex $F \in \mathbf{D_{qc}}(X)$ is *perfect* if there exists an open cover $X = \cup_i U_i$ such that, for each U_i, the restriction map $u_i^* : \mathbf{D_{qc}}(X) \longrightarrow \mathbf{D_{qc}}(U_i)$ takes F to an object $u_i^*(F)$ isomorphic in $\mathbf{D_{qc}}(U_i)$ to a bounded complex of vector bundles.

6. Assume X is coherent. The objects of $\mathbf{D}^b_{\mathbf{coh},Z}(X) \subset \mathbf{D_{qc}}_{,Z}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Theorem (1, continued)

Now let R be an associative ring. Then the category $\mathbf{D}^b(R\text{–proj})$ admits an intrinsic metric [up to equivalence], so that

$$\mathfrak{S}\left[\mathbf{D}^b(R\text{–proj})\right] = \mathbf{D}^b(R\text{–mod}).$$

If we further assume that R is coherent then there is on $\left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}}$ an intrinsic metric [again up to equivalence], such that

$$\mathfrak{S}\left(\left[\mathbf{D}^b(R\text{–mod})\right]^{\mathrm{op}}\right) = \left[\mathbf{D}^b(R\text{–proj})\right]^{\mathrm{op}} .$$

Theorem (1, continued)

Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. There is an intrinsic equivalence class of metrics on $\mathbf{D}_Z^{\mathrm{perf}}(X)$ for which

$$\mathfrak{S}\left[\mathbf{D}_Z^{\mathrm{perf}}(X)\right] = \mathbf{D}^b_{\mathbf{coh},Z}(X) \ .$$

Now assume that X is a coherent scheme. Then the category $\left[\mathbf{D}^b_{\mathbf{coh},Z}(X)\right]^{\mathrm{op}}$ can be given intrinsic metrics [up to equivalence], so that

$$\mathfrak{S}\left(\left[\mathbf{D}^b_{\mathbf{coh},Z}(X)\right]^{\mathrm{op}}\right) = \left[\mathbf{D}_Z^{\mathrm{perf}}(X)\right]^{\mathrm{op}} \ .$$

Projective resolutions

Suppose we are given an object $F^* \in \mathbf{D}(R)$, meaning a cochain complex

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^{0} \longrightarrow F^{1} \longrightarrow F^{2} \longrightarrow \cdots$$

Assume $F^* \in \mathbf{D}(R)^{\leq 0}$, meaning

$$H^i(F^*) = 0 \qquad \text{for all } i > 0.$$

Then F^* has a projective resolution. We can produce a cochain map

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^{0} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^{0} & \longrightarrow & F^{1} & \longrightarrow & F^{2} & \longrightarrow \cdots
\end{array}$$

inducing an isomorphism in cohomology, and so that the P^i are projective.

Projective resolutions—a different perspective

We have found in $\mathbf{D}(R)$ an isomorphism $P^* \longrightarrow F^*$. Now consider

$$
\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

This gives in $\mathbf{D}(R)$ triangles

$$E_n^* \longrightarrow F^* \longrightarrow D_n^* \longrightarrow$$

with $D_n^* \in \mathbf{D}(R)^{\leq -n-1}$ and E_n^* not too complicated.

Reminder of standard notation

Let $\mathcal{T}$ be a triangulated category, possibly with coproducts, and let $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be full subcategories. We define the full subcategories

- $$\mathcal{A} * \mathcal{B} \quad = \quad \left\{ x \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{there exists a triangle } a \longrightarrow x \longrightarrow b \\ \text{with } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \right\}$$
- $\text{add}(\mathcal{A})$: all finite coproducts of objects of $\mathcal{A}$. [slightly nonstandard]
- Assume $\mathcal{T}$ has coproducts. Define $\text{Add}(\mathcal{A})$: all coproducts of objects of $\mathcal{A}$. [slightly nonstandard]
- $\text{smd}(\mathcal{A})$: all direct summands of objects of $\mathcal{A}$.

Measuring the effort

Let $\mathcal{T}$ be a triangulated category, possibly with coproducts, let $\mathcal{A} \subset \mathcal{T}$ be a full subcategory and let $m \leq n$ be integers. We define the full subcategories

- $\mathcal{A}[m,n] = \cup_{i=m}^{n} \mathcal{A}[-i]$
- $\langle \mathcal{A} \rangle_1^{[m,n]} = \text{smd}\Big[\text{add}\big(\mathcal{A}[m,n]\big)\Big]$
- $\overline{\langle \mathcal{A} \rangle}_1^{[m,n]} = \text{smd}\Big[\text{Add}\big(\mathcal{A}[m,n]\big)\Big]$ [assumes coproducts exist]
-
-

Measuring the effort

Let $\mathcal{T}$ be a triangulated category, possibly with coproducts, let $\mathcal{A} \subset \mathcal{T}$ be a full subcategory and let $m \leq n$ be integers. We define the full subcategories

- $\mathcal{A}[m,n] = \cup_{i=m}^{n} \mathcal{A}[-i]$
- $\langle \mathcal{A} \rangle_1^{[m,n]} = \text{smd}\Big[\text{add}\big(\mathcal{A}[m,n]\big)\Big]$
- $\overline{\langle \mathcal{A} \rangle}_1^{[m,n]} = \text{smd}\Big[\text{Add}\big(\mathcal{A}[m,n]\big)\Big]$ [assumes coproducts exist]

Now let $\ell > 0$ be an integer, and assume $\langle \mathcal{A} \rangle_k^{[m,n]}$ and $\overline{\langle \mathcal{A} \rangle}_k^{[m,n]}$ have been defined for all $1 \leq k \leq \ell$. We continue with

- $\langle \mathcal{A} \rangle_{\ell+1}^{[m,n]} = \text{smd}\Big[\langle \mathcal{A} \rangle_1^{[m,n]} * \langle \mathcal{A} \rangle_\ell^{[m,n]}\Big]$
- $\overline{\langle \mathcal{A} \rangle}_{\ell+1}^{[m,n]} = \text{smd}\Big[\overline{\langle \mathcal{A} \rangle}_1^{[m,n]} * \overline{\langle \mathcal{A} \rangle}_\ell^{[m,n]}\Big]$ [assumes coproducts exist]

Measuring the effort

Still with $\mathcal{T}$ be a triangulated category, possibly with coproducts, with $\mathcal{A} \subset \mathcal{T}$ a full subcategory and with $m \leq n$ integers, we set

- $\langle \mathcal{A} \rangle^{[m,n]}$ is the smallest full subcategory $\mathcal{B} \subset \mathcal{T}$ satisfying

$$\mathcal{A}[m,n] \subset \mathcal{B}, \qquad \mathcal{B} * \mathcal{B} \subset \mathcal{B}, \qquad \mathrm{smd}\big[\mathrm{add}(\mathcal{B})\big] = \mathcal{B}.$$

-

Measuring the effort

Still with $\mathcal{T}$ be a triangulated category, possibly with coproducts, with $\mathcal{A} \subset \mathcal{T}$ a full subcategory and with $m \leq n$ integers, we set

- $\langle \mathcal{A} \rangle^{[m,n]}$ is the smallest full subcategory $\mathcal{B} \subset \mathcal{T}$ satisfying

$$\mathcal{A}[m,n] \subset \mathcal{B}, \qquad \mathcal{B} * \mathcal{B} \subset \mathcal{B}, \qquad \text{smd}\big[\text{add}(\mathcal{B})\big] = \mathcal{B}.$$

- $\overline{\langle \mathcal{A} \rangle}^{[m,n]}$ is the smallest full subcategory $\mathcal{B} \subset \mathcal{T}$ satisfying

$$\mathcal{A}[m,n] \subset \mathcal{B}, \qquad \mathcal{B} * \mathcal{B} \subset \mathcal{B}, \qquad \text{smd}\big[\text{Add}(\mathcal{B})\big] = \mathcal{B}.$$

Example (back to $\mathbf{D}(R)$—the version with finite coproducts)

Let $\mathcal{A} = \{R\}$ be the full subcategory of $\mathbf{D}(R)$ with a single object. Then

- $\langle R \rangle_1^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \xrightarrow{0} \cdots \xrightarrow{0} P^{-1} \xrightarrow{0} P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i finitely generated and projective.

- $\langle R \rangle_{n+1}^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i finitely generated and projective.

Example (back to $\mathbf{D}(R)$—the version with infinite coproducts)

Let $\mathcal{A} = \{R\}$ be the full subcategory of $\mathbf{D}(R)$ with a single object. Then

- $\overline{\langle R\rangle}_1^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \xrightarrow{0} \cdots \xrightarrow{0} P^{-1} \xrightarrow{0} P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i projective.

- $\overline{\langle R\rangle}_{n+1}^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i projective.

Definition (formal definition of approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if there exist a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ so that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}^{[-A,A]}$.
- The category $\mathcal{T}$ is approximable if, in the triangle $E \longrightarrow F \longrightarrow D$ of (2), we may assume $E \in \overline{\langle G \rangle}_A^{[-A,A]}$.

Example (the category $\mathbf{D}(R)$)

Let R be a ring. The object $R \in \mathbf{D}(R)$ is a compact generator, the t–structure we take is the standard one, and we set $A = 1$.

Example (the category $\mathbf{D}(R)$)

Let R be a ring. The object $R \in \mathbf{D}(R)$ is a compact generator, the t–structure we take is the standard one, and we set $A = 1$.

It's clear that $R^\perp$ contains $\mathbf{D}(R)^{\leq -1} \cup \mathbf{D}(R)^{\geq 1}$.

Example (the category $\mathbf{D}(R)$)

Let R be a ring. The object $R \in \mathbf{D}(R)$ is a compact generator, the t–structure we take is the standard one, and we set $A = 1$.

It's clear that $R^\perp$ contains $\mathbf{D}(R)^{\leq -1} \cup \mathbf{D}(R)^{\geq 1}$.

Given an object $F \in \mathbf{D}(R)^{\leq 0}$ we first replace F by a projective resolution, then form the triangle $E \longrightarrow F \longrightarrow D$ below

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

with $D \in \mathbf{D}(R)^{\leq -1}$ and $E \in \overline{\langle R \rangle}_1^{[0,0]} \subset \overline{\langle R \rangle}_1^{[-1,1]}$.

The main theorems—sources of more examples

1. If $\mathcal{T}$ has a compact generator G so that $\mathrm{Hom}\big(G, G[i]\big) = 0$ for all $i \geq 1$, then $\mathcal{T}$ is approximable.
2. Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then the category $\mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable.
3. Let X be a quasicompact, separated scheme. Then the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable.
4. [Joint with Jesse Burke and Bregje Pauwels]: Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathcal{S} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \mathcal{T}$$

   with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and that any compact object $H \in \mathcal{S}$ has the property that $\mathrm{Hom}\big(H, H[i]\big) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

The main theorems—sources of more examples

1. If $\mathcal{T}$ has a compact generator G so that $\mathrm{Hom}\big(G, G[i]\big) = 0$ for all $i \geq 1$, then $\mathcal{T}$ is approximable.
2. Let X be a quasicompact, quasiseparated scheme, and let $Z \subset X$ be a closed subset with quasicompact complement. Then the category $\mathbf{D}_{\mathbf{qc},Z}(X)$ is weakly approximable.
3. Let X be a quasicompact, separated scheme. Then the category $\mathbf{D}_{\mathbf{qc}}(X)$ is approximable.
4. [Joint with Jesse Burke and Bregje Pauwels]: Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \rightleftarrows \mathcal{S} \rightleftarrows \mathcal{T}$$

with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and that any compact object $H \in \mathcal{S}$ has the property that $\mathrm{Hom}\big(H, H[i]\big) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

Alexander A. Beĭlinson, *The derived category of coherent sheaves on* $\mathbf{P}^n$, Selecta Mathematica Sovietica, vol. 3, 1983/84, Selected translations, pp. 233–237.

Alexander A. Beĭlinson, *The derived category of coherent sheaves on* $\mathbf{P}^n$, Selecta Mathematica Sovietica, vol. 3, 1983/84, Selected translations, pp. 233–237.

Dmitri O. Orlov, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.

Let R be a commutative ring. The short exact sequence

$$0 \longrightarrow R[x] \xrightarrow{\ x\ } R[x] \longrightarrow R \longrightarrow 0$$

gives a quasi-isomorphism of R with the complex

$$0 \longrightarrow R[x] \xrightarrow{\ x\ } R[x] \longrightarrow 0$$

Let R be a commutative ring. The short exact sequence

$$0 \longrightarrow R[x] \xrightarrow{\ x\ } R[x] \longrightarrow R \longrightarrow 0$$

gives a quasi-isomorphism of R with the complex

$$0 \longrightarrow R[x] \xrightarrow{\ x\ } R[x] \longrightarrow 0$$

Tensoring together $n+1$ of these we deduce a quasi-isomorphism of R with the Koszul complex

$$\bigotimes_{i=0}^{n} \left(R[x_i] \xrightarrow{x_i} R[x_i] \right)$$

Applying Proj to this, we obtain a quasi-isomorphism of $\mathcal{O}(1)$ with a complex

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

Applying Proj to this, we obtain a quasi-isomorphism of $\mathcal{O}(1)$ with a complex

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

Tensoring this with itself $\ell > 0$ times yields a quasi-isomorphism of $\mathcal{O}(\ell)$ with some complex

$$\cdots \longrightarrow \oplus\mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

Applying Proj to this, we obtain a quasi-isomorphism of $\mathcal{O}(1)$ with a complex

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

Tensoring this with itself $\ell > 0$ times yields a quasi-isomorphism of $\mathcal{O}(\ell)$ with some complex

$$\cdots \longrightarrow \oplus\mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

which has a brutal truncation

$$0 \longrightarrow \oplus\mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

Applying Proj to this, we obtain a quasi-isomorphism of $\mathcal{O}(1)$ with a complex

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

Tensoring this with itself $\ell > 0$ times yields a quasi-isomorphism of $\mathcal{O}(\ell)$ with some complex

$$\cdots \longrightarrow \oplus\mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

which has a brutal truncation

$$0 \longrightarrow \oplus\mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

And this brutal truncation must be quasi-isomorphic to $\mathcal{O}(\ell) \oplus \mathcal{V}[n]$ for some vector bundle $\mathcal{V}$.

Applying the functor $(-)^\vee = \mathcal{RH}om(-, \mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell) \oplus \mathcal{V}^\vee[-n]$ with

$$0 \longrightarrow \oplus\mathcal{O} \longrightarrow \oplus\mathcal{O}(1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(n-1) \longrightarrow \oplus\mathcal{O}(n) \longrightarrow 0$$

Applying the functor $(-)^\vee = \mathcal{RH}om(-, \mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell) \oplus \mathcal{V}^\vee[-n]$ with

$$0 \longrightarrow \oplus\mathcal{O} \longrightarrow \oplus\mathcal{O}(1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(n-1) \longrightarrow \oplus\mathcal{O}(n) \longrightarrow 0$$

Thus with $G = \oplus_{i=0}^{n}\mathcal{O}(i)$, we have that

$$\mathcal{O}(-\ell) \in \langle G \rangle_{n+1}^{[0,n]} \qquad \text{for all } \ell > 0.$$

Applying the functor $(-)^\vee = \mathcal{RH}om(-, \mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell) \oplus \mathcal{V}^\vee[-n]$ with

$$0 \longrightarrow \oplus\mathcal{O} \longrightarrow \oplus\mathcal{O}(1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(n-1) \longrightarrow \oplus\mathcal{O}(n) \longrightarrow 0$$

Thus with $G = \oplus_{i=0}^n \mathcal{O}(i)$, we have that

$$\mathcal{O}(-\ell) \in \langle G \rangle_{n+1}^{[0,n]} \qquad \text{for all } \ell > 0.$$

Now every $F \in \mathbf{D_{qc}}(X)^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$, with E a coproduct of $\mathcal{O}(-\ell)$ and with $D \in \mathbf{D_{qc}}(X)^{\leq -1}$.

Applying the functor $(-)^\vee = \mathcal{RH}om(-, \mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell) \oplus \mathcal{V}^\vee[-n]$ with

$$0 \longrightarrow \oplus\mathcal{O} \longrightarrow \oplus\mathcal{O}(1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(n-1) \longrightarrow \oplus\mathcal{O}(n) \longrightarrow 0$$

Thus with $G = \oplus_{i=0}^n \mathcal{O}(i)$, we have that

$$\mathcal{O}(-\ell) \in \langle G \rangle_{n+1}^{[0,n]} \qquad \text{for all } \ell > 0.$$

Now every $F \in \mathbf{D_{qc}}(X)^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$, with E a coproduct of $\mathcal{O}(-\ell)$ and with $D \in \mathbf{D_{qc}}(X)^{\leq -1}$. By the above we have that $E \in \overline{\langle G \rangle}_{n+1}^{[-n-1,n+1]}$.

Applying the functor $(-)^\vee = \mathcal{RH}om(-,\mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell)\oplus\mathcal{V}^\vee[-n]$ with

$$0 \longrightarrow \oplus\mathcal{O} \longrightarrow \oplus\mathcal{O}(1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(n-1) \longrightarrow \oplus\mathcal{O}(n) \longrightarrow 0$$

Thus with $G = \oplus_{i=0}^n \mathcal{O}(i)$, we have that

$$\mathcal{O}(-\ell) \in \langle G\rangle_{n+1}^{[0,n]} \qquad \text{for all } \ell > 0.$$

Now every $F \in \mathbf{D_{qc}}(X)^{\leq 0}$ admits a triangle $E \longrightarrow F \longrightarrow D$, with E a coproduct of $\mathcal{O}(-\ell)$ and with $D \in \mathbf{D_{qc}}(X)^{\leq -1}$. By the above we have that $E \in \overline{\langle G\rangle}_{n+1}^{[-n-1,n+1]}$.

And $G^\perp$ clearly contains $\mathbf{D_{qc}}(X)^{\leq -n-1} \cap \mathbf{D_{qc}}(X)^{\geq n+1}$.

It's time to come to the applications to algebraic geometry. Before stating the next two we remind the reader what the terms used in the theorems mean.

Some old definitions

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object.

- G is a classical generator if $\mathcal{S} = \langle G \rangle^{(-\infty,\infty)}$.
-

It's time to come to the applications to algebraic geometry. Before stating the next two we remind the reader what the terms used in the theorems mean.

Some old definitions

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object.

- G is a classical generator if $\mathcal{S} = \langle G \rangle^{(-\infty,\infty)}$.
- G is a strong generator if there exists an integer $\ell > 0$ with $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$. The category $\mathcal{S}$ is strongly generated if there exists a strong generator $G \in \mathcal{S}$.

The main theorems

1. Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.

2.

The main theorems

1. Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.

2. Let X be a finite-dimensional, separated, quasiexcellent noetherian scheme. Then the category $\mathbf{D}^{b}_{\mathbf{coh}}(X)$ is strongly generated.

- Ko Aoki, *Quasiexcellence implies strong generation*, J. Reine Angew. Math. (published online 14 August 2021, 6 pages), see also https://arxiv.org/abs/2009.02013.
- Amnon Neeman, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^{b}_{\mathbf{coh}}(X)$*, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

What was known before about strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$

- If X is an affine scheme, the theorem goes back to a 1965 article by Max Kelly. What's more the proof is easy, we will give it later in the slides.
-
-

What was known before about strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$

- If X is an affine scheme, the theorem goes back to a 1965 article by Max Kelly. What's more the proof is easy, we will give it later in the slides.
- If X is smooth over a field k, the theorem may be found in a 2003 article by Bondal and Van den Bergh.
-

What was known before about strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$

- If X is an affine scheme, the theorem goes back to a 1965 article by Max Kelly. What's more the proof is easy, we will give it later in the slides.
- If X is smooth over a field k, the theorem may be found in a 2003 article by Bondal and Van den Bergh.
- If X is regular and of finite type over a field, the theorem may be deduced from either a 2008 result of Rouquier, or a 2016 theorem of Orlov.

Alexei I. Bondal and Michel Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

G. Maxwell Kelly, *Chain maps inducing zero homology maps*, Proc. Cambridge Philos. Soc. **61** (1965), 847–854.

Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.

What was known before about strong generators in $\mathbf{D}^b_{\mathbf{coh}}(X)$

- If X is regular and finite-dimensional then $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^b_{\mathbf{coh}}(X)$, and the result follows easily from the work on $\mathbf{D}^{\mathrm{perf}}(X)$ mentioned on previous slides.
-
-

What was known before about strong generators in $\mathbf{D}^b_{\mathbf{coh}}(X)$

- If X is regular and finite-dimensional then $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^b_{\mathbf{coh}}(X)$, and the result follows easily from the work on $\mathbf{D}^{\mathrm{perf}}(X)$ mentioned on previous slides.
- If X is of finite type over a perfect field k, the theorem may be found in a 2008 article by Rouquier.
-

What was known before about strong generators in $\mathbf{D}^b_{\mathbf{coh}}(X)$

- If X is regular and finite-dimensional then $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^b_{\mathbf{coh}}(X)$, and the result follows easily from the work on $\mathbf{D}^{\mathrm{perf}}(X)$ mentioned on previous slides.

- If X is of finite type over a perfect field k, the theorem may be found in a 2008 article by Rouquier.

- The generalization to X of finite type over an arbitrary field may be found in a 2008 preprint by Keller and Van den Bergh. [The article appeared in 2011, with an appendix by Murfet, but with the result relevant to us here omitted.] A different proof may be found in a 2010 paper by Lunts.

- Bernhard Keller and Michel Van den Bergh, *On two examples by Iyama and Yoshino*, (e-print http://arXiv.org/abs/0803.0720v1).
- Valery A. Lunts, *Categorical resolution of singularities*, J. Algebra **323** (2010), no. 10, 2977–3003.
- Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.

What was known before (continued)

- Suppose X is affine—the question was studied in several papers by Takahashi and coathors. The union of the results says: $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated as long as either X is essentially of finite type over a field, or else it is the spectrum of an equicharacteristic complete local ring.

What was known before (continued)

- Suppose X is affine—the question was studied in several papers by Takahashi and coathors. The union of the results says: $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated as long as either X is essentially of finite type over a field, or else it is the spectrum of an equicharacteristic complete local ring.

Note the contrast:

If X is finite-dimensional, regular and affine, the strong generation of $\mathbf{D}^b_{\mathbf{coh}}(X) = \mathbf{D}^{\mathrm{perf}}(X)$ is easy and goes back to a 1965 theorem by Max Kelly. If X is still affine, but we allow singularities, the strong generation of $\mathbf{D}^b_{\mathbf{coh}}(X)$ is decidedly non-trivial.

- Takuma Aihara and Ryo Takahashi, *Generators and dimensions of derived categories of modules*, Comm. Algebra **43** (2015), no. 11, 5003–5029.
- Abdolnaser Bahlekeh, Ehsan Hakimian, Shokrollah Salarian, and Ryo Takahashi, *Annihilation of cohomology, generation of modules and finiteness of derived dimension*, Q. J. Math. **67** (2016), no. 3, 387–404.
- Hailong Dao and Ryo Takahashi, *The radius of a subcategory of modules*, Algebra Number Theory **8** (2014), no. 1, 141–172.
- Srikanth B. Iyengar and Ryo Takahashi, *Annihilation of cohomology and strong generation of module categories*, Int. Math. Res. Not. IMRN (2016), no. 2, 499–535.

Recall: a strong generator in $\mathcal{S}$ is an object $G \in \mathcal{S}$ such that, for some integer $\ell > 0$, we have $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$. One can ask for estimates on ℓ. This leads to the definitions

-
-
-

Recall: a strong generator in $\mathcal{S}$ is an object $G \in \mathcal{S}$ such that, for some integer $\ell > 0$, we have $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$. One can ask for estimates on ℓ. This leads to the definitions

Bounds on the integer ℓ

- Given objects $G, F \in \mathcal{S}$, the G–level of F is the smallest integer ℓ such that $F \in \langle G \rangle_\ell^{(-\infty,\infty)}$. [This notion is due to Avramov, Buchweitz and Iyengar].
-
-

Recall: a strong generator in $\mathcal{S}$ is an object $G \in \mathcal{S}$ such that, for some integer $\ell > 0$, we have $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$. One can ask for estimates on ℓ. This leads to the definitions

Bounds on the integer ℓ

- Given objects $G, F \in \mathcal{S}$, the G–level of F is the smallest integer ℓ such that $F \in \langle G \rangle_\ell^{(-\infty,\infty)}$. [This notion is due to Avramov, Buchweitz and Iyengar].
- Let G be an object of $\mathcal{S}$. The Orlov dimension of G is the smallest ℓ for which $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$.
-

Recall: a strong generator in $\mathcal{S}$ is an object $G \in \mathcal{S}$ such that, for some integer $\ell > 0$, we have $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$. One can ask for estimates on ℓ. This leads to the definitions

Bounds on the integer ℓ

- Given objects $G, F \in \mathcal{S}$, the G–level of F is the smallest integer ℓ such that $F \in \langle G \rangle_\ell^{(-\infty,\infty)}$. [This notion is due to Avramov, Buchweitz and Iyengar].
- Let G be an object of $\mathcal{S}$. The Orlov dimension of G is the smallest ℓ for which $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$.
- The Rouquier dimension of $\mathcal{S}$ is the smallest integer ℓ such that there exists a G with $\mathcal{S} = \langle G \rangle_\ell^{(-\infty,\infty)}$.

There are several conjectures, and many papers estimating these numbers—all in the equal charateristic case, after all until recently it wasn't known to be finite in mixed characteristic. One can ask if the theorems surveyed in this article give good bounds in mixed characteristic.

There are several conjectures, and many papers estimating these numbers—all in the equal charateristic case, after all until recently it wasn't known to be finite in mixed characteristic. One can ask if the theorems surveyed in this article give good bounds in mixed characteristic.

The short answer is No. In more detail:

No good bounds in mixed characteristic

- If we assume that X is regular and quasiprojective, then the proof of strong generation is effective. It gives an explicit upper bound on the the Rouquier dimension of $\mathbf{D}^b_{\mathbf{coh}}(X)$. But the bound is dreadful.
-

There are several conjectures, and many papers estimating these numbers—all in the equal charateristic case, after all until recently it wasn't known to be finite in mixed characteristic. One can ask if the theorems surveyed in this article give good bounds in mixed characteristic.

The short answer is No. In more detail:

No good bounds in mixed characteristic

- If we assume that X is regular and quasiprojective, then the proof of strong generation is effective. It gives an explicit upper bound on the Rouquier dimension of $\mathbf{D}^b_{\mathbf{coh}}(X)$. But the bound is dreadful.
- If we drop the quasiprojectivity hypothesis, and/or if we allow singularities, then the proof becomes ineffective. It proves the existence of an integer $\ell > 0$ and a generator G with $\mathbf{D}^b_{\mathbf{coh}}(X) = \langle G \rangle_\ell^{(-\infty,\infty)}$, but there is no estimate on ℓ.

Reminder: finite homological functors

Let R be a commutative, noetherian ring, and let $\mathcal{S}$ be an R-linear triangulated category. An R–linear homological functor $H : \mathcal{S} \longrightarrow R\text{–}\mathrm{Mod}$ is *finite* if, for all objects $C \in \mathcal{S}$, the R–module $\oplus_i H^i(C)$ is finite.

Reminder: a key application of strong generation

Theorem

Let R be a commutative, noetherian ring, and let $\mathcal{S}$ be an R-linear triangulated category. Assume

1. *The category $\mathcal{S}$ has a strong generator.*
2. *For any pair of objects $X, Y \in \mathcal{S}$ we have that $\mathrm{Hom}(X, Y)$ is a finite R-module, and $\mathrm{Hom}(X, Y[n])$ vanishes for all but finitely many n.*

Then every finite homological functor $F : \mathcal{S} \longrightarrow R\text{–}\mathrm{mod}$ is representable.

Alexei I. Bondal and Michel Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

Raphaël Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.

Thank you!